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A unique continuation theorem for exterior differential forms on Riemannian manifolds

By N. Aronszajn, A. Krzywicki and J. Szarski (1)

Introduction

The aim of the present paper is to prove a theorem which may be stated as follows (a more precise formulation will be given later):

Let $\mathfrak{M}^n$ be a Riemannian manifold with a metric of class $C^{0,1}$ (2). *If an exterior differential form u of rank p has a zero of infinite order at a point P_0 of the manifold, and if in each compact part of any coordinate patch the components of the differential du and of the codifferential δu are majorated by a constant times the sum of absolute values of components of u, then $u=0$ on the whole manifold.*

This theorem is of the type known as *strong* unique continuation theorems which extend the classical property of analytic functions to other classes of functions (3). Such theorems establish the basic characteristic property of quasi-analytic classes of functions of one real variable (Carleman [5], Mandelbrojt [11]). This property was proved by Carleman [6] for solutions of elliptic systems of two linear equations of first order in two variables. In 1954–1955 it was proved by C. Müller [12] and E. Heinz [8] for solutions of differential inequalities of the type

$$|\Delta u(x)| \leqslant M\left[|u(x)| + \sum_{k=1}^{n}\left|\frac{d\,u(x)}{d\,x^k}\right|\right]$$

in n variables, Δ being the usual Laplacian, and M a constant.

One of the present authors was able to replace the Laplacian in the inequality by a general elliptic operator of second order with coefficients in class $C^{2,1}$ [1, 2]. Soon afterwards, Cordes [7] proved a theorem in the same case but with coefficients only in C^2. Some other unique continuation theorems were proved by Pederson [13].

In most of the proofs of strong unique continuation theorems for the different cases considered, the essential part was the establishment of an inequality, the

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(2) In general $C^{m,\lambda}$ is the class of functions with continuous derivatives of orders $\leqslant m$ and with mth derivatives satisfying a Hölder condition with exponent λ. Hence, $C^{0,1}$ means "continuous and Lipschitzian".

(3) In a *weak* unique continuation theorem one assumes that the form vanishes in an open non-empty set instead of merely having a single zero of infinite order.

general idea of which was introduced by Carleman, and we shall call these "Carleman type" inequalities.

Considerable progress has been made recently in the investigation of the question of the uniqueness of solutions of Cauchy problem ([1]), a question very much akin to that of the unique continuation. Most of the recent proofs of uniqueness for Cauchy problem also rely on an inequality of Carleman type.

Every case in which the non-uniqueness in the Cauchy problem is established is one in which unique continuation (weak or strong) is not valid.

In a recent paper, L. Hörmander [9] proved that for systems of linear operators with constant coefficients a Carleman type inequality cannot hold if there are multiple characteristics of multiplicity >2 ([2]). Quite recently, A. Pliś constructed several examples in which uniqueness in the Cauchy problem does not hold (these examples are not as yet published). Among these examples we will mention two:

(a) An elliptic equation in R^3, of order 4 with real C^∞ coefficients ([3]).

(b) An elliptic equation in R^3 of order 2 (hence with simple characteristics) with coefficients in C^∞ outside of a plane and of class $C^{0,\lambda}$ in a neighborhood of this plane, for every λ, $0<\lambda<1$.

The first example shows that the unique continuation theorem is not true for all elliptic systems (and it is only for such systems that it may be true). It is therefore of interest to prove the unique continuation for special types of systems.

The second example shows that in our theorem, the regularity requirement that the metric a_{ij} be of class $C^{0,1}$ cannot, in general, be lowered to $C^{0,\lambda}$ with $0<\lambda<1$. This follows from the fact that our results imply (see Remark 3, § 5) the unique continuation theorem for solutions of the corresponding Beltrami–Laplace equation

$$\delta d f \equiv \frac{1}{\sqrt{a}} \frac{\partial}{\partial x^i} \sqrt{a}\, a^{ij} \frac{\partial f}{\partial x^j} = 0.$$

However it will be shown in Remark 6, § 5 that the regularity requirement can be considerably lowered at the zero of infinite order.

There exists a general method for transforming an arbitrary linear differential system into a system of operators of first order. Whenever this transformation can be used, it gives rise to a considerable lowering of the regularity requirements and also a much simpler presentation of the proofs. This method transforms, in general, a well-determined system into an over-determined system. However, it transforms an elliptic system into an elliptic system. ([4])

This leads to a general investigation of the unique continuation for over-determined elliptic systems of first order. The present paper can be considered as a pilot investigation in this direction. We consider the components of a p-form u on an n-dimensional Riemannian manifold as forming (locally) a system

(1) See, for example, A. P. Calderon [4] and L. Hörmander [9, 10].

(2) Hörmander, however, considers inequalities of rather special type, adapted to the Cauchy problem. In this connection see footnote in Remark 2, § 5.

(3) To be published in Communications on Pure and Applied Mathematics.

(4) We call an arbitrary system of linear differential operators elliptic if the sum of absolute squares of their characteristic polynomials is positive definite.

of $\binom{n}{p}$ functions. The components of the differential du and of the codifferential δu form a system of $\left[\binom{n}{p+1}+\binom{n}{p-1}\right]$ linear operators. This system is elliptic (even minimal elliptic, i.e. when a single operator is taken away, the system ceases to be elliptic). The reasons for our choice of this system are that 1° the operators d and δ are of importance in many domains of mathematics and 2° the investigation is made easier by readily available tools of the theory of exterior differential forms.

In § 1 we give the notations and formulas concerning exterior differential forms on Riemannian manifolds which are to be used in this paper.

In § 2 we first state our theorem in a precise form and then proceed to a series of reductions and transformations of the theorem. First we reduce it to the case $n=2p$ (Proposition 1). In further transformations we reduce it to the case when, in a neighborhood of P_0—the zero of infinite order—the geodesic spheres with center P_0 coincide with concentric Euclidean spheres (or ellipsoids). This reduction allows us to manage with a metric only of class $C^{0,1}$. A final transformation brings in the reduction to a Carleman type inequality (2.13).

In § 3 we prove the inequality, but our proof is valid only for $n=2p$, which explains the necessity for the reduction of our theorem to this case.

In order to simplify the presentation of the arguments in §§ 2 and 3, we have relegated certain proofs to § 4, especially those of all the evaluations needed in the preceding sections. In certain instances, particularly in parts III and VI, we have had to prove some properties of exterior differential forms which do not seem to be readily available in the literature. Having in mind the extensions of our results (indicated in § 5), we have made an effort to get quite precise evaluations.

§ 5 contains six remarks treating the following subjects: 1, the best constant in the Carleman type inequality; 2, the Carleman type inequality for $n \neq 2p$; 3, on solutions of elliptic inequalities of second order; 4, the geometric significance of our inequality of Carleman type; 5, on sets of zeros of p-form satisfying differential inequalities; 6, on weakening the hypotheses in Theorem I.

§ 1. Notations and formulas

For notation concerning differential forms on Riemannian manifolds we refer the reader to the book of de Rham [14]. In some instances, however, we introduce special notations which we believe more convenient for our present purposes, and for the convenience of the reader the formulas from the theory of exterior differential forms which we will use in this paper are now given in this new notation.

On a manifold of dimension n, an exterior differential form u of rank p is given in a coordinate patch by a system of components $u_{(i)}$. Here, (i) stands for a system of indices, $i_1, i_2, \ldots, i_p$, where $1 \leqslant i_\mu \leqslant n$ for $\mu=1, 2, \ldots, p$, and the indices i_μ are *strictly increasing*.

We introduce the following notations. If (i) is a system of p indices, then $(i)^c$ is the "complementary" system, i.e., the system of $n-p$ indices j_ν between 1 and n which do not figure in (i), arranged in strictly increasing order.

If $j \in (i)^c$ then $(i) \cup j$ denotes the sequence (i) increased by the index j and arranged increasingly. If $i_\mu \in (i)$ then $(i) \setminus i_\mu$ is the sequence (i) with i_μ deleted from it. If $j_1, j_2, \ldots, j_q$ is any sequence of indices all different, then $\varepsilon [j_1, j_2, \ldots, j_q] = \pm 1$, depending on the parity of the permutation of the q indices j_ν which puts them in increasing order. We will need the following relation:

$$\varepsilon [(i)^c, (i)] = (-1)^{p(n-p)} \varepsilon [(i), (i)^c]. \tag{1.1}$$

For a p-form u we can write the components of the differential du in the given coordinate patch as

$$(du)_{(j)} = \sum_{\lambda=1}^{p+1} (-1)^{\lambda-1} \frac{\partial u_{(j)\setminus j\lambda}}{\partial x^{j_\lambda}} \tag{1.2}$$

Here, (j) is a sequence of $p+1$ indices.

Let a_{ij} be a metric tensor on the manifold. As usual, we denote by a^{ij} the corresponding contravariant tensor (the matrix $\{a^{ij}\}$ is the inverse of the matrix $\{a_{ij}\}$), by a the determinant of the matrix $\{a_{ij}\}$ so that $\sqrt{a}\,dx$ is the corresponding invariant measure on the manifold.

For two p-sequences (i) and (j) we denote by $A_{(i),(j)}$ the determinant of the minor of the matrix $\{a_{ij}\}$ with rows given by the indices i_μ and columns given by the indices j_ν. Similarly, $A^{(i),(j)}$ is formed from $\{a^{ij}\}$. Obviously,

$$A_{(1\ldots n),(1\ldots n)} = a, \quad A^{(1\ldots n),)1\ldots n)} = a^{-1}.$$

For a p-form u the components of $*u$, which is an $(n-p)$-form, are given in our notation by

$$(*u)_{(h)} = \sqrt{a}\, \varepsilon [(h)^c, (h)] \sum_{(i)} A^{(h)^c,(i)} u_{(i)}. \tag{1.3}$$

where (h) is a sequence of $n-p$ indices and (i) in the sum runs through all sequences of p indices.

We will often use the two well-known formulas for the square of the star operator and for the codifferential δ of a p-form

$$**u = (-1)^{p(n-p)} u, \tag{1.4}$$

and

$$\delta u = (-1)^{n(p+1)+1} *d*u. \tag{1.5}$$

For a p-form u the function defined on the manifold by $*(u \wedge *u)$ will be denoted by $Q(u)$. In our notation it can be written as

$$Q(u) = \sum_{(i),(j)} A^{(i),(j)} u_{(i)} u_{(j)}. \tag{1.6}$$

This is a quadratic form of u at each point of the manifold, positive definite, and its square root can be considered as a norm of the p-form u at a point of the manifold. (It is obviously independent of the coordinate patch.) The corresponding bilinear form in two p-forms u and v is

$$Q(u, v) = \sum_{(i),(j)} A^{(i),(j)} u_{(i)} v_{(j)}. \tag{1.6'}$$

The following formulas are easy to check.

$$Q(*u, *v) = Q(u, v), \quad Q(*u) = Q(u). \tag{1.7}$$

From (1.5) and (1.7) it follows that

$$Q(\delta u) = Q(d * u). \tag{1.8}$$

We have

$$\int Q(u, dv) \sqrt{a}\, dx = \int Q(\delta u, v) \sqrt{a}\, dx \tag{1.9}$$

for any two forms u, v of rank p and $p-1$ respectively, of which one at least vanishes outside of a compact. The last formula represents the fact that the operators d and δ are adjoint to one another.

In order to prevent a cumbersome number of indices we omit the index for Q (there will be no ambiguity) to indicate the rank of forms u to which Q is applied.

§ 2. The main theorem, its reductions and transformations

We now state our main theorem precisely, giving all the hypotheses in their weakest form.

Our hypotheses concerning the manifold and the metric are as follows:

(i) *$\mathfrak{M}^n$ is a manifold of class $C^{1,1}$. The metric tensor a_{ij} is locally in class $C^{0,1}$.*

Obviously we have to assume the manifold of at least class $C^{1,1}$ in order to give a meaning to $C^{0,1}$. A tensor is locally in $C^{0,1}$ if for each point of the manifold there exists a coordinate patch containing it where the components of the tensor are of class $C^{0,1}$.

We give now our hypotheses about the p-form u.

(ii) *u is locally L^2 with strong L^2-derivatives of first order.*

(iii) *At a point $P_0 \in \mathfrak{M}^n$, u has a zero of infinite order in 1-mean.*

In a Euclidean space R^n, a function f defined a.e. in a neighborhood of a point x_0 has a zero of infinite order at x_0 in q-mean ($q \geq 1$) if

$$\int_{|x-x_0|<r} |f(x)|^q\, dx = o(r^\alpha)$$

for all $\alpha > 0$ when $r \to 0$.

Obviously a zero of infinite order in q-mean, for any $q > 1$, is a zero of infinite order in 1-mean. Hypothesis (iii) means that in a coordinate patch containing P_0 all components of u have a zero of infinite order in 1-mean. (This is independent of the choice of the coordinate patch.) Even for continuous u

(iii) is weaker than the usual notion of zero of infinite order. However, (iii) jointly with (ii) implies that u has at P_0 a zero of infinite order in 2-mean (1).

Finally, our hypothesis concerning the majoration of du and δu by u may be written in the invariant form:

(iv) *For every compact* $\mathcal{K} \subset \mathcal{M}^n$ *there exists a constant* $M_{\mathcal{K}}$ such that $Q(du) + Q(\delta u) \leqslant M_{\mathcal{K}} Q(u)$ *a.e. on* $\mathcal{K}$.

Because of the weakness of hypotheses (i) and (ii), u, du, and δu exist and are well determined only a.e. on $\mathcal{M}^n$; hence we can assume the majoration in (iv) only a.e. Besides this change, hypothesis (iv) is equivalent to the corresponding hypothesis in the introduction.

We can now state our theorem as follows.

Theorem 1. *If hypotheses* (i)–(iv) *are satisfied,* $u = 0$ *a.e. on* $\mathcal{M}^n$.

Remark 1. Hypothesis (ii) could be replaced by a more restrictive one, namely:

(ii′) u *is locally in* P^1, i.e. *in each coordinate patch the components are locally potentials of order* 1 (2).

A p-form u satisfying (ii′) is defined everywhere on $\mathcal{M}^n$ except on a set of 2-capacity 0; hence it is much more precisely defined than a u satisfying (ii). However, each u satisfying (ii) is a.e. equal to a u' satisfying (ii′), so that hypothesis (ii′) is not essentially more restrictive. Replacing (ii) by (ii′) allows a more precise conclusion in Theorem 1, namely that u vanishes everywhere on $\mathcal{M}^n$ except on a set of 2-capacity 0.

We give now a series of reductions and transformations of Theorem 1.

Proposition 1. *Theorem 1 is equivalent to the special case where the dimension* n *equals twice the rank* p *of the form* u.

Proof. We first notice that in hypothesis (iv) the inequality can be written $Q(du) + Q(d{*}u) \leqslant M_{\mathcal{K}} Q(u) = M_{\mathcal{K}} Q({*}u)$, (by virtue of (1.5) and (1.7)). It is therefore obvious that the hypotheses and the conclusion of Theorem 1 are equivalent when stated for u and ${*}u$. Since ${*}u$ is an $(n-p)$-form, and $0 \leqslant p \leqslant n$, there is no loss of generality in assuming that the rank p is $\geqslant n/2$. Assume that $p > n/2$, and replace the manifold $\mathcal{M}^n$ by $\overline{\mathcal{M}}^{2p} = \mathcal{M}^n \times R^{2p-n}$. In this product manifold we consider coordinate patches which are products of coordinate patches in $\mathcal{M}^n$ by R^{2p-n}, and in these coordinate patches we define the metric tensor $\bar{a}_{ij}$ as equal to a_{ij} for i and $j \leqslant n$ and $= \delta_{ij}$ when one of the i, j is $> n$. For any q-form v in $\mathcal{M}^n$ we define its extension $\bar{v}$ (of the same rank) to $\overline{\mathcal{M}}^{2p}$ by putting $\bar{v}_{(i)}(x_1, \ldots, x_n, \ldots, x_{2p}) = v_{(i)}(x_1, \ldots, x_n)$ when the sequence (i) contains only indices $\leqslant n$ and $\bar{v}_{(i)} = 0$ when the sequence (i) contains at least one index $> n$.

