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Quadratic set-valued functions

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A real-valued function $L(x)$ which is defined on a real linear space R will be called in accordance with S. Kurepa [2] a *quadratic functional* if

$$L(x+y)+L(x-y)=2L(x)+2L(y)$$

holds true for every pair x, y in R.

The main objective of this note will be to investigate cases in which for set-valued functionals the relation $M(t+s)+M(t-s)=2M(t)+2M(s)$ will be equivalent to $M(t)=t^2M(1)$ where t, s are any real numbers.

Let E^n denote an n-dimensional linear space, A, B, C, M ... are subsets of E^n. By $A+B$ we denote the sumset of A and B, i.e. the set of all vectors $a+b$ with a in A and b in B. By tM we denote the set of all tm where m is in M.

We will now proceed to prove a theorem which is related to results obtained by H. Rådström [1]. The notion of continuity for set-valued functions will be used in the sense defined in that paper.

Theorem. *Let $M(t)$ be a function which to each real number assigns a compact set in E^n in such a way that*

$$M(t+s)+M(t-s)=2M(t)+2M(s). \tag{1}$$

If $M(t)$ is a continuous function of t, then the function $M(t)$ has the form $M(t)=t^2M(1)$.

If $M(t)$ in addition is assumed to be convex the converse is also true.

Proof. The sufficient condition will be proven by use of induction.

Let $s=t=0$ in $M(t+s)+M(t-s)=2M(t)+2M(s)$. It follows that $M(0)+M(0)=2[M(0)+M(0)]$ which shows that $M(0)+M(0)$, being bounded, is equal to $\{0\}$. Hence $M(0)=\{0\}$.

Next let $s=0$. Then $M(t)+M(t)=2M(t)$ which shows that $M(t)$ is convex since it is closed.

The relation $M(nt)=n^2M(t)$ is therefore true for $n=0$ and is trivially true for $n=1$. Assume now that $M(nt)=n^2M(t)$ holds for all $n\leqslant m$. We will show that the relation holds also for $n=m+1$. We make use of the fact that if M is any convex set in a linear space R and α, β are non-negative real numbers then

$$(\alpha+\beta)M=\alpha M+\beta M.$$

Let $t = sm$ in (1): then $M[(m+1)s] + M[(m-1)s] = 2M(ms) + 2M(s)$. This gives $M[(m+1)s] = [2m^2 + 2 - (m-1)^2]M(s) = (m+1)^2 M(s)$. Therefore $M(nt) = n^2 M(t)$ for every natural number n. Replacing t by t/n in $M(nt) = n^2 M(t)$ we obtain $M(t) = n^2 M(t/n)$. Thus

$$M[(n/m)t] = n^2 M(t/m) = (n/m)^2 M(t).$$

Since, as we shall see below, $M(t) = M(-t)$ it follows that for every real number t and for every rational number r

$$M(rt) = r^2 M(t).$$

Also $M(y) = y^2 M(1)$ for any *real number* y. Since $M(t)$ is a continuous function by hypotheses then $r_n \to y$ and

$$M(y) = \lim M(r_n) = \lim r_n^2 M(1) = y^2 M(1).$$

We prove $M(s) = M(-s)$ by putting $t = 0$ in (1). Then $M(s) + M(-s) = 2M(s)$. Here the left member does not change if s is replaced by $-s$. Therefore the sufficiency of the theorem is proven.

For the converse, assume $M(t) = t^2 M$ where M is convex. Then

$$\begin{aligned} M(t+s) + M(t-s) &= (t+s)^2 M + (t-s)^2 M \\ &= [(t+s)^2 + (t-s)^2] M \\ &= (2t^2 + 2s^2) M = 2M(t) + 2M(s). \end{aligned}$$ Q.E.D.

REFERENCES

1. H. RÅDSTRÖM, One-parameter semigroups of subsets of a real linear space.
2. S. KUREPA, On the functional equation $f(x+y) \cdot f(x) f(y) \cdot g(x) g(y)$.

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