Denote by $\bar{*}$, $\bar{d}$, $\bar{\delta}$, $\bar{Q}$, the operators and the forms Q on $\overline{\mathcal{M}}^{2p}$. By using the formulas from § 1 we check immediately that $\bar{d}\bar{v} = \overline{(dv)}$, $\bar{Q}(\bar{d}\bar{v}) = Q(dv)$ and $\bar{Q}(\bar{\delta}\bar{v}) = \bar{Q}(\bar{d}\bar{*}\bar{v}) = Q(d{*}v) = Q(\delta v)$.

It follows that hypothesis (iv) for $\bar{u}$ on $\overline{\mathcal{M}}^{2p}$ is implied by the same hypothesis for u on $\mathcal{M}^n$. On the other hand, hypotheses (i)–(iii) for $\bar{a}_{ij}$ and $\bar{u}$ are

(1) See § 4, I.

(2) For the definition and properties of classes P^{α}, see Aronszajn and Smith [3].

obviously implied by the corresponding hypotheses for a_{ij} and u (the point $\bar{P}_0$ can be chosen as $\{P_0, 0\}$). Hence if the theorem is true for $n=2p$, we get $\bar{u}=0$ a.e. on $\mathfrak{M}^{2p}$ and thus $u=0$ a.e. on $\mathfrak{M}^n$.

Remark 2. We use this reduction of the theorem only at the end of our proof in § 3. It will then become clear why it is necessary for the proof to impose the restriction $n=2p$.

We might restrict ourselves to coordinate patches U on $\mathfrak{M}^n$ whose images in R^n are not the whole of R^n. We can transfer the Euclidean metric from R^n to U, and then consider the distance of $P \in U$ to the boundary of U (in R^n). This distance will be denoted by $2r_0(P)$.

Proposition 2. *Let the coordinate patches U_k form a locally finite covering of $\mathfrak{M}^n$. Theorem* 1 *is equivalent to the following statement: If hypotheses* (i), (ii) *and* (iv) *hold, then for each U_k there exists a continuous positive function $r_1(s)$ for $s>0$ ($r_1(s)$ depends on U_k) such that $r_1(s) \leqslant s$ and whenever* (iii) *is valid with $P_0 \in U_k$, u vanishes a.e. in a Euclidean sphere around P_0 with radius $r_1(r_0(P_0))$.*

Proof. We have only to prove that the statement implies Theorem 1. If the hypotheses of Theorem 1 are satisfied, there exists an open neighborhood of P_0 where u vanishes a.e. Consider the largest open set G where u vanishes a.e. If $G=\mathfrak{M}^n$, Theorem 1 is proved. On the other hand, if G were not equal to $\mathfrak{M}^n$, there would be a boundary point P_0' of G in $\mathfrak{M}^n$. P_0' lies in some U_k, and for arbitrarily small ε there will exist a $P_0'' \in G \cap U_k$ with Euclidean distance $<\varepsilon$ from P_0'. Therefore, $|r_0(P_0') - r_0(P_0'')| \leqslant \varepsilon/2$. Since hypothesis (iii) is satisfied at P_0'', it is clear from our statement that u must vanish a.e. in the Euclidean sphere with radius $r_1(r_0(P_0''))$ around P_0'', and for ε small enough, by continuity of $r_1(s)$, this sphere will contain P_0' and thus $P_0' \in G$, which is impossible.

Proposition 2 allows us to restrict ourselves to a single coordinate patch on $\mathfrak{M}^n$. We can even choose the coordinate patch to be a relatively compact subdomain of another coordinate patch and transfer our considerations to its image in R^n. We can further restrict ourselves to the case where the image of the coordinate patch is the sphere $|x|<2r_0$, the image of P_0 being 0. We are led therefore to the following statement which implies Theorem 1.

Theorem 2. *There exists a positive function r_1, $r_1 \equiv r_1(r_0, \Lambda_1, \Lambda_2, \Lambda, M, p, n)$, defined for positive r_0, Λ_1, Λ_2, Λ, and M with $\Lambda_1 \leqslant \Lambda_2$, and integers p and n with $0 \leqslant p \leqslant n$, which is continuous in r_0, Λ_1, Λ_2, Λ, and M, and which has the following property: if a_{ij} is a metric tensor and u is a p-form defined in $[|x|<2r_0] \subset R^n$, then $u=0$* a.e. *in $|x| \leqslant r_1$ if the following conditions are satisfied:*

There exist positive constants Λ_1 and Λ_2 such that for every

$$x \in [|x|<2r_0], \quad \Lambda_1 |\xi|^2 \leqslant a_{ij}(x)\, \xi^i \xi^j \leqslant \Lambda_2 |\xi|^2 \, (^1). \tag{2.1}$$

On all straight segments contained in $[|x|<2r_0]$ the a_{ij} are absolutely continuous and there exists a positive constant Λ such that

(1) We use the usual summation convention for tensors. $|\xi|$ is the Euclidean norm of ξ.

$$\left|\frac{\partial a_{ij}}{\partial e}\xi^i\xi^j\right| \leqslant \Lambda\,|\xi|^2 \tag{2.1'}$$

for every vector ξ *and every vector* e (1) *whenever all* $\partial a_{ij}/\partial e$ *exist.*

$$u \textit{ is } L^2 \textit{ and has strong } L^2 \textit{ first derivatives in } |x| < 2r_0. \tag{2.2}$$

$$u \textit{ has a zero of infinite order in 2-mean at } 0. \tag{2.3}$$

There exists a constant M *such that a.e. in* $|x| < 2r_0$,

$$Q(du) + Q(\delta u) \leqslant M\,Q(u). \tag{2.4}$$

Our manifold is now the Euclidean sphere $|x| < 2r_0$. In all the hypotheses, the local conditions have become global. Properties (2.1) and (2.1′) give a quantitative form for hypothesis (i). In fact (2.1) means that the matrix $a_{ij}(x)$ is uniformly bounded and uniformly positive definite in $|x| < 2r_0$, whereas (2.1′) means that the derivatives $\partial a_{ij}/\partial e$ are uniformly bounded wherever they exist and this in turn, with the absolute continuity of a_{ij} is equivalent to a_{ij} being $C^{0,1}$ in $|x| < 2r_0$.

Proof of Theorem 2. We assume that properties (2.1), (2.1′), (2.2), (2.3), and (2.4) hold. Put

$$r \equiv r(x) = (a_{ij}(0)\,x^i x^j)^{\frac{1}{2}}, \tag{2.5}$$

$$\tilde{a}_{ij}(x) = a_{ij}(x)\left[a^{kl}(x)\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l}\right] \tag{2.6}$$

$\tilde{a}_{ij}$ is a new metric tensor in $|x| < 2r_0$ for which we have the following evaluations: (2)

$$\tilde{\Lambda}_1|\xi|^2 \leqslant \tilde{a}_{ij}(x)\,\xi^i\xi^j \leqslant \tilde{\Lambda}_2|\xi|^2 \quad \textit{for every vector } \xi \tag{2.7}$$

with
$$\tilde{\Lambda}_1 = \frac{\Lambda_1^2}{\Lambda_2} \quad \textit{and} \quad \tilde{\Lambda}_2 = \frac{\Lambda_2^2}{\Lambda_1}.$$

$$\textit{a.e. in } [|x| < 2r_0], \quad \left|\frac{\partial \tilde{a}_{ij}}{\partial e}\xi^i\xi^j\right| \leqslant \tilde{\Lambda}\,|\xi|^2 \quad \textit{for every } e \tag{2.7'}$$

and ξ *with*
$$\tilde{\Lambda} = \frac{6\Lambda\Lambda_2^2}{\Lambda_1^2}\sqrt{\frac{\Lambda_2}{\Lambda_1}}.$$

Introducing the corresponding $\tilde{A}$, $\tilde{\delta}$, and $\tilde{Q}$, we deduce from (2.4) the evaluation

$$\textit{a.e. in } [|x| < 2r_0], \quad \tilde{Q}(du) + \tilde{Q}(\tilde{\delta}u) \leqslant \tilde{M}\,\tilde{Q}(u). \tag{2.8}$$

(1) $\partial/\partial e$ is the derivative in the direction of the unit vector e.

(2) The proof of these evaluations is given in § 4, II.

In § 4, III, we give $\tilde{M}$ as function of M, Λ_1, Λ_2, Λ, n, and p. Below we give the value for $n=2p$ (see (4.III.2)):

$$\tilde{M} = \frac{\Lambda_2}{\Lambda_1} M \quad \textit{for } n=2p. \tag{2.8'}$$

We now introduce a positive function θ (to be determined at the end of § 3) $\theta \equiv \theta(r_0, \Lambda_1, \Lambda_2, \Lambda, p, n)$ which is continuous in r_0, Λ_1, Λ_2, Λ.

We put
$$\tilde{\tilde{a}}_{ij}(x) = \tilde{a}_{ij}(x)\, e^{-2\theta r}, \tag{2.9}$$

$$R \equiv R(x) = \frac{1}{\theta}(1 - e^{-\theta r}). \tag{2.10}$$

$\tilde{\tilde{a}}_{ij}$ is again a metric tensor in $[|x| < 2r_0]$ and we have the following evaluations (1).

$$\tilde{\tilde{\Lambda}}_1 |\xi|^2 \leqslant \tilde{\tilde{a}}_{ij}(x)\, \xi^i \xi^j \leqslant \tilde{\tilde{\Lambda}}_2 |\xi|^2 \quad \textit{for every } \xi \tag{2.11}$$

with
$$\tilde{\tilde{\Lambda}}_1 = \frac{\Lambda_1^2}{\Lambda_2} e^{-4\theta r_0} \quad \textit{and} \quad \tilde{\tilde{\Lambda}}_2 = \frac{\Lambda_2^2}{\Lambda_1}.$$

$$\textit{a.e. in } [|x| < 2r_0], \quad \left| \frac{\partial \tilde{\tilde{a}}_{ij}}{\partial e} \xi^i \xi^j \right| \leqslant \tilde{\tilde{\Lambda}} |\xi|^2 \quad \textit{for every } e \tag{2.11'}$$

and ξ *with*
$$\tilde{\tilde{\Lambda}} = \frac{6\Lambda\Lambda_2^2}{\Lambda_1^2}\sqrt{\frac{\Lambda_2}{\Lambda_1}} + \frac{2\theta\Lambda_2^2\sqrt{\Lambda_2}}{\Lambda_1}$$

$$\textit{a.e. in } [|x| < 2r_0], \quad \tilde{\tilde{Q}}(du) + \tilde{\tilde{Q}}(\tilde{\tilde{\delta}} u) \leqslant \tilde{\tilde{M}}\, \tilde{\tilde{Q}}(u) \tag{2.12}$$

The expression of $\tilde{\tilde{M}}$ is given in § 4, III. For $n=2p$ we have (see (4.III.3))

$$\tilde{\tilde{M}} = \frac{\Lambda_2}{\Lambda_1} e^{4\theta r_0} M \quad \textit{for } n=2p. \tag{2.12'}$$

Our proof is now achieved by using the following lemma which gives an inequality of Carleman type and which will be proved in § 3.

Lemma. *Let the continuous metric tensor* a_{ij} *satisfy* (2.1) *and* (2.1′). *There exists a positive function* R_0 *depending only on* r_0, Λ_1, Λ_2, Λ, p *and* n, *continuous in* r_0, Λ_1, Λ_2, Λ, *with the following property: for every* $s \leqslant R_0$ *and for every p-form* v *satisfying* (2.2) *and* (2.3) *and vanishing outside of a compact contained in* $[R(x) < s]$, *we have*

(1) The proof is similar to the one for (2.7), (2.7′), and (2.8) and is given in § 4, II–III.

$$s^2 \int\limits_{R(x)<s} R^{-2\alpha} [\dot{\tilde{Q}}(d\,v) + \dot{\tilde{\tilde{Q}}}(\tilde{\tilde{\delta}}\,v)] \sqrt{\tilde{\tilde{a}}}\,d\,x \geqslant \int\limits_{R(x)<s} R^{-2\alpha} \dot{\tilde{Q}}(v) \sqrt{\tilde{\tilde{a}}}\,d\,x \tag{2.13}$$

for every $\alpha \geqslant 0$.

Remark 3. Actually this lemma will be proved only for $n=2p$ and it is for this reason that we need the reduction of Theorem 1 to the case $n=2p$ (see Proposition 1). However, since the need for this restriction will become apparent only at the end of § 3, we will proceed now as if no such restriction were imposed. We could consider the inequality (2.13) with a positive constant C multiplying the left-hand side of it. In § 3 we prove the inequality for $n=2p$ with the constants $C=1$, $\theta=(2p-1)\omega$, and $R_0 = \dfrac{2}{5(2p-1)\omega}$, where

$$\omega = 30\,\Lambda\,\Lambda_2^{\frac{5}{2}} \sqrt{\Lambda_2}/\Lambda_1^{\frac{7}{2}}.$$

In § 5, Remark 1, we will indicate that the constant $C=1$ could be replaced by any $C > 1/j_{01}^2 = 0.1729\ldots$ with a suitable choice of R_0; here j_{01} is the first positive zero of the Bessel function $J_0(z)$.

The proof of Theorem 2, based on the lemma, is a standard procedure in unique continuation theorems. We will give a brief sketch of this proof since we wish to determine the function r_1 which is asked for in our theorem.

We notice first that by (2.1) written for $x=0$ and by (2.5) and (2.10) we have

$$\Lambda_1^{\frac{1}{2}} |x| \leqslant r(x) \leqslant \Lambda_2^{\frac{1}{2}} |x|, \quad \Omega = \frac{1 - e^{-2\theta r_0 \sqrt{\Lambda_2}}}{2\theta r_0 \sqrt{\Lambda_2}} \leqslant \frac{R(x)}{r(x)} \leqslant 1. \tag{2.14}$$

Hence
$$\Lambda_1^{\frac{1}{2}} \Omega |x| \leqslant R(x) \leqslant \Lambda_2^{\frac{1}{2}} |x|. \tag{2.15}$$

We shall prove that our theorem holds with r_1 given by

$$r_1 = \min \left[\left(\frac{\Lambda_1}{\Lambda_2}\right)^{\frac{1}{2}} \Omega\, r_0,\ \frac{R_0}{\sqrt{\Lambda_2}},\ \frac{1}{\sqrt{\tilde{\tilde{M}} \Lambda_2}} \right]. \tag{2.16}$$

We must show that the form u vanishes a.e. in the sphere $|x| < r_1$. By (2.15) it is enough to show that u vanishes a.e. in $[R(x) < s]$ for

$$s < \min \left[\Omega\, r_0 \sqrt{\Lambda_1}, R_0, 1/\sqrt{\tilde{\tilde{M}}} \right].$$

From (2.15) it is also clear that the set $[R(x) < s] \subset [|x| < r_0]$.

Consider any positive s_1 and s_2 with $s_1 < s_2 < s$. Let φ be a function in C^∞, $=1$ for $[R(x) < s_2]$ and vanishing outside of a compact in $[R(x) < s]$. Then $v = \varphi u$ is a p-form satisfying the hypothesis of the lemma, and using the inequalities (2.13), and (2.12) for $x \in [R(x) < s_2]$ where $v = u$, we arrive at the inequality

$$(1-\tilde{\tilde{M}}s^2)\int_{R(x)<s_1} R^{-2\alpha}\,\tilde{\tilde{Q}}(u)\sqrt{\tilde{\tilde{a}}}\,dx \leqslant s^2 \int_{s_1\leqslant R(x)<s} R^{-2\alpha}\,[\tilde{\tilde{Q}}(dv)+\tilde{\tilde{Q}}(\delta v)]\sqrt{\tilde{\tilde{a}}}\,dx,$$

which is valid for arbitrarily large α. Therefore $\tilde{\tilde{Q}}(u)=0$ a.e. in $[R(x)<s_1]$. $u=0$ a.e. in the same set and since s_1 can be taken arbitrarily near s, the proof is achieved.

Remark 4. In the case $n=2p$ we get a value of r_1 by putting $\theta=(2p-1)\omega$, $R_0=2/5(2p-1)\omega$ and $\tilde{\tilde{M}}=(\Lambda_2/\Lambda_1)e^{4\theta r_0}M$ (see Remark 3 and (2.12′)). This value depends on Λ_1, Λ_2, Λ, M, r_0 and p. If we want a value of r_1 for $n\neq 2p$, we have first to extend the manifold, the metric and the form as in the proof of Proposition 1. If we extend the metric by $a_{ij}=\Gamma\,\delta_{ij}$, when one of the i, j is $>n$, with a constant Γ between Λ_1 and Λ_2, all the relations in the proof of Proposition 1 will stay unchanged and in addition the inequalities (2.1) and (2.1′) will hold for the extended metric with the same constants Λ_1, Λ_2 and Λ. It follows that for $n\neq 2p$ we get the same expression for r_1 as in the case $n=2p$ provided we replace p by $\max\,[p, n-p]$.

§ 3. Proof of the inequality

We remark first that the lemma of the preceding section will be proved in all generality if we show that it holds under the following assumptions: (1)

1° The metric tensor a_{ij} is defined and of class C^∞ in the sphere $|x|<3r_0/2$ and satisfies there conditions (2.1) and (2.1′).

2° The p-form v is defined and of class C^∞ in the sphere $|x|<3r_0/2$ and vanishes outside of a compact lying in $[0<\varepsilon<R(x)<s]$ with ε depending on v.

From now on we assume that the above conditions are satisfied for a_{ij} and v.

The metric tensors $\tilde{a}_{ij}$ and $\tilde{\tilde{a}}_{ij}$ are then obviously of class C^∞ in the sphere $|x|<3r_0/2$ except at the origin, where in general they will be only of class $C^{0,1}$.

We will prove that $r(x)$ is the geodesic distance from 0 to x in the metric $\tilde{a}_{ij}$ and that in this metric the geodesic lines passing through the origin satisfy the following system of ordinary differential equations:

$$\frac{dx^i}{d\sigma}=\tilde{a}^{ij}\frac{\partial r}{\partial x^j},\quad i=1, 2, \ldots, n. \tag{3.1}$$

We notice first that the ellipsoid

$$S\equiv S_{\tilde{r}_0}=[r(x)<\tilde{r}_0], \tag{3.2}$$

where $\tilde{r}_0=r_0\sqrt{\Lambda_1}$ is contained in the sphere $|x|<r_0$. Take a point t on the boundary Σ of this ellipsoid and solve the system (3.1) with the initial condition

$$x(\tilde{r}_0)=t. \tag{3.3}$$

(1) This will be shown in § 4, IV.

As is well known, there exists a unique solution $x(\sigma)$ of (3.1) and (3.3) defined and of class C^∞ in an open interval of σ containing $\tilde{r}_0$ and which approaches the boundary of $|x|<3r_0/2$, or the origin, when σ approaches the endpoints of the interval(1). By (2.6) we have

$$\frac{dr(x(\sigma))}{d\sigma}=\frac{\partial r}{\partial x^i}\frac{dx^i}{d\sigma}=\tilde{a}^{ij}\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}=\left(a^{ij}\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}\right)\left(a^{kl}\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l}\right)^{-1}=1.$$

Hence $r(x(\sigma))=\sigma+c$; $c=0$ since $r(x(\tilde{r}_0))=r(t)=\tilde{r}_0$, and thus

$$r(x(\sigma))=\sigma. \tag{3.4}$$

This means that the interval where $x(\sigma)$ is defined starts at zero and when we consider the solutions $x(\sigma)$ in the interval $(0, \tilde{r}_0]$ for all points t on Σ we get a system of simple arcs joining zero to t, mutually disjoint and filling out the whole ellipsoid $\tilde{S}$.

The points of the arc joining 0 to t can be denoted by $x(\sigma)\equiv x(t;\sigma)$ and if we choose local coordinates $t^1, \ldots, t^{n-1}$ on Σ we can write $x(t^1, \ldots, t^{n-1}; \sigma)$. The manifold $\tilde{S}-(0)$ now becomes the product $\Sigma\times(0, \tilde{r}_0)$. We now introduce the polar coordinates $t^1, \ldots, t^{n-1}, r$; they correspond to the point $x(t^1, \ldots t^{n-1}; r)$ with $r\equiv r(x)$.

Writing the metric tensor in these coordinates, we get

$$\tilde{a}_{ij}\,dx^i\,dx^j=\tilde{a}_{ij}\left(\frac{\partial x^i}{\partial t_\mu}dt^\mu+\frac{\partial x^i}{\partial r}dr\right)\left(\frac{\partial x^j}{\partial t^\nu}dt^\nu+\frac{\partial x^j}{\partial r}dr\right),$$

where μ and ν run from 1 to $n-1$. Noticing that

$$\tilde{a}_{ij}\frac{\partial x^i}{\partial r}\frac{\partial x^j}{\partial r}=\tilde{a}_{ij}\tilde{a}^{ik}\frac{\partial r}{\partial x^k}\tilde{a}^{jl}\frac{\partial r}{\partial x^l}=\tilde{a}^{kl}\frac{\partial r}{\partial x^k}\frac{\partial r}{\partial x^l}=1$$

and
$$\tilde{a}_{ij}\frac{\partial x^i}{\partial t^\mu}\frac{\partial x^j}{\partial r}=\tilde{a}_{ij}\frac{\partial x^i}{\partial t^\mu}\tilde{a}^{jl}\frac{\partial r}{\partial x^l}=\frac{\partial x^l}{\partial t^\mu}\frac{\partial r}{\partial x^l}=\frac{\partial r}{\partial t^\mu}=0,$$

we obtain the following form for our metric

$$\tilde{a}_{ij}\,dx^i\,dx^j=r^2\tilde{b}_{\mu\nu}\,dt^\mu\,dt^\nu+dr^2 \quad \textit{with} \quad \tilde{b}_{\mu\nu}=\frac{1}{r^2}\tilde{a}_{ij}\frac{\partial x^i}{\partial t^\mu}\frac{\partial x^j}{\partial t^\nu}. \tag{3.5}$$

This formula indicates that the line t = constant is a geodesic line and that $\tilde{b}_{\mu\nu}(t;r)$, for r = const. is the restriction of the metric $\tilde{a}_{ij}$ to the concentric hypersurface Σ_r up to the factor r^2. It also shows that the polar coordinates $t^1 \ldots t^{n-1}, r$ are geodesic relative to the metric $\tilde{a}_{ij}$ and hence r is the geodesic distance from 0 to x.

At a point $t\in\Sigma$ consider a fixed contravariant vector τ tangential to Σ. The

(1) It should be noticed that the right-hand side of (3.1) is not continuous at the origin even that it is of class C^∞ everywhere else in $|x|<3r_0/2$.

quadratic form $\tilde{b}_{\mu\nu}\tau^\mu\tau^\nu$ considered as function of the parameter r satisfies the following inequality: (¹)

$$\left|\frac{\partial \tilde{b}_{\mu\nu}}{\partial r}\tau^\mu\tau^\nu\right| \leqslant \omega\,(\tilde{b}_{\mu\nu}\tau^\mu\tau^\nu) \quad \textit{with} \quad \omega = \frac{30\,\Lambda\,\Lambda_2^5\sqrt{\Lambda_2}}{\Lambda_1^7}. \tag{3.6}$$

If we now change the coordinate r into $R=(1/\theta)(1-e^{-\theta r})$ the manifold $S-(0)$ becomes $\Sigma\times(0,\tilde{R}_0)$, where $\tilde{R}_0=(1/\theta)(1-e^{-\theta\tilde{r}_0})$.

Consider the metric $\tilde{\tilde{a}}_{ij}$. By (2.9) we have

$$\tilde{\tilde{a}}_{ij}\,dx^i\,dx^j = e^{-2\theta r}\,(r^2\tilde{b}_{\mu\nu}\,dt^\mu\,dt^\nu + dr^2) = R^2\,b_{\mu\nu}\,dt^\mu\,dt^\nu + dR^2, \tag{3.7}$$

where we have put

$$b_{\mu\nu} = \frac{r^2\,e^{-2\theta r}}{R^2}\tilde{b}_{\mu\nu} = \Phi\,(\theta r)^{-2}\tilde{b}_{\mu\nu} \quad \textit{with} \quad \Phi(\sigma) = \frac{e^\sigma - 1}{\sigma}. \tag{3.8}$$

Formula (3.7) shows that $t^1, \ldots, t^{n-1}, R$ are polar geodesic coordinates for the metric $\tilde{\tilde{a}}_{ij}$ and hence R is the new geodesic distance from 0 to x.

We shall write the two sides of inequality (2.13) in terms of the last polar coordinates.

Since $b_{\mu\nu}(t^1, \ldots, t^{n-1}, R)$ for R = const. is a metric tensor on the corresponding hypersurface Σ_R, we will give the subscript R to all operators, etc. of § 1 referring to this metric on Σ_R (e.g. $*_R$, δ_R, Q_R, etc.).

From now on the notation (i), (j) will refer only to increasing systems of indices, all $\leqslant n-1$. If we want to consider such a system with one index $=n$, we will write $(i)\cup n$. For two systems of q indices we define $B^{(i),(j)}$ similarly to $A^{(i),(j)}$.

By virtue of (3.7) and (3.8) we have the following relations:

$$\sqrt{\tilde{\tilde{a}}} = R^{n-1}\sqrt{b} = R^{n-1}\,\Phi\,(\theta r)^{-(n-1)}\sqrt{\tilde{b}}. \tag{3.9}$$

If (i) and (j) are q-sequences, then

$$\tilde{\tilde{A}}^{(i),(j)} = R^{-2q}\,B^{(i),(j)} = R^{-2q}\,\Phi\,(\theta r)^{2q}\,\tilde{B}^{(i),(j)}. \tag{3.10}$$

For two $(q-1)$-sequences (i), (j), we have

$$\tilde{\tilde{A}}^{(i)\cup n,(j)\cup n} = R^{-2(q-1)}\,B^{(i),(j)} = R^{-2(q-1)}\,\Phi\,(\theta r)^{2(q-1)}\,\tilde{B}^{(i),(j)}, \tag{3.10'}$$

and finally for a q-sequence (i) and a $(q-1)$-sequence (j), we have

$$\tilde{\tilde{A}}^{(i),(j)\cup n} = \tilde{\tilde{A}}^{(j)\cup n,(i)} = 0. \tag{3.10''}$$

Let w be a q-form on $S-(0)$. Its components in the polar coordinates decompose into two classes: those correspoding to q-sequences (i) and those cor-

(¹) This property will be proved in § 4, V.

responding to sequences $(i') \cup n$ with (i') a $(q-1)$-sequence. On each hypersurface Σ_R, the former constitute a q-form $\hat{w}$, and the latter, a $(q-1)$-form $\hat{\hat{w}}$. These two forms are invariant under the change of local coordinates $t^1, \ldots, t^{n-1}$ and can be considered as forms defined on Σ, depending on the parameter R. By using formulas (3.10), (3.10′), and (3.10″) we see immediately that

$$\tilde{\tilde{Q}}(w) = Q_R(R^{-q}\hat{w}) + Q_R(R^{-(q-1)}\hat{\hat{w}}). \tag{3.11}$$

For the differential dw we verify (by (1.2)) the following formulas where (k) is a $(q+1)$-sequence and (k') a q-sequence:

$$(dw)\hat{}_{(k)} = (d\hat{w})_{(k)}, \tag{3.12}$$

$$(dw)\hat{\hat{}}_{(k')} = (d\hat{\hat{w}})_{(k')} + (-1)^q \left(\frac{\partial \hat{w}}{\partial R}\right)_{(k')}. \tag{3.12′}$$

For the operator $\tilde{\tilde{*}}w$ we get from (1.3), (3.10), (3.10′), and (3.10″), with (h) an $(n-q)$-sequence and (h') an $(n-q-1)$-sequence,

$$(\tilde{\tilde{*}}w)\hat{}_{(h)} = (-1)^{(n-q)} R^{n-2q+1} (*_R \hat{\hat{w}})_{(h)}, \tag{3.13}$$

$$(\tilde{\tilde{*}}w)\hat{\hat{}}_{(h')} = R^{n-2q-1} (*_R \hat{w})_{(h')}, \tag{3.13′}$$

Combining (3.12), (3.12′), (3.13) and (3.13′) we obtain, for (m) an $(n-q+1)$-sequence and for (m') an $(n-q)$-sequence,

$$(d\tilde{\tilde{*}}w)\hat{}_{(m)} = (-1)^{n-q} R^{n-2q+1} (d*_R \hat{\hat{w}})_{(m)}, \tag{3.14}$$

$$(d\tilde{\tilde{*}}w)\hat{\hat{}}_{(m')} = R^{n-2q-1} (d*_R \hat{w})_{(m')} + \left(\frac{\partial}{\partial R}(R^{n-2q+1} *_R \hat{\hat{w}})\right)_{(m')}. \tag{3.14′}$$

Applying the above formulas to our p-form v, we notice that $\hat{v}$ and $\hat{\hat{v}}$ vanish for R outside of a closed interval contained in $(0, s)$. Assuming $s \leqslant \tilde{R}_0$ we can write the inequality (2.13) in polar coordinates in the following form

$$\begin{aligned} s^2 \int_0^s \int_\Sigma R^{-2\alpha} \Bigg[& Q_R(R^{-p-1} d\hat{v}) + Q_R\left(R^{-p}\left(d\hat{\hat{v}} + (-1)^p \frac{\partial \hat{v}}{\partial R}\right)\right) + Q_R(R^{-p} d*_R \hat{\hat{v}}) \\ & + Q_R\left(R^{-p-1}\left(d*_R \hat{v} + R^2 \frac{\partial}{\partial R}(*_R \hat{v}) + (n-2p+1) R *_R \hat{v}\right)\right)\Bigg] R^{n-1}\sqrt{b}\, dt\, dR \\ & \geqslant \int_0^s \int_\Sigma R^{-2\alpha} [Q_R(R^{-p}\hat{v}) + Q_R(R^{-(p-1)}\hat{\hat{v}})] R^{n-1}\sqrt{b}\, dt\, dR. \end{aligned} \tag{3.15}$$

We now change the variable R into ϱ by

$$R = e^{-\varrho}, \tag{3.16}$$

which transforms the manifold $S-(0)$ into $\Sigma\times(\varrho_0, \infty)$, with $\varrho_0=-\log(\tilde{R}_0)$. We put

$$\beta=\alpha-\frac{n}{2}+1, \tag{3.17}$$

and then write the forms $\hat{v}$ and $\hat{\hat{v}}$ in the following way:

$$\hat{v}=e^{-(\beta+p)\varrho}V, \quad \hat{\hat{v}}=e^{-(\beta+p-1)\varrho}U. \tag{3.18}$$

Denoting $-\log s$ by σ and replacing the subscript R in the operators by the corresponding value $\varrho=-\log R$, we write inequality (3.15) (which is to be proved) in the form:

$$\begin{aligned}e^{-2\sigma}\int_\sigma^\infty\int_\Sigma [&Q_\varrho(d\,V)+Q_\varrho(d\,U+(-1)^p(\beta+p)\,V-(-1)^p\,V')\\ &+Q_\varrho(d{*_\varrho}U)+Q_\varrho(d{*_\varrho}V+(\beta+n-p){*_\varrho}U-({*_\varrho}U)')]\sqrt{b}\,dt\,d\varrho\\ &\geqslant\int_\sigma^\infty\int_\Sigma e^{-2\varrho}[Q_\varrho(V)+Q_\varrho({*_\varrho}U)]\sqrt{b}\,dt\,d\varrho.\end{aligned} \tag{3.19}$$

In the above inequality the primes indicate differentiation with respect to ϱ.

For the proof we will develop the left-hand side, transform it by partial integration, with respect to ϱ, or over the concentric hypersurfaces Σ_ϱ (using (1.9)), and then group the terms suitably. To simplify all these operations we will make use of a symmetry in the integrand of the left-hand side. The terms in the first line of the square bracket transform into the corresponding terms of the second line if we replace U by ${*_\varrho}V$, V by $(-1)^p{*_\varrho}U$, and then p by $n-p$. Also if we make the same substitution in the second line, it will be transformed into the first one (using (1.4)). In order to take full advantage of this symmetry we maintain, in all transformations, the separation of the terms into two kinds, those of the first being transformed into the corresponding terms of the second by the above substitutions. It will be enough to perform the transformations for half of the terms; the remaining terms will be written just by using the substitutions. We put

$$\varphi=\left(\log\sqrt{b}\right)'. \tag{3.20}$$

By developing the first line of the integral in the left-hand side of (3.19) we obtain

$$\begin{aligned}\int_\sigma^\infty\int_\Sigma [&Q_\varrho(d\,V)+Q_\varrho(d\,U)+(\beta+p)^2\,Q_\varrho(V)+Q_\varrho(V')+2(-1)^p(\beta+p)\,Q_\varrho(d\,U,\,V)\\ &+2(-1)^{p+1}\,Q_\varrho(d\,U,\,V')-2(\beta+p)\,Q_\varrho(V,\,V')]\sqrt{b}\,dt\,d\varrho.\end{aligned} \tag{3.21}$$

By partial integration with respect to ϱ we transform the last term as follows:

$$-2(\beta+p)\iint Q_\varrho(V, V')\sqrt{b}\,dt\,d\varrho=(\beta+p)\iint [Q'_\varrho(V)+\varphi Q_\varrho(V)]\sqrt{b}\,dt\,d\varrho, \tag{3.22}$$

where Q'_ϱ is obtained by considering Q_ϱ as a quadratic form defined for exterior differential forms on the manifold Σ but depending on the parameter ϱ, and taking the derivative with respect to ϱ. A similar meaning will be given to $*'_\varrho$.

We now combine the next to the last term in (3.21) with the corresponding term of second kind and put half of this sum among terms of first kind and half among terms of second kind. The corresponding integral in the first part is then transformed as follows:

$$\begin{aligned}
&\iint [(-1)^{p+1}Q_\varrho(dU, V')-Q_\varrho(d*_\varrho V, (*_\varrho U)')]\sqrt{b}\,dt\,d\varrho\\
&\quad=\iint (-1)^{p+1}Q_\varrho(dU, V')-Q_\varrho(d*_\varrho V, *_\varrho U')-Q_\varrho(d*_\varrho V, *'_\varrho U)]\sqrt{b}\,dt\,d\varrho\\
&\quad=\iint [(-1)^{p+1}Q_\varrho(dU, V')+(-1)^{p+1}Q_\varrho(dU', V)]\sqrt{b}\,dt\,d\varrho\\
&\qquad\qquad\qquad\qquad-\iint Q_\varrho(d*_\varrho V, *'_\varrho U)\sqrt{b}\,dt\,d\varrho\,(^1).
\end{aligned}$$

In the first integral we apply integration by parts with respect to ϱ and thus the whole expression becomes

$$(-1)^p\iint [Q'_\varrho(dU, V)+\varphi Q_\varrho(dU, V)]\sqrt{b}\,dt\,d\varrho-\iint Q_\varrho(d*_\varrho V, *'_\varrho U)\sqrt{b}\,dt\,d\varrho. \tag{3.23}$$

The first integral is put among the terms of first kind, the second among terms of second kind which means that the corresponding term

$$\begin{aligned}
&-\iint Q_\varrho(d*_\varrho(-1)^{n-p}*_\varrho U, *'_\varrho *_\varrho V)\sqrt{b}\,dt\,d\varrho\\
&\qquad\qquad=-(-1)^{p(n-p)}\iint Q_\varrho(dU, *'_\varrho *_\varrho V)\sqrt{b}\,dt\,d\varrho
\end{aligned}\tag{3.24}$$

is put among terms of first kind. In this way, the term which is next to the last in (3.21) is replaced by sum of three terms; this changes the expression (3.21) but does not change the total sum of terms of first and second kind.

We now introduce the operator H_ϱ which transforms forms into forms of the same rank, and is defined by

$$Q'_\varrho(X, Y)=Q_\varrho(X, H_\varrho Y). \tag{3.25}$$

We prove in § 4, VI, that (see (4.VI.6))

$$(-1)^{np+1}*'_\varrho *_\varrho=\varphi I+H_\varrho. \tag{3.26}$$

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(1) We use here (1.4), (1.5), (1.7) and (1.9) and the symmetry of Q_ϱ.

Using this relation we see that expression (3.22) becomes

$$(-1)^{np+1}(\beta+p)\iint Q_\varrho(\ast_\varrho'\ast_\varrho V,\,V)\sqrt{b}\,dt\,d\varrho,$$

and the first integral in (3.23) added to the integral in (3.24) give

$$2(-1)^{np+1+p}\iint Q_\varrho(dU,\,\ast_\varrho'\ast_\varrho V)\sqrt{b}\,dt\,d\varrho.$$

Putting these expressions in place of the last two terms in (3.21) allows us to write the left-hand integral in (3.19) in the form

$$\begin{aligned}&\int_\sigma^\infty\int_\Sigma Q_\varrho((\beta+p)(-1)^p V+dU+(-1)^{np+1+p}\ast_\varrho'\ast_\varrho V)\sqrt{b}\,dt\,d\varrho\\ &\quad+\int_\sigma^\infty\int_\Sigma(-1)^{np}(\beta+p)\,Q_\varrho(\ast_\varrho'\ast_\varrho V,\,V)\sqrt{b}\,dt\,d\varrho-\int_\sigma^\infty\int_\Sigma Q_\varrho(\ast_\varrho'\ast_\varrho V)\sqrt{b}\,dt\,d\varrho\\ &\quad+\int_\sigma^\infty\int_\Sigma Q_\varrho(dV)\sqrt{b}\,dt\,d\varrho+\int_\sigma^\infty\int_\Sigma Q_\varrho(V')\sqrt{b}\,dt\,d\varrho\\ &\quad+\textit{corresponding terms of second kind.}\end{aligned}\tag{3.27}$$

In § 4, VI, we prove that the symmetric operator $(-1)^{np}\ast_\varrho'\ast_\varrho$ for p-forms lies between the following bounds

$$\begin{aligned}e^{-\varrho}e^{\theta r}\left[(2p-n+1)\,\theta\frac{\Phi'(\theta r)}{\Phi(\theta r)}-\frac{n-1}{2}\,\omega\right]I&\leqslant(-1)^{np}\ast_\varrho'\ast_\varrho\\ &\leqslant e^{-\varrho}e^{\theta r}\left[(2p-n+1)\,\theta\frac{\Phi'(\theta r)}{\Phi(\theta r)}+\frac{n-1}{2}\,\omega\right]I.\end{aligned}\tag{3.28}$$

Here ω is the constant introduced in (3.6), $\Phi(\sigma)$ is the function $(e^\sigma-1)/\sigma$. We check immediately that

$$\frac{\Phi'(\sigma)}{\Phi(\sigma)}\ \textit{is increasing for}\ \sigma\geqslant 0,\quad \frac{1}{2}\leqslant\frac{\Phi'(\sigma)}{\Phi(\sigma)}\leqslant 1.\tag{3.29}$$

To achieve our proof we have to find, *in the first place,* a positive value of θ such that the second integral in (3.27) as well as the corresponding integral of second kind be positive. By (3.28) such choice of θ is possible for the integral of first kind if $2p-n+1>0$ and, for the integral of second kind, if $2(n-p)-n+1=n+1-2p>0$. These two condititions can be satisfied if and only if $n=2p$. If $n=2p$, a suitable choice of θ (in wiev of (3.29)) is $\theta\geqslant(n-1)\,\omega=$ $=(2p-1)\,\omega$. We assume therefore from now on

$$n = 2p, \quad \theta \geqslant (2p-1)\,\omega. \tag{3.30}$$

It follows that: $\beta + p = \alpha + 1$ (see (3.17)); $(-1)^{np} = 1$; the form $*U$, which in terms of second kind corresponds to $(-1)^p V$, is also a p-form and hence whatever we get for terms of first kind is true also for corresponding terms of second kind. Furthermore

$$Q_\varrho(*_\varrho' *_\varrho V,\ V) = -[Q_\varrho'(V) + \varphi\, Q_\varrho(V)] \geqslant 0. \tag{3.31}$$

In the second place, we require for our proof that the difference between the second and third integrals in (3.27) be $\geqslant 0$. This is achieved if $Q_\varrho(*_\varrho' *_\varrho w) \leqslant (\alpha+1)\, Q_\varrho(*_\varrho' *_\varrho w, w)$ for any p-form w, i.e. if

$$|*_\varrho' *_\varrho| \leqslant \alpha + 1. \tag{3.32}$$

Assuming this inequality valid over the whole domain of integration $\varrho > \sigma$, i.e. $R < s$, we proceed to the proof of our inequality (3.19). The integral in the left-hand side, which is the expression (3.27), is now

$$\geqslant \int_\sigma^\infty \int_\Sigma Q_\varrho(V')\sqrt{b}\, dt\, d\varrho + \int_\sigma^\infty \int_\Sigma Q_\varrho((*_\varrho U)')\sqrt{b}\, dt\, d\varrho.$$

For the first term of the right-hand side we obtain by partial integration relative to ϱ and by (3.31)

$$\begin{aligned}
\int_\sigma^\infty \int_\Sigma e^{-2\varrho} Q_\varrho(V)\sqrt{b}\, dt\, d\varrho &= \frac{1}{2}\int_\sigma^\infty \int_\Sigma e^{-2\varrho}\,[2\,Q_\varrho(V,\ V') + Q_\varrho'(V) + \varphi\, Q_\varrho(V)]\sqrt{b}\, dt\, d\varrho \\
&\leqslant \int_\sigma^\infty \int_\Sigma e^{-2\varrho}\, Q_\varrho(V,\ V')\sqrt{b}\, dt\, d\varrho \\
&\leqslant \left\{\int_\sigma^\infty \int_\Sigma e^{-2\varrho}\, Q_\varrho(V)\sqrt{b}\, dt\, d\varrho \int_\sigma^\infty \int_\Sigma e^{-2\varrho}\, Q_\varrho(V')\sqrt{b}\, dt\, d\varrho\right\}^{\frac{1}{2}},
\end{aligned}$$

$$\int_\sigma^\infty \int_\Sigma e^{-2\varrho}\, Q_\varrho(V)\sqrt{b}\, dt\, d\varrho \leqslant \int_\sigma^\infty \int_\Sigma e^{-2\varrho}\, Q_\varrho(V')\sqrt{b}\, dt\, d\varrho \leqslant e^{-2\sigma}\int_\sigma^\infty \int_\Sigma Q_\varrho(V')\sqrt{b}\, dt\, d\varrho.$$

We proceed similarly with the second term of the right-hand side and thus the inequality is proved.

It remains to check on the validity of (3.32). By (3.28) it is enough to have (see (3.29))

$$e^{-\varrho} e^{\theta r}\left(\theta + \frac{2p-1}{2}\,\omega\right) \leqslant \alpha + 1.$$

We may want to have it for all possible domains of integration contained in $S=[r(x)\leqslant\tilde{r}_0=r_0\sqrt{\Lambda_1}]$. Since by (2.10) and (3.16) $e^{-\varrho}e^{\theta r}=\Phi(\theta r)\,r$, we get a condition for α

$$\alpha+1\geqslant\Phi(\theta\tilde{r}_0)\,\tilde{r}_0\left(\theta+\frac{2p-1}{2}\omega\right)$$

and the smallest bound is obtained for the smallest choice of θ, namely, $\theta=(2p-1)\,\omega$, and then

$$\alpha+1\geqslant\frac{3}{2}\,\Phi\,(\tilde{r}_0\,\omega\,(2p-1))\,\tilde{r}_0\,\omega\,(2p-1).$$

However, in the lemma of § 2 we want the inequality to be valid for all $\alpha\geqslant 0$. Writing now $e^{-\varrho}e^{\theta r}=R/(1-\theta R)$ we have to satisfy

$$\frac{R}{1-\theta R}\left(\theta+\frac{2p-1}{2}\omega\right)\leqslant 1,\quad R\leqslant\frac{1}{2\theta+\dfrac{2p-1}{2}\omega},$$

and the least restrictive bound for R is obtained again by $\theta=(2p-1)\,\omega$ and the desired bound R_0 is then

$$R_0=\frac{2}{5\,(2p-1)\,\omega}. \tag{3.33}$$

§ 4. Evaluations

I. We assume that the p-form u satisfies (ii) and (iii) of § 2 and prove that

For $n\geqslant 2$, u has at P_0 a zero of infinite order in q-mean, for every $q<2+4/(n-2)$. (4.I.1)

For $n=1$, u is equivalent to a continuous function which has at $P_0=0$ a zero of infinite order in the classical sense, i.e. $|u(x)|=o(|x|^{\alpha})$ for any $\alpha>0$. (4.I.2)

Proof of (4.I.1). Take a coordinate patch U containing P_0. We can transfer our considerations to the image $G\subset R^n$ of U. We may assume that P_0 corresponds to 0 and consider each component $u_{(i)}$ of u separately. By (ii) and the Young–Soboleff theorem, $u_{(i)}$ is locally in L^{q_1} for any $q_1<2+4/(n-2)$. Take $q<q_1<2+4/(n-2)$, $\tau=(q_1-q)/(q_1-1)$, $\theta=q_1(q-1)/(q_1-1)$ (hence $\tau+\theta=q$) and apply Hölder's inequality:

$$\int|u_{(i)}|^q\,dx=\int|u_{(i)}|^{\tau}\,|u_{(i)}|^{\theta}\,dx\leqslant\left[\int|u_{(i)}|\,dx\right]^{\tau}\left[\int|u_{(i)}|^{q_1}\,dx\right]^{1-\tau}.$$

Proof of (4.I.2). We can assume $0<x<a<\infty$ and replace u by the equivalent continuous function $u(x)=u(0)+\int_0^x u'(y)\,dy$ and, by (iii), $u(0)=0$. Furthermore $|u(x_2)-u(x_1)| \leqslant A^{\frac{1}{2}}|x_2-x_1|^{\frac{1}{2}}$ with $A=\int_0^a |u'|^2\,dx$. Hence $(1/A)\,|u(x)|^2 \leqslant x$ and in the interval $0 \leqslant x-(1/A)\,|u(x)|^2 < y < x$ we have $|u(y)| \geqslant |u(x)| - A^{\frac{1}{2}}(x-y)^{\frac{1}{2}} > 0$. Thus

$$o(|x|^{3\alpha}) = \int_0^x |u(y)|\,dy \geqslant \int_{x-\frac{1}{A}|u(x)|^2}^{x} [|u(x)| - A^{\frac{1}{2}}(x-y)^{\frac{1}{2}}]\,dy = \frac{1}{3A}|u(x)|^3.$$

II. We assume a_{ij} to satisfy (2.1) and (2.1′). We will use Hilbert space notations, the scalar product being $(\xi,\eta)=\sum \xi^i\eta^i$, $|\xi|=(\xi,\xi)^{\frac{1}{2}}$. The matrix $\{a_{ij}\}$ then represents a linear operator $T \equiv T_x$ depending on the vector (point) x. Formula (2.1) means then that T is positive definite, $\Lambda_1 I \leqslant T \leqslant \Lambda_2 I$, with I denoting the identity. We introduce following notation for differential relative to a vector $w \neq 0$,

$$D w T \equiv D w T_x = \lim_{\tau\to 0} \frac{T_{x+\tau w}-T_x}{\tau}. \tag{4.II.1}$$

Whenever DwT exists, $D(\tau\omega)T=\tau DwT$ for real $\tau\neq 0$. When the Fréchet differential exists at x (and it does a.e., since T_x is $C^{0,1}$), DwT is linear in w. DwT is a symmetric operator and the inequality in (2.1′) means that whenever DwT exists its bound $|DwT| \leqslant \Lambda |w|$.

The matrix $\{a^{ij}\}$ corresponds to T^{-1} and (2.5) can be written

$$r(x) = (T_0 x, x)^{\frac{1}{2}} = |T_0^{\frac{1}{2}} x|. \tag{4.II.2}$$

Hence, the vector $\{\partial r/\partial x^k\}$ is $(1/r)\,T_0 x$ and we can put

$$\tilde{\Psi} \equiv \tilde{\Psi}(x) = \left(a^{ij}\frac{\partial r}{\partial x^i}\frac{\partial r}{\partial x^j}\right) = \left(T^{-1}\frac{1}{r}T_0 x, \frac{1}{r}T_0 x\right)$$
$$= (T_0 x, x)^{-1}(T_0 T^{-1} T_0 x, x), \quad \text{for } x \neq 0, \quad \tilde{\Psi}(0)=1. \tag{4.II.3}$$

For $x \neq y$, putting $w=y-x$, we have $T_y - T_x = \int_0^1 DwT_{x+\tau w}\,d\tau$ and

$$|T_y - T_x| \leqslant \Lambda|y-x|, \quad |T_y^{-1}-T_x^{-1}| = |T_y^{-1}(T_x-T_y)T_x^{-1}| \leqslant \frac{\Lambda|y-x|}{\Lambda_1^2}. \tag{4.II.4}$$

For $x \neq 0$, we have

$$|\tilde{\Psi}(x)-\tilde{\Psi}(0)| = |(T_0(T_0^{-1}-T^{-1})T_0 x, x)|\,(T_0 x, x)^{-1}$$
$$= \frac{(T_0^{\frac{1}{2}}(T_0^{-1}-T^{-1})T_0^{\frac{1}{2}}T_0^{\frac{1}{2}}x, T_0^{\frac{1}{2}}x)}{|T_0^{\frac{1}{2}}x|^2}$$
$$\leqslant \frac{\Lambda\Lambda_2|x|}{\Lambda_1^2}.$$

For $x \neq y$, both $\neq 0$, we put $w = y - x$ and obtain

$$|\tilde{\Psi}(y) - \tilde{\Psi}(x)| = \left| \int_0^1 Dw\, \tilde{\Psi}(x + \tau w)\, d\tau \right|.$$

Since by (4.II.3) $Dw\,\tilde{\Psi}(x)$ exists whenever $Dw\,T_x$ exists, it follows from (2.1′) that $\tilde{\Psi}$ will be proved to be in $C^{0,1}$ if we show that $|Dw\,\tilde{\Psi}(x)| \leqslant C|w|$ with C independent of w and x. We assume that $Dw\,T_x$ exists and obtain

$$\begin{aligned}
Dw\,\tilde{\Psi}(x) &= Dw \frac{(T_0 T_x^{-1} T_0 x, x)}{(T_0 x, x)} \\
&= \frac{(T_0 (Dw\,T_x^{-1}) T_0 x, x)}{(T_0 x, x)} + \frac{2(T_0 T_x^{-1} T_0 x, w)}{(T_0 x, x)} - \frac{2(T_0 x, w)(T_0 T_x^{-1} T_0 x, x)}{(T_0 x, x)^2} \\
&= \frac{(T_0 (Dw\,T_x^{-1}) T_0 x, x)}{(T_0 x, x)} + \frac{2(T_0 x, x)(T_0 (T_x^{-1} - T_0^{-1}) T_0 x, w)}{(T_0 x, x)^2} \\
&\quad - \frac{2(T_0 x, w)(T_0 (T_x^{-1} - T_0^{-1}) T_0 x, x)}{(T_0 x, x)^2} \\
&= \frac{(T_0^{\frac{1}{2}} (Dw\,T_x^{-1}) T_0^{\frac{1}{2}} T_0^{\frac{1}{2}} x, T_0^{\frac{1}{2}} x)}{|T_0^{\frac{1}{2}} x|^2} + \frac{2(T_0 (T_x^{-1} - T_0^{-1}) T_0^{\frac{1}{2}} T_0^{\frac{1}{2}} x, w)}{|T_0^{\frac{1}{2}} x|^2} \\
&\quad - \frac{2(T_0 x, w)(T_0^{\frac{1}{2}} (T_x^{-1} - T_0^{-1}) T_0^{\frac{1}{2}} T_0^{\frac{1}{2}} x, T_0^{\frac{1}{2}} x)}{|T_0^{\frac{1}{2}} x|^4}
\end{aligned}$$

Using the relations

$$Dw\,T_x^{-1} = -T_x^{-1}(Dw\,T_x)\,T_x^{-1}, \quad |T_0^{\frac{1}{2}}\, Dw\,T_x^{-1}\, T_0^{\frac{1}{2}}| \leqslant \frac{\Lambda\Lambda_2}{\Lambda_1^2}|w|,$$

$$|T_x^{-1} - T_0^{-1}| \leqslant \frac{\Lambda|x|}{\Lambda_1^2}, \quad |T_0 (T_x^{-1} - T_0^{-1}) T_0^{\frac{1}{2}}| \leqslant \frac{\Lambda\Lambda_2^{\frac{3}{2}}}{\Lambda_1^2}|x|,$$

$$|T_0^{\frac{1}{2}} x| \geqslant \sqrt{\Lambda_1}\,|x|, \quad |(T_0 x, w)| \leqslant |T_0^{\frac{1}{2}} x|\sqrt{\Lambda_2}\,|w|,$$

$$|T_0^{\frac{1}{2}} (T_x^{-1} - T_0^{-1}) T_0^{\frac{1}{2}}| \leqslant \frac{\Lambda\Lambda_2}{\Lambda_1^2}|x|,$$

we get
$$|Dw\,\tilde{\Psi}(x)| \leqslant \frac{\Lambda\Lambda_2}{\Lambda_1^2}\left[1 + 4\sqrt{\frac{\Lambda_2}{\Lambda_1}}\right]|w| \leqslant \frac{5\Lambda\Lambda_2}{\Lambda_1^2}\sqrt{\frac{\Lambda_2}{\Lambda_1}}\,|w|. \tag{4.II.5}$$

The metric tensor $\tilde{a}_{ij} = \tilde{\Psi}\, a_{ij}$ corresponds to the operator $\tilde{\Psi}\,T$. Formula (2.7) comes from

$$\frac{\Lambda_1}{\Lambda_2} \leqslant \tilde{\Psi}(x) \leqslant \frac{\Lambda_2}{\Lambda_1}$$

and (2.7′) from

$$|Dw(\tilde{\Psi}(x)T_x)| \leqslant |Dw\tilde{\Psi}(x)||T_x| + \tilde{\Psi}(x)|DwT_x|$$
$$\leqslant \frac{5\Lambda\Lambda_2}{\Lambda_1^2}\sqrt{\frac{\Lambda_2}{\Lambda_1}}|w|\Lambda_2 + \frac{\Lambda_2}{\Lambda_1}\Lambda|w| < \frac{6\Lambda\Lambda_2^2}{\Lambda_1^2}\sqrt{\frac{\Lambda_2}{\Lambda_1}}|w|.$$

The metric tensor $\tilde{\tilde{a}}_{ij}$ corresponds to the operator $\tilde{\tilde{\Psi}}T$ with $\tilde{\tilde{\Psi}} = \tilde{\Psi}\exp(-2\theta|T_0^{\frac{1}{2}}x|)$ We proceed with $\tilde{\tilde{\Psi}}$ as with $\tilde{\Psi}$. We calculate

$$Dw\ \exp(-2\theta|T_0^{\frac{1}{2}}x|) = -2\theta\frac{(T_0^{\frac{1}{2}}x,\ T_0^{\frac{1}{2}}w)}{|T_0^{\frac{1}{2}}x|}\exp(-2\theta|T_0^{\frac{1}{2}}x|).$$

The evaluations (2.11) are immediate whereas (2.11′) results from

$$|Dw\tilde{\tilde{\Psi}}T| \leqslant |Dw\tilde{\Psi}|e^{-2\theta r}|T| + |Dwe^{-2\theta r}||\tilde{\Psi}||T| + \tilde{\Psi}e^{-2\theta r}|DwT|$$
$$\leqslant \left[\frac{6\Lambda\Lambda_2^2}{\Lambda_1^2}\sqrt{\frac{\Lambda_2}{\Lambda_1}} + \frac{2\theta\Lambda_2^2\sqrt{\Lambda_2}}{\Lambda_1}\right]|w|.$$

III. To obtain evaluations (2.8) and (2.12) we deduce first the following inequality for u, a 1-form, and v, a p-form $(p<n)$:

$$Q(u\wedge v) \leqslant Q(u)\,Q(v).\,^{(1)} \tag{4.III.1}$$

We write first $\quad (u\wedge v)_{(k)} = \sum_{\mu\in(k)} u_\mu v_{(k\setminus)\mu}\varepsilon[\mu,\ (k)\setminus\mu].$

We then choose local coordinates such that at the point of the manifold in question we have $a_{ij} = \delta_{ij}$. Thus $A^{(i),(j)} = \delta^{(i),(j)}$ and

$$Q(u) = \sum_{\mu=1}^{n} u_\mu^2, \quad Q(v) = \sum_{(i)} v_{(i)}^2,$$
$$Q(u\wedge v) = \sum_{(k)} (u\wedge v)_{(k)}^2.$$

Denoting by (i') a $(p-1)$-system we now obtain

$$Q(u\wedge v) = \sum_{(i)} v_{(i)}^2 \sum_{\mu\in(i)^c} u_\mu^2 + \sum_{(i')} \sum_{\substack{\mu\in(i')^c\\ \nu\in(i')^c\\ \mu\neq\nu}} v_{(i')\cup\mu}v_{(i')\cup\nu}u_\mu u_\nu \varepsilon[\mu,\ (i')\cup\nu]\varepsilon[\nu,\ (i')\cup\mu]$$
$$= \sum_{(i)} v_{(i)}^2 \sum_{\mu=1}^{n} u_\mu^2 - \sum_{(i')}\Big\{\sum_{\mu\in(i')^c} v_{(i')\cup\mu}u_\mu\varepsilon[(i'),\ \mu]\Big\}^2 \leqslant Q(u)\,Q(v).$$

(1) This inequality is the best possible. When u is a q-form $(p+q\leqslant n)$, a similar inequality can be proved with a constant factor C in the right-hand side $\left(C\text{ can be taken as }\binom{n-p}{q}\text{ or }\binom{n-q}{p}\text{ or }\binom{p+q}{p}\right)$. But it seems difficult to find the best possible C in this general case.

This proof actually gives the identity

$$Q(u \wedge v) + Q(u \wedge *v) = Q(u)\,Q(v). \tag{4.III.1'}$$

Consider now the metric tensor $\tilde{a}_{ij} = \tilde{\Psi} a_{ij}$. It is clear that

$$\tilde{a}^{ij} = \tilde{\Psi}^{-1} a^{ij}, \quad \tilde{a} = \tilde{\Psi}^{n} a,$$

$$\tilde{A}^{(i),(j)} = \tilde{\Psi}^{-p} A^{(i),(j)} \quad \textit{for } p\textit{-systems } (i) \textit{ and } (j),$$

$$\tilde{*}\, u = \tilde{\Psi}^{(n/2-p)} * u \quad \textit{for a } p\textit{-form } u,$$

$$\tilde{Q}(du) = \tilde{\Psi}^{-p-1} Q(du), \quad \tilde{Q}(u) = \tilde{\Psi}^{-p} Q(u),$$

$$\tilde{Q}(d\tilde{*}u) = \tilde{\Psi}^{-(n-p+1)} Q(d(\tilde{\Psi}^{(n/2-p)} * u)).$$

Since $d(\tilde{\Psi}^{(n/2-p)} * u) = (d\,\tilde{\Psi}^{(n/2-p)}) \wedge *u + \tilde{\Psi}^{(n/2-p)}\, d*u$

$$= (n/2-p)\,\tilde{\Psi}^{(n/2-p-1)}\,(d\tilde{\Psi}) \wedge *u + \tilde{\Psi}^{(n/2-p)}\, d*u,$$

we get, by (4.III.1)

$$\begin{aligned}\tilde{Q}(du) + \tilde{Q}(d\tilde{*}u) &= \tilde{\Psi}^{-p-1} Q(du) + \tilde{\Psi}^{-p-1} Q\left(d*u + \left(\frac{n}{2}-p\right)\tilde{\Psi}^{-1} d\tilde{\Psi} \wedge *u\right)\\ &\leqslant 2\,\tilde{\Psi}^{-p-1}\left[Q(du) + Q(d*u) + \left(\frac{n}{2}-p\right)^2 Q(\tilde{\Psi}^{-1} d\tilde{\Psi})\,Q(u)\right]\\ &\leqslant 2\,\tilde{\Psi}^{-1}\left[M + \left(\frac{n}{2}-p\right)^2 Q(\tilde{\Psi}^{-1} d\tilde{\Psi})\right]\tilde{Q}(u).\end{aligned}$$

At each point x where T_x has a Fréchet differential, the covariant vector $d\tilde{\Psi}$ coincides with the linear functional $D w \tilde{\Psi}$, and therefore our evaluations in II give (see (4.II.5)):

$$Q(d\tilde{\Psi}) = (T^{-1} d\tilde{\Psi}, d\tilde{\Psi}) \leqslant \frac{1}{\Lambda_1} |d\tilde{\Psi}|^2 \leqslant \frac{25\,\Lambda^2 \Lambda_2^3}{\Lambda_1^6}.$$

$$\tilde{M} = 2\frac{\Lambda_2}{\Lambda_1}\left[M + \left(\frac{n}{2}-p\right)^2 \frac{25\,\Lambda^2\Lambda_2^5}{\Lambda_1^8}\right].$$

In the specially important case $p = n/2$ we get a much simpler formula

$$\tilde{Q}(du) + \tilde{Q}(d\tilde{*}u) = \tilde{\Psi}^{-p-1}\left[Q(du) + Q(d*u)\right] \leqslant \frac{\Lambda_2}{\Lambda_1} M\, \tilde{Q}(u). \tag{4.III.2}$$

Similarly for $\tilde{\tilde{a}}_{ij} = \tilde{\tilde{\Psi}} a_{ij}$, we get

$$\tilde{\tilde{\Psi}}^{-1} d\tilde{\tilde{\Psi}} = \tilde{\Psi}^{-1} d\tilde{\Psi} + e^{2\theta r} d\, e^{-2\theta r},$$

$$e^{2\theta r} d\, e^{-2\theta r} = -2\theta \frac{T_0 x}{|T_0^{\frac{1}{2}} x|}, \quad Q(e^{2\theta r} d\, e^{-2\theta r}) \leqslant 4\theta^2 \frac{\Lambda_2}{\Lambda_1},$$

$$\tilde{\tilde{M}} = 2\frac{\Lambda_2}{\Lambda_1} e^{4\theta r_0}\left[M + \left(\frac{n}{2}-p\right)^2 \left(\frac{5\,\Lambda\,\Lambda_2^2 \sqrt{\Lambda_2}}{\Lambda_1^4} + 2\theta\sqrt{\frac{\Lambda_2}{\Lambda_1}}\right)^2\right].$$

Again, in the special case $p=n/2$, we get

$$\tilde{\tilde{Q}}(du)+\tilde{\tilde{Q}}(d\tilde{\tilde{*}}u)\leqslant\frac{\Lambda_2}{\Lambda_1}e^{4\theta r_0}M\tilde{\tilde{Q}}(u). \tag{4.III.3}$$

IV. We shall prove more than what is stated at the beginning of § 3.

Suppose that the inequality (2.13) *holds for a fixed real* α, *for all* s *with* $0<s\leqslant R_0$, C (*see Remark* 3, § 2) *and* R_0 *possibly depending also on* α, *for all metric tensors* a_{ij} *satisfying* 1° *of* § 3 *and for all p-forms* v *satisfying* 2° *of* § 3. *Then* (2.13) *holds with the same* C *and* R_0 *for all* a_{ij} *satisfying* (2.1) *and* (2.1′) *and all* v *satisfying* (2.2), *vanishing outside of a compact in* $R(x)<s$ *and such that*

$$\int\limits_{|x|<\varrho}\tilde{\tilde{Q}}(v)\sqrt{\tilde{\tilde{a}}}\,dx=o(\varrho^{2\alpha+2}). \tag{4.IV.1}$$

Proof. (1) Suppose first that a_{ij} satisfies only (2.1) and (2.1′) but v satisfies the conditions in 2° of § 3. We regularize a_{ij} as usual by convoluting it with a regularizing function $\tau^{-n}\,\mathrm{e}(x/\tau)$ for $\tau\searrow 0$. The resulting tensors $a_{ij}(\tau)$ still satisfy (2.1) and (2.1′) with the same constants and are C^∞. Hence by our assumption inequality (2.13) is valid for them and for the form v. Since the integrals in (2.13) now extend over $\varepsilon<R(x)<s$, the integrands are uniformly bounded (all the evaluations of § 2 are valid for $r(\tau)$, $R(\tau)$, $\tilde{a}_{ij}(\tau)$ and $\tilde{\tilde{a}}_{ij}(\tau)$) and converge pointwise a.e. to the corresponding integrands for a_{ij} and v when $\tau\searrow 0$. Therefore (2.13) is valid for a_{ij} and v.

(2) Suppose now that a_{ij} satisfies (2.1) and (2.1′) and that v satisfies only (2.2) but still vanishes outside of a compact lying in $\varepsilon<R(x)<s$. Regularising the components of v we obtain a form $v(\tau)$ which for $\tau\searrow 0$ converges pointwise a.e. together with its first derivatives to v and its derivatives. In addition, $v(\tau)$ and its derivatives are dominated by fixed L^2 functions. Since for τ sufficiently small, $v(\tau)$ vanishes for $|x|\leqslant\varepsilon/2$, we can apply (2.13) to a_{ij} and $v(\tau)$ and the integrals in (2.13) converge for $\tau\searrow 0$ to their values for a_{ij} and v.

(3) Finally, suppose that a_{ij} satisfy (2.1) and (2.1′) and that v satisfies (2.2), vanishes outside of a compact in $R(x)<s$ and that (4.IV.1) holds. Take a function in C^∞ depending only on $|x|$ and non-decreasing as function of $|x|$ and such that $\varphi(x)=0$ for $|x|\leqslant\frac{1}{2}$ and $\varphi(x)=1$ for $|x|\geqslant 1$. We may assume that the left-hand side integral in (2.13) for a_{ij} and v is finite (otherwise the inequality holds trivially) (1). Consider now the form $v(\tau)=\varphi(x/\tau)\,v$ for $\tau\searrow 0$. By the preceding proofs we can apply the inequality to a_{ij} and $v(\tau)$. The right-hand side converges increasingly to its value for v. It is enough therefore to show that the left-hand side integral converges to its value for v. For $|x|>\tau$, $v(\tau)=v$ and for $|x|<\tau/2$, $v(\tau)=0$; hence we must show that the integral restricted to $\tau/2<|x|<\tau$ converges to 0. The integrand (by (4.III.1)) is

(1) Our present hypotheses on v do not make it sure that this integral is finite.

$$R^{-2\alpha}\left[\tilde{\tilde{Q}}\left(d\left(\varphi\left(\frac{x}{\tau}\right)v\right)\right)+\tilde{\tilde{Q}}\left(d\left(\tilde{\tilde{*}}\varphi\left(\frac{x}{\tau}\right)v\right)\right)\right]\sqrt{\tilde{\tilde{a}}}$$

$$\leqslant R^{-2\alpha}\,2\left[\tilde{\tilde{Q}}\left(d\varphi\left(\frac{x}{\tau}\right)\right)\tilde{\tilde{Q}}(v)+\varphi^2\left(\frac{x}{\tau}\right)\tilde{\tilde{Q}}(dv)+\tilde{\tilde{Q}}\left(d\varphi\left(\frac{x}{\tau}\right)\right)\tilde{\tilde{Q}}(\tilde{\tilde{*}}v)+\varphi^2\left(\frac{x}{\tau}\right)\tilde{\tilde{Q}}(d\tilde{\tilde{*}}v)\right]\sqrt{\tilde{\tilde{a}}}$$

$$=4R^{-2\alpha}\tilde{\tilde{Q}}\left(d\varphi\left(\frac{x}{\tau}\right)\right)\tilde{\tilde{Q}}(v)\sqrt{\tilde{\tilde{a}}}+2R^{-2\alpha}\varphi^2\left(\frac{x}{\tau}\right)[\tilde{\tilde{Q}}(dv)+\tilde{\tilde{Q}}(d\tilde{\tilde{*}}v)]\sqrt{\tilde{\tilde{a}}}.$$

The second part here, integrated over $\tau/2<|x|<\tau$, obviously converges to 0 (by our assumption that the integral for v is finite). In view of (2.14) we can replace R by $|x|$ in the first part and consider

$$\int\limits_{\tau/2<|x|<\tau} 4|x|^{-2\alpha}\tilde{\tilde{Q}}\left(d\varphi\left(\frac{x}{\tau}\right)\right)\tilde{\tilde{Q}}(v)\sqrt{\tilde{\tilde{a}}}\,dx.$$

Noticing that $|d\varphi(x/\tau)|<Hr^{-1}$ with constant H, and hence $\tilde{\tilde{Q}}(d\varphi(x/\tau))=O(\tau^{-2})$ we get by partial integration and by (4.IV.1)

$$4\tau^{-2}\int\limits_{\tau/2<|x|<\tau}|x|^{-2\alpha}\tilde{\tilde{Q}}(v)\sqrt{\tilde{\tilde{a}}}\,dx$$

$$=4\tau^{-2}\left[\tau^{-2\alpha}\int\limits_{|x|<\tau}\tilde{\tilde{Q}}(v)\sqrt{\tilde{\tilde{a}}}\,dx-\left(\frac{\tau}{2}\right)^{-2\alpha}\int\limits_{|x|<\tau/2}\tilde{\tilde{Q}}(v)\sqrt{\tilde{\tilde{a}}}\,dx\right]$$

$$+8\alpha\tau^{-2}\int\limits_{\tau/2}^{\tau}\varrho^{-2\alpha-1}\int\limits_{|x|<\varrho}\tilde{\tilde{Q}}(v)\sqrt{\tilde{\tilde{a}}}\,dx\,d\varrho$$

$$=o(1).$$

Remark. If we accept the natural definition that v has a zero of order β in p-mean if $\int\limits_{|x|<\varrho}|v|^p\,dx=O(\varrho^{p\beta+n})$, then (4.IV.1) means that v has a zero of order $\alpha+1-n/2$ in 2-mean.

V. To prove the evaluation (3.6) we write the system (3.1), and the initial condition (3.3) in the notations of (II), where T now stands for the operator corresponding to the matrix $\{\tilde{a}_{ij}\}$ and all the constants are proved with $\tilde{}$.

$$\frac{dx}{dr}=\frac{1}{r}T_x^{-1}T_0x \quad \textit{with} \quad r^2=(T_0x,x), \tag{4.V.1}$$

$$x=x(t;r),\ x(t;\tilde{r}_0)=t \quad \textit{with} \quad (T_0t,t)=\tilde{r}_0^2. \tag{4.V.1'}$$

On the hypersurface Σ we take any local coordinates $(t^1,\ldots,t^{n-1})=t$, and introduce the vectors

$$y_\alpha = \frac{1}{r}\frac{\partial x}{\partial t^\alpha}. \tag{4.V.2}$$

We have

$$\frac{\partial y_\alpha}{\partial r} = -\frac{1}{r}y_\alpha + \frac{1}{r^2}\frac{\partial}{\partial t^\alpha}(T_x^{-1}T_0 x) = -\frac{1}{r}y_\alpha + \frac{1}{r}[(D y_\alpha T_x^{-1})T_0 x + T_x^{-1}T_0 y_\alpha]$$

$$= \frac{-1}{r}T_x^{-1}(D y_\alpha T_x)T_x^{-1}T_0 x + \frac{1}{r}(T_x^{-1} - T_0^{-1})T_0 y_\alpha.$$

Also, if we put $y_\tau = y_\alpha \tau^\alpha$ for any fixed tangential vector $\tau = (\tau^1, \ldots, \tau^{n-1})$ at the point t of Σ, we have by linearity

$$\frac{\partial y_\tau}{\partial r} = \frac{-1}{r}T_x^{-1}(D y_\tau T_x)T_x^{-1}T_0 x + \frac{1}{r}(T_x^{-1} - T_0^{-1})T_0 y_\tau. \tag{4.V.3}$$

By (3.5) we have

$$\tilde{b}_{\mu\nu}\tau^\mu\tau^\nu = \tilde{a}_{ij}y_\tau^i y_\tau^j = (T_x y_\tau, y_\tau). \tag{4.V.4}$$

Hence
$$\frac{\partial}{\partial r}\tilde{b}_{\mu\nu}\tau^\mu\tau^\nu = \left(\left(\frac{\partial}{\partial r}T_x\right)y_\tau, y_\tau\right) + 2\left(T_x\frac{\partial y_\tau}{\partial r}, y_\tau\right).$$

Since
$$\left|\frac{\partial}{\partial r}T_x\right| = \left| D\frac{\partial x}{\partial r}T_x\right| \leqslant \tilde{\Lambda}\left|\frac{\partial x}{\partial r}\right| = \tilde{\Lambda}\left|\frac{1}{r}T_x^{-1}T_0 x\right| \leqslant \frac{\tilde{\Lambda}\sqrt{\tilde{\Lambda}_2}}{\tilde{\Lambda}_1},$$

$$T_x\frac{\partial y_\tau}{\partial r} = \frac{-1}{r}(D y_\tau T_x)T_x^{-1}T_0 x + \frac{1}{r}(T_0 - T_x)y_\tau,$$

$$\left|T_x\frac{\partial y_\tau}{\partial r}\right| \leqslant \frac{\tilde{\Lambda}\sqrt{\tilde{\Lambda}_2}}{\tilde{\Lambda}_1}|y_\tau| + \frac{\tilde{\Lambda}}{\sqrt{\tilde{\Lambda}_1}}|y_\tau|,$$

it follows

$$\left|\frac{\partial}{\partial r}\tilde{b}_{\mu\nu}\tau^\mu\tau^\nu\right| \leqslant \left(3\frac{\tilde{\Lambda}\sqrt{\tilde{\Lambda}_2}}{\tilde{\Lambda}_1} + 2\frac{\tilde{\Lambda}}{\sqrt{\tilde{\Lambda}_1}}\right)|y_\tau|^2 \leqslant \frac{\tilde{\Lambda}}{\tilde{\Lambda}_1^2}\left(3\sqrt{\tilde{\Lambda}_2} + 2\sqrt{\tilde{\Lambda}_1}\right)(T_x y_\tau, y_\tau)$$
$$\leqslant \omega\tilde{b}_{\mu\nu}\tau^\mu\tau^\nu,$$

with

$$\omega = \frac{5\tilde{\Lambda}\sqrt{\tilde{\Lambda}_2}}{\tilde{\Lambda}_1^2} = \frac{30\Lambda\Lambda_2^5\sqrt{\Lambda_2}}{\Lambda_1^7}. \tag{4.V.5}$$

VI. To prove the statements made at the end of § 3 we start by some general considerations.

Let $b_{\mu\nu}$ be a metric tensor on a $(n-1)$-dimensional manifold (in our case Σ) depending on a parameter r. Assume that for any tangential vector τ at a point t we have

$$\left|\frac{\partial b_{\mu\nu}}{\partial r}\tau^{\mu}\tau^{\nu}\right| \leqslant \hat{\omega}\, b_{\mu\nu}\tau^{\mu}\tau^{\nu}, \tag{4.VI.1}$$

with a positive $\hat{\omega}$, independent of τ, but which may depend on t and r. The manifold, and tensor $b_{\mu\nu}$ and its dependence on r are assumed to be C^{∞}.

Fixing local coordinates around t and considering the corresponding Euclidean scalar product on the tangential space at t we can write $b_{\mu\nu}\tau^{\mu}\tau^{\nu} = (B\tau, \tau)$, $(\partial b_{\mu\nu}/\partial r)\tau^{\mu}\tau^{\nu} = ((\partial B/\partial r)\tau, \tau)$. Then (4.IV.1) means that $|B^{-\frac{1}{2}}(\partial B/\partial r)B^{-\frac{1}{2}}| \leqslant \hat{\omega}$.

Considering the determinant b of $\{b_{\mu\nu}\}$ it is easy to check directly that

$$\frac{\partial}{\partial r}\log\sqrt{b} = \frac{1}{2b}\frac{\partial b}{\partial r} = \frac{1}{2}\operatorname{trace}\left\{\frac{\partial B}{\partial r}B^{-1}\right\} = \frac{1}{2}\operatorname{trace}\left\{B^{-\frac{1}{2}}\frac{\partial B}{\partial r}B^{-\frac{1}{2}}\right\},$$

$$\left|\frac{\partial}{\partial r}\log\sqrt{b}\right| \leqslant \frac{1}{2}\hat{\omega}\,(n-1). \tag{4.VI.2}$$

Since $B^{\frac{1}{2}}((\partial/\partial r)B^{-1})B^{\frac{1}{2}} = -B^{-\frac{1}{2}}(\partial B/\partial r)B^{-\frac{1}{2}}$, (4.VI.1) gives

$$\left|\frac{\partial b^{\mu\nu}}{\partial r}\tau_{\mu}\tau_{\nu}\right| \leqslant \hat{\omega}\, b^{\mu\nu}\tau_{\mu}\tau_{\nu}. \tag{4.VI.3}$$

Take now $Q(u)$ for a p-form v independent of r:

$$Q(v) = \sum_{(i),(j)} B^{(i),(j)} v_{(i)} v_{(j)}.$$

For convenience, we fix the local coordinates temporarily so that for a given r and t the tensor $b_{\mu\nu}$ be Euclidean, i.e. that $b_{\mu\nu} = \delta_{\mu\nu}$. Then $b^{\mu\nu} = \delta^{\mu\nu}$ and $B^{(i),(j)} = \delta^{(i),(j)}$. Furthermore an inspection of $(\partial/\partial r)B^{(i),(j)}$ gives

$$\frac{\partial}{\partial r}B^{(i),(j)} = \sum_{\mu=1}^{p}\sum_{\nu=1}^{p}(-1)^{\mu+\nu}\left(\frac{\partial}{\partial r}b^{i_\mu, j_\nu}\right)B^{(i)\setminus i_\mu,(j)\setminus j_\nu}.$$

Hence
$$\frac{\partial}{\partial r}B^{(i),(i)} = \sum_{\mu=1}^{n}\frac{\partial}{\partial r}b^{i_\mu, i_\mu},$$

$$\frac{\partial}{\partial r}B^{(i')\cup k,(i')\cup l} = \varepsilon[(i'),k]\,\varepsilon[(i'),l]\frac{\partial b^{k,l}}{\partial r},$$

for a $(p-1)$-system (i'), $k \neq l$ and $(k,l) \subset (i')^{c}$,

$$\frac{\partial}{\partial r}B^{(i),(j)} = 0$$

in all other cases. It follows that

$$\frac{\partial}{\partial r} Q(v) = \sum_{(i),(j)} \frac{\partial B^{(i),(j)}}{\partial r} v_{(i)} v_{(j)}$$

$$= \sum_{(i')} \sum_{(k,l) \subset (i')^c} \frac{\partial b^{k,l}}{\partial r} \varepsilon[(i'), k] v_{(i') \cup k} \varepsilon[(i'), l] v_{(i') \cup l},$$

$$\left| \frac{\partial}{\partial r} Q(v) \right| \leqslant \hat{\omega} \sum_{(i')} \sum_{k \in (i')^c} |v_{(i') \cup k}|^2,$$

$$\left| \frac{\partial}{\partial r} Q(v) \right| \leqslant \hat{\omega}\, p\, Q(v).\ (^1) \tag{4.VI.4}$$

Consider now the operators $*$ transforming the space of p-forms u at t onto the space of $(n-1-p)$ forms v at t. By (1.3), with (h) an $(n-1-p)$-system and (i) a p-system

$$(*u)_{(h)} = \sqrt{b}\, \varepsilon[(h)^c, (h)] \sum_{(i)} B^{(h)^c,(i)} u_{(i)},$$

$$\left(\frac{\partial *}{\partial r} u\right)_{(h)} = \left(\frac{\partial}{\partial r} \log \sqrt{b}\right) (*u)_{(h)} + \sqrt{b}\, \varepsilon[(h)^c, (h)] \sum_{(i)} \left(\frac{\partial}{\partial r} B^{(h)^c,(i)}\right) u_{(i)}.$$

If we again use the local coordinates as above it becomes clear that $*(\partial */\partial r)u$ is a p-form given by

$$\left(* \frac{\partial *}{\partial r} u\right)_{(j)} = \left(\frac{\partial}{\partial r} \log \sqrt{b}\right) (-1)^{np} u_{(j)} + (-1)^{np} \sum_{(i)} \left(\frac{\partial}{\partial r} B^{(j),(i)}\right) u_{(i)}.$$

It follows that in these coordinates, with another p-form v

$$Q\left(* \frac{\partial *}{\partial r} u, v\right) = \sum_{(j)} \left(* \frac{\partial *}{\partial r} u\right)_{(j)} v_{(j)} = (-1)^{np} \left[\frac{\partial \log \sqrt{b}}{\partial r} Q(u, v) + \frac{\partial Q}{\partial r}(u, v)\right].\ (^2)$$

If we introduce $Q(u)$ as square of a norm in the space of p-forms at the point t of the manifold and for the value r of the parameter, we can write $(\partial Q/\partial r)(u, v) = Q(u, Hv)$ (as in (3.25)) and the above formula shows that

$$(-1)^{np} * \frac{\partial *}{\partial r} = \frac{\partial \log \sqrt{b}}{\partial r} I + H,$$

where I is the identity.

Since, by (1.4) $** = (-1)^{np} I$, differentiation gives

$$* \frac{\partial *}{\partial r} = -\frac{\partial *}{\partial r} *, \tag{4.VI.5}$$

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(1) Under the hypothesis (4.VI.1) the inequalities (4.VI.2–4) are the best possible. Example: $b_{\mu\nu} = \psi(t; r)\, \bar{b}_{\mu\nu}$ with $\bar{b}_{\mu\nu}$ independent of r; here $\hat{\omega} = (1/\psi)\, |\partial \psi / \partial r|$.

(2) This relation could also be obtained by differentiating the formula $Q(u, v) = *(u \wedge *v)$.

and the above formula gives

$$(-1)^{np+1}\frac{\partial *}{\partial r} * = \frac{\partial \log \sqrt{b}}{\partial r} I + H. \tag{4.VI.6}$$

Since H is obviously symmetric (rel. to norm $\sqrt{Q(u)}$), $(\partial */\partial r)*$ is symmetric too and commutes with H. From (4.VI.2) and (4.VI.4) we could already obtain an evaluation of $|(\partial */\partial r)*|$, but we shall obtain a slightly better bound as follows. Differentiating the equation $Q(*u, *u) = Q(u, u)$ we get

$$\frac{\partial Q}{\partial r}(*u, *u) + Q\left(\frac{\partial *}{\partial r}u, *u\right) + Q\left(*u, \frac{\partial *}{\partial r}u\right) = \frac{\partial Q}{\partial r}(u, u),$$

$$(-1)^{np}\left[Q\left(*\frac{\partial *}{\partial r}u, u\right) + Q\left(u, *\frac{\partial *}{\partial r}u\right)\right] = \frac{\partial Q}{\partial r}(u, u) - \frac{\partial Q}{\partial r}(*u, *u),$$

$$(-1)^{np}Q\left(\frac{\partial *}{\partial r}*u, u\right) = \frac{1}{2}\left[\frac{\partial Q}{\partial r}(*u, *u) - \frac{\partial Q}{\partial r}(u, u)\right]. \tag{4.VI.7}$$

Since $(\partial */\partial r)*$ is symmetric it follows by (4.VI.4) for $(n-1-p)$-forms $*u$ and for p-forms u

$$\left|\frac{\partial *}{\partial r}*\right| \leqslant \frac{n-1}{2}\hat{\omega}.\text{(1)} \tag{4.VI.8}$$

We recall now the situation treated in § 3 where, on Σ, the metric tensors $b_{\mu\nu}$ depend on ϱ. We know from (3.8) that $b_{\mu\nu}$ is related to $\tilde{b}_{\mu\nu}$ and from this relation we get

$$*_\varrho = \Phi(\theta r)^{-(n-1)+2p}\,\tilde{*}_r \quad \textit{for p-forms}, \tag{4.VI.9}$$

$$*_\varrho = \Phi(\theta r)^{(n-1)-2p}\,\tilde{*}_r \quad \textit{for } (n-1-p)\textit{-forms}. \tag{4.VI.9'}$$

The parameters ϱ and r are related by $R = e^{-\varrho} = (1-e^{-\theta r})/\theta$, hence $(\partial/\partial\varrho) = -e^{-\varrho}e^{\theta r}(\partial/\partial r)$. By differentiating (4.VI.9') we get

$$*_\varrho' = -e^{-\varrho}e^{\theta r}\Phi(\theta r)^{n-1-2p}\left[(n-1-2p)\,\theta\frac{\Phi'(\theta r)}{\Phi(\theta r)}\tilde{*}_r + \frac{\partial\tilde{*}_r}{\partial r}\right].$$

Composing this with (4.VI.9) gives

$$*_\varrho' *_\varrho = -e^{-\varrho}e^{\theta r}\left[(-1)^{np}(n-1-2p)\,\theta\frac{\Phi'(\theta r)}{\Phi(\theta r)}I + \frac{\partial\tilde{*}_r}{\partial r}\tilde{*}_r\right].$$

Since for p-forms, $Q_\varrho(u) = \Phi(\theta r)^{2p}\tilde{Q}_r(u)$, the bounds of an operator with respect to the norms $\sqrt{Q_\varrho(u)}$ and $\sqrt{\tilde{Q}_r(u)}$ are the same. By (3,6) and (4.VI.8) considered for the metric $\tilde{b}_{\mu\nu}$ we obtain

(1) The constant $(n-1)/2$ is the best possible.

$$e^{-\varrho}\, e^{\theta r}\left[(2\,p-n+1)\,\theta\,\frac{\Phi'(\theta\,r)}{\Phi(\theta\,r)}-\frac{n-1}{2}\,\omega\right] I \leqslant (-1)^{np} *_{\varrho}' *_{\varrho}$$

$$\leqslant e^{-\varrho}\, e^{\theta r}\left[(2\,p-n+1)\,\theta\,\frac{\Phi'(\theta\,r)}{\Phi(\theta\,r)}+\frac{n-1}{2}\,\omega\right] I. \tag{4.VI.10}$$

§ 5. Final remarks

Remark 1. *The best constant in the Carleman type inequality.* The constant $C=1$ with which we prove our inequality (see Remark 3 in § 2) is not the best possible. It can be proved that if $j_{0,1}$ denotes the smallest positive zero of the Bessel function $J_0(z)$ the inequality holds for any $C>1/j_{0,1}^2$ with a proper choice of R_0, whereas the inequality is not true for $C<1/j_{0,1}^2$ (with any choice of R_0) and a suitably chosen p-form. It is not clear if $1/j_{0,1}^2=0.1729\ldots$ is always an admissible value for C. To prove this assertion we notice that from the proof in § 3 it follows that the quantity

$$C^*=\limsup_{\sigma\to\infty\;\; V}\frac{\int\limits_{\sigma}^{\infty}\int\limits_{\Sigma} e^{-2\varrho}\, Q_{\varrho}(V)\sqrt{b}\; dt\, d\varrho}{e^{-2\sigma}\int\limits_{\sigma}^{\infty}\int\limits_{\Sigma} Q_{\varrho}(V')\sqrt{b}\, dt\, d\varrho}$$

separates the admissible from the non-admissible values of C.

By using (3.6) and (4.VI.4) we prove immediately for a fixed p-form v and point t on Σ that for $0<r_1<r_2$

$$e^{-p\omega(r_2-r_1)}\,\tilde{Q}_{r_2}(v)\leqslant \tilde{Q}_{r_1}(v)\leqslant e^{p\omega(r_2-r_1)}\,\tilde{Q}_{r_2}(v),$$

hence, for the corresponding ϱ_1 and ϱ_2, $\varrho_1>\varrho_2$,

$$e^{-p\omega(r_2-r_1)}\,\Phi(\theta\, r_2)^{-2p}\,Q_{\varrho_2}(v)\leqslant \Phi(\theta\, r_1)^{-2p}\,Q_{\varrho_1}(v)\leqslant e^{p\omega(r_2-r_1)}\,\Phi(\theta\, r_2)^{-2p}\,Q_{\varrho_2}(v).$$

This shows that when $\varrho\to+\infty$ (i.e. $r\to 0$), $Q_{\varrho}(v)\to Q_{\infty}(v)$ and

$$e^{-p\omega r}\,(\Phi(\theta\, r))^{-2p}\,Q_{\varrho}(v)\leqslant Q_{\infty}(v)\leqslant e^{p\omega r}\,(\Phi(\theta\, r))^{-2p}\,Q_{\varrho}(v).$$

Hence in the expression for C^* we can replace Q_{ϱ} by Q_{∞}. Then by choosing an orthonormal system $\{u^k\}$ of p-forms relative to the quadratic norm

$$\|u\|^2=\int\limits_{\Sigma} Q_{\infty}(u)\sqrt{b_{\infty}}\, dt,$$

we develop

$$V=\sum_{1}^{\infty} f_k(\varrho)\, u^k$$

and find that

$$C^* = \sup_f \int_0^\infty e^{-2\varrho} |f(\varrho)|^2 d\varrho \Big/ \int_0^\infty |f'(\varrho)|^2 d\varrho,$$

for $f \in C^\infty$ vanishing outside of a closed interval on the positive half-axis.

One then shows that C^* is the reciprocal of the smallest eigenvalue of the eigenvalue problem $f''(\varrho) + \lambda e^{-2\varrho} f(\varrho) = 0$ for $0 \leqslant \varrho \leqslant \infty$ with "boundary conditions" $f(0) = 0$ and $f' \in L^2[0, \infty]$.

The eigenvalues of this problem are $\lambda = j_{0,m}^2$ and a corresponding eigenfunction is $J_0(j_{0,m} e^{-\varrho})$, where $j_{0,m}$ is the mth positive root of $J_0(z)$.

Remark 2. *The Carleman type inequality for* $n \neq 2p$. The idea which allowed the reduction of Theorem 1, § 2, to the case $n = 2p$ can be applied to deduce a Carleman type inequality for $n \neq 2p$ from the inequality which was proved for $n = 2p$.

To simplify our considerations we assume that the local coordinates are chosen so that $a_{ij}(0)$ is the Euclidean metric, i.e. $r(x) = |x|$ and that $a^{ij}(\partial r/\partial x^i)(\partial r/\partial x^j) = 1$ holds; this last property was obtained in § 2 by passing from a_{ij} to $\tilde{a}_{ij}$. Hence the geodesic distance from zero coincides with the Euclidean distance. Let u be a p-form satisfying (2.2) and (2.3) and vanishing outside of a compact contained in $|x| < s$. To simplify, we assume that $p > n/2$. The manifold is extended by taking its product with R^{2p-n}. The metric a_{ij} extended to the $2p$-dimensional manifold (as in the proof of proposition 1, § 2) will be denoted by $\tilde{a}_{ij}$ with i, j varying now from 1 to $2p$. For this metric the geodesic distance from the origin is also equal to the Euclidean distance.

The form u, extended as in the proof of proposition 1, to the new manifold no longer vanishes outside of a compact there. Denote by $\tilde{x}$ the point with coordinates $x^1 \dots x^n$, $x^{n+1} \dots x^{2p}$, by $r' \equiv r'(\tilde{x}) = \left(\sum_{k=n+1}^{2p} |x^k|^2 \right)^{\frac{1}{2}}$, by $\varphi(r')$ a non-increasing function in C^∞, with $\varphi(r') = 1$ for $r' \leqslant 1$ and $\varphi(r') = 0$ for $r' \geqslant 2$. Then by putting $\tilde{u} = \varphi((1/s) r') u$, we obtain now a p-form in the $2p$-dimensional manifold vanishing outside of a compact in $|\tilde{x}| < 3s$.

For the metric $\tilde{a}_{ij}$ we construct the corresponding metric $\tilde{\tilde{a}}$ with a suitable θ such that inequality (2.13) is valid. We use this inequality for the form $\tilde{u}$ and taking advantage of the relations:

$$r = |\tilde{x}|, \quad R = \frac{1}{\theta}(1 - e^{-\theta|\tilde{x}|}), \quad \sqrt{\tilde{\tilde{a}}} = \sqrt{\tilde{a}}\, e^{-2\theta p|\tilde{x}|},$$

$$\tilde{\tilde{Q}}(\tilde{u}) = e^{2\theta p|\tilde{x}|} \tilde{Q}(\tilde{u}), \quad \tilde{\tilde{Q}}(d\tilde{u}) = e^{2\theta(p+1)|\tilde{x}|} \tilde{Q}(d\tilde{u}),$$

$$\tilde{\tilde{\ast}} = \tilde{\ast} \text{ for } p\text{-forms}, \quad \tilde{\tilde{Q}}(d \tilde{\tilde{\ast}} \tilde{u}) = e^{2\theta(p+1)|\tilde{x}|} \tilde{Q}(d \tilde{\ast} \tilde{u}).$$

Hence the inequality can now be written (for small s, say $s \leqslant 1/3\theta$)

$$(3s)^2 \int\limits_{|\tilde{x}|<3s} \left[\frac{1}{\theta}(1-e^{-\theta|\tilde{x}|})\right]^{-2\alpha} e^{2\theta|\tilde{x}|}\,[\tilde{Q}(d\tilde{u})+\tilde{Q}(d\tilde{\ast}\tilde{u})]\sqrt{\tilde{a}}\,d\tilde{x}$$
$$\geqslant \int\limits_{|\tilde{x}|<3s} \left[\frac{1}{\theta}(1-e^{-\theta|\tilde{x}|})\right]^{-2\alpha} \tilde{Q}(\tilde{u})\sqrt{\tilde{a}}\,d\tilde{x}.$$

We notice now that

$$\tilde{a}=a,\quad \tilde{Q}(\tilde{u})=\varphi^2\left(\frac{1}{s}r'\right)Q(u),\quad d\tilde{u}=d\varphi\left(\frac{1}{s}r'\right)\wedge u+\varphi\left(\frac{1}{s}r'\right)du,$$
$$\tilde{Q}(d\tilde{u})=\frac{1}{s^2}\varphi'\left(\frac{1}{s}r'\right)^2 Q(u)+\varphi^2\left(\frac{1}{s}r'\right)Q(du),$$
$$d\tilde{\ast}\tilde{u}=\varphi\left(\frac{1}{s}r'\right)d\tilde{\ast}u,\quad \tilde{Q}(d\tilde{\ast}\tilde{u})=\varphi^2\left(\frac{1}{s}r'\right)Q(d\ast u).$$

Therefore, by integrating in the above inequality with respect to the variables $x^{n+1}\,\dots\,x^{2p}$ and introducing for $0<t\leqslant s$ the expressions

$$R(t,s,\alpha)=\int\limits_{r'<2s}\left[\frac{1}{\theta}(1-e^{-\theta\sqrt{t^2+r'^2}})\right]^{-2\alpha}\varphi^2\left(\frac{1}{s}r'\right)dx^{n+1}\,\dots\,dx^{2p}.$$
$$\bar{R}(t,s,\alpha)=\int\limits_{s<r'<2s}\left[\frac{1}{\theta}(1-e^{-\theta\sqrt{t^2+r'^2}})\right]^{-2\alpha}\varphi'^2\left(\frac{1}{s}r'\right)dx^{n+1}\,\dots\,dx^{2p},$$

we deduce

$$9s^2e^{6\theta s}\int\limits_{|x|<s}R(|x|,s,\alpha)\,[Q(du)+Q(d\ast u)]\sqrt{a}\,dx$$
$$+9e^{6\theta s}\int\limits_{|x|<s}\bar{R}(|x|,s,\alpha)\,Q(u)\sqrt{a}\,dx$$
$$\geqslant\int\limits_{|x|<s}R(|x|,s,\alpha)\,Q(u)\sqrt{a}\,dx. \tag{A}$$

This inequality is *not* of the usual Carleman type. However, if we remark that: R and $\bar{R}$ are decreasing functions of t; that for $\alpha>p-n/2$, $\lim R=+\infty$ for $t\to 0$; that for $t=s<1$, $s^{2\alpha-2p+n}R$ is bounded above and below by two positive constants independent of s and α; that $s^{2\alpha-2p+n}\bar{R}(t,s,\alpha)$, is bounded above and below by two constants independent of t, s, and α; and finally, that for $t^1<t^2$ and $\alpha\to+\infty$, $R(t^2,s,\alpha)/R(t^1,s,\alpha)\to 0$, it becomes clear that the inequality (A) can be used for a proof of a unique continuation theorem in the same way as (2.13). We can say that (A) is a Carleman type inequality of a generalized kind and hence we obtain this kind of inequality for p-forms on n-manifolds without assuming $n=2p$ (1).

(1) It would be of interest to investigate the generalized kind of inequality in view of the limitations in the applicability of the usual kind as described recently by L. Hörmander [9].

Remark 3. *On solutions of elliptic inequalities of second order.* We mentioned in the introduction that we can now prove the unique continuation for such solutions under much weaker hypotheses than was done before. This result is obtained by using our inequality rather than the theorem.

If a^{ij} are the leading coefficients of the elliptic linear operator in question—we assume that these are in class $C^{0,1}$—the differential inequality can be written in the form

$$Q(\delta d f) \leqslant M\,[|f|^2 + Q(df)], \quad f \text{ being a function.}$$

We consider now the $(n-1)$-form $u = \star d f$. We assume, as in the preceding Remark, that the metric a_{ij} is already transformed and the local coordinates chosen so that the geodesic distance from the origin equals the Euclidean distance. We extend the manifold to a $(2n-2)$-dimensional one and the metric to $\tilde{a}_{ij}$ and then define a suitable metric $\tilde{\tilde{a}}_{ij}$ with geodesic distance R from the origin as in the preceding Remark.

The form u extended to the new manifold as in the proof of Proposition 1, § 2 has a zero of infinite order at the origin. Thus we could proceed with the proof as given at the end of § 2 if it were not for the change in our differential inequality which now appears on the original manifold as

$$Q(du) \leqslant M\,[Q(u) + |f|^2], \quad d\star u = 0, \quad u = \star d f.$$

On the extended manifold in the metric $\tilde{a}_{ij}$ these relations become

$$\tilde{\tilde{Q}}(u) = e^{2\theta(n-1)|\tilde{x}|}\,\tilde{Q}(u) = e^{2\theta(n-1)|\tilde{x}|}\,Q(u) = e^{2\theta(n-1)|\tilde{x}|}\,Q(df)$$
$$= e^{2\theta(n-1)|\tilde{x}|}\,\tilde{Q}(df) = e^{2\theta(n-2)|\tilde{x}|}\,\tilde{\tilde{Q}}(df).$$

We notice that $\tilde{\tilde{Q}}(df) \geqslant |\partial f/\partial R|^2$ and obtain an immediate evaluation

$$\int_{R<s_1} R^{-2\alpha}|f|^2 \sqrt{\tilde{\tilde{a}}}\,d\tilde{x} \leqslant \frac{s_1^2\,\Lambda^0}{2(2\alpha-2n+4)} \int_{R<s_1} R^{-2\alpha}\left|\frac{\partial f}{\partial R}\right|^2 \sqrt{\tilde{\tilde{a}}}\,d\tilde{x}$$
$$\leqslant \frac{s_1^2\,\Lambda'}{2(2\alpha-2n+4)} \int_{R<s_1} R^{-2\alpha}\,\tilde{\tilde{Q}}(u)\sqrt{\tilde{\tilde{a}}}\,d\tilde{x}.$$

for every $s_1 < R_0$, the constants Λ^0, Λ' depending only on R_0 and the constants Λ_1, Λ_2, and Λ corresponding to the metric a_{ij}.

Obviously we can now eliminate the obstructing term, $\int R^{-2\alpha}|f|^2\sqrt{\tilde{\tilde{a}}}\,d\tilde{x}$, and finish the proof in the same way as at the end of § 2.

Remark 4: *Geometric significance of our inequality of Carleman type.* Inequality (2.13) has an intrinsic meaning for a p-form u on a $2p$-dimensional Riemannian manifold with the metric $\tilde{\tilde{a}}_{ij}$, where R is the geodesic distance from a point O of the manifold. Our proof in § 3 can be interpreted solely in terms of the metric $\tilde{\tilde{a}}_{ij}$ without reference to the metric a_{ij} from which it originated and without the change of variables $R = e^{-\varrho}$. The operator $*_R$ defined on the surfaces Σ_R of concentric geodesic spheres with center O has the property: $(\partial *_R/\partial R) *_R$ is negative semi-definite relative to the norm $Q_R(u)$ for p-forms u on Σ_R. It should be noted that in order to form the derivative of $*_R$ we transfer the p-form along the geodesics issued from O, *not* by the parallel shift on the Riemannian manifold. The correspondence is given by polar geodesic coordinates with pole at O. The inequality is true for every sphere $R < s$ and for every $\alpha \geqslant 0$ in each sphere $R < s$, in which in addition, the operator $(\partial *_R/\partial R) *_R$ has a bound $\leqslant 1/R$. In general we shall have to assume that the metric has sufficient regularity (say at least $C^{1,1}$) to assure the existence of polar geodesic spheres in which the geodesics issued from O do not intersect. A number of questions arise in this connection:

(1) Are the above conditions necessary for the validity of our inequality for any form u vanishing outside of a compact in $[R < s]$ and with a zero of infinite order at O?

(2) What is the connection of the above properties to the usual properties of Riemannian manifolds?

(3) If we assume the above properties to hold on surfaces of geodesic spheres for every center O on the manifold, what kind of manifolds will be obtained?

It is not clear if the answers to (2) and (3) can be given in terms of usual properties of Riemannian manifolds. It seems probable that the answer to (3) will have to do with the positive curvature of the metric.

Remark 5: *On sets of zeros of p-forms satisfying differential inequalities.* Our theorem and the inequality allows us to obtain statements concerning the sets of zeros (not of infinite order) of a p-form u satisfying a differential inequality of type (iv) of § 2.

We have already introduced the notion of a zero of order β in 2-mean (see § 4, IV, Remark) which means that

$$\int\limits_{|x|<\varrho} \sum_{(i)} |v_{(i)}|^2 \, dx = O(\varrho^{2\beta+n}), \tag{$*$}$$

where we consider the Euclidean metric in any system of local coordinates with origin at the zero O in question and where n is the dimension of the manifold.

u and $*u$ have the same zeros of the same order. We make the construction as in the proof of proposition 1, § 2, and extend the form u (or $*u$) to the extended manifold of dimension $2p$ (or $2(n-p)$). We obtain a form v on the extended manifold which has a zero of order β at each point of $O \times R^{|2p-n|}$ if and only if the original form u has a zero of order β at O.

We can therefore restrict ourselves to the study of p-forms on a $2p$-dimensional manifold, satisfying a differential inequality of type (iv), § 2.

The statement in § 4, IV, tells us that at every zero of the form v of order $> \beta$, our inequality is valid with $\alpha \leqslant \beta + n/2 - 1$, and from the inequality we deduce immediately that for all such zeros contained in a compact subset K of the manifold there exist constants Γ_K and R_K such that the integral of $\tilde{\tilde{Q}}(v)$

over every geodesic sphere (in the metric $\tilde{\tilde{a}}_{ij}$) centered at such a zero and with radius $\varrho \leqslant R_K$ is $\leqslant \Gamma_K \varrho^{2\beta+n-2}$.

This fact gives rise immediately to the following statement:

(*a*) *A point p of the manifold which is a limit of a sequence of zeros of v of orders increasing to ∞ is a zero of infinite order.*

For the next statement we introduce some definitions. Consider a point p which is a limit point of a set P on the manifold. For each $\varepsilon > 0$ denote by $\gamma(\varepsilon)$ the infimum of all ϱ's such that the geodesic spheres of radius ϱ with centers in P cover the whole geodesic sphere with center p and radius ε. We have, obviously, $0 < \gamma(\varepsilon) \leqslant \varepsilon$. By a familiar argument from the theory of Vitali coverings we deduce that the sphere $S(p, \varepsilon)$ can be covered by a finite number of spheres $S(p_k, 3\gamma(\varepsilon))$, $k = 1, 2, \ldots, N$, $p_k \in P$, with the number N of order $(\varepsilon/\gamma(\varepsilon))^n$.

If $\gamma(\varepsilon) = O(\varepsilon^\tau)$, we will say that the density of P at p is of order τ. If the density is of every order $\tau > 1$ we say that it is of infinite order (1). We now state:

(*b*) *If at a point p of the manifold the set P_β of zeros of v of order $\beta > 1$ has a density of infinite order, then v has at p a zero of infinite order.*

From (*a*) and (*b*) and Theorem 1 we get the following consequences:

(*A*) *If v does not vanish identically, then on every compact subset of the manifold the zeros of v have uniformly bounded orders.*

(*B*) *If v does not vanish identically, the set P_β of zeros of v of order $\beta > 1$ does not have a density of infinite order at any of its limit points on the manifold.*

Statement (*B*) seems rather weak. It is possible that actually the density cannot be of higher order than 1 and that the statement remains true if we take zeros of any positive order β.

It should be noticed that statements similar to (*A*) and (*B*) can be obtained for any system of differential inequalities with linear operators of any orders for which a Carleman type inequality with a single pole is valid.

Remark 6*: On weakening the hypotheses in Theorem* I. In view of Pliś's counter-example we cannot expect to be able to lower the global hypothesis on the metric a_{ij} very markedly, for instance, to replace the class $C^{0,1}$ by a Hölderian class $C^{0,\lambda}$ with $0 < \lambda < 1$.

However, in this counter-example, the lack of Lipschitzian property appears on a whole hypersurface (2). It therefore seems possible, a priori, that the Lipschitz condition could be relaxed at a single point, namely, the zero of infinite order of u, without losing the theorem.

(1) To illustrate these notions take the point $p = 0$ in the plane of the complex variable ζ. If P is the set of concentric circumferences $|\zeta| = m^{-\delta}$, $0 < \delta < 1$, $m = 1, 2, \ldots$, its density at 0 is of order $1 + 1/\delta$; if P is an enumerable set distributed on $|\zeta| = 1/\log m$, $m = 2, 3, \ldots$, the part of P on $|\zeta| = 1/\log m$ being formed by m equidistant points, the density of P at 0 is of infinite order.

(2) It is actually a counter-example to the uniqueness of Cauchy problem and the hypersurface is the one where the Cauchy data are prescribed.

We place ourselves in the conditions of Theorem II, § 2, where we may assume $n=2p$. We still assume the continuity everywhere of the a_{ij}; hence Λ_1 and Λ_2 are constant. However, we shall now consider Λ as a function of r, increasing to $+\infty$ when $r\searrow 0$. It is easy to check the evaluations in § 4, II–III, in order to see that $\tilde{\Lambda}_1$ and $\tilde{\Lambda}_2$ and $\tilde{M}$ are the same as before ($\tilde{M}$ being so because $n=2p$). But $\tilde{\Lambda}$ is now a function of r increasing to $+\infty$ with $1/r$ (1).

To pass to the metric $\tilde{\tilde{a}}_{ij}$ we must replace the constant θ by a suitable function $\theta(r)$ increasing to $+\infty$ with $1/r$. We put

$$\tilde{\tilde{a}}_{ij}=e^{-2\theta(r)r}\,\tilde{a}_{ij},\qquad R=\int_0^r e^{-\theta(s)s}\,ds,\qquad \Phi_1(r)=\frac{e^{\theta(r)r}}{r}R.$$

The evaluations for $\tilde{\tilde{\Lambda}}_1$, $\tilde{\tilde{\Lambda}}_2$ and $\tilde{\tilde{M}}$ are again the same, and for $\tilde{\tilde{\Lambda}}$ we obtain a function increasing to $+\infty$ with $1/r$. Everything will now be settled if the inequality is proved.

The ω of (3.6) is evaluated as in § 4, V, and we obtain a function $\omega(r)\nearrow+\infty$ with $1/r$. The crucial point in the proof in § 3 is the fact that $\ast_\varrho'\ast_\varrho>0$ and $|\ast_\varrho'\ast_\varrho|<1$. The expression for $\ast_\varrho'\ast_\varrho$ is now

$$\ast_\varrho'\ast_\varrho=R\,e^{\theta(r)r}\left[\frac{1}{\Phi_1(r)}\frac{d\Phi_1(r)}{dr}I-\frac{\partial\tilde{\ast}_r}{\partial r}\tilde{\ast}_r\right],\quad\text{with}\quad\left|\frac{\partial\tilde{\ast}_r}{\partial r}\tilde{\ast}_r\right|\leqslant\frac{2p-1}{2}\omega(r).$$

Hence everything depends on the possibility of a right choice of $\theta(r)$ so that for r sufficiently small

$$\frac{1}{\Phi_1(r)}\frac{d\Phi_1(r)}{dr}>\frac{2p-1}{2}\omega(r)\quad\text{and}\quad 2r\frac{d\Phi_1(r)}{dr}<1.$$

Such a choice of $\theta(r)$ can be made if $\omega(r)$ and therefore $\Lambda(r)$ do not increase too rapidly to $+\infty$ for $r\searrow 0$.

As an example consider $\Lambda(r)=\hat{\Lambda}\,r^{-\lambda}$, $0<\lambda<1$, $\hat{\Lambda}$ constant. This corresponds to a Hölder condition for a_{ij} at 0 of order $1-\lambda$. Evaluating, we get $\omega(r)=\hat{\omega}\,r^{-\lambda}$ with constant $\hat{\omega}$. We choose $\theta(r)=\hat{\theta}\,r^{-\lambda}$ with $\hat{\theta}=(2p-1)\hat{\omega}\,(2-\lambda)/(1-\lambda)^2$. Using the series development of the exponential, one shows that the conditions are satisfied, since

$$R=r-\frac{\hat{\theta}}{2-\lambda}r^{2-\lambda}+\frac{\hat{\theta}^2}{2(3-2\lambda)}r^{3-2\lambda}-\dots,$$

$$\Phi_1(r)=1+\frac{\hat{\theta}(1-\lambda)}{2-\lambda}r^{1-\lambda}+\frac{\hat{\theta}^2(1-\lambda)^2}{(2-\lambda)(3-2\lambda)}r^{2-2\lambda}+\dots,$$

$$\frac{d\Phi_1(r)}{dr}=\frac{\hat{\theta}(1-\lambda)^2}{2-\lambda}r^{-\lambda}+\dots.$$

(1) The expression of $\tilde{\Lambda}$ also now depends on the modulus of continuity of the a_{ij} because of the evaluation of $|T_x-T_0|$ in § 4.II.

Before concluding this remark we notice that—similarly to other cases of unique continuation for differential inequalities (e.g. [2])—we can replace the constant M in inequality (2.4) by the function $M r^{-\gamma}$ with $0 < \gamma < 2$.

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