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On the asymptotic distribution of sums of independent identically distributed random variables

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1. Introduction

Let X_i $(i=1, 2, \ldots)$ be independent random variables with the common distribution function $F(x)$. Let $F_n(x)$ be the d.f. of the sum $S_n = X_1 + X_2 + \cdots + X_n$. We define the probabilities

$$a_n = \text{Prob}\,(S_n < 0), \quad n = 1, 2, \ldots .$$

Let I_n be intervals on the x-axis. Theorem 1 is concerned with the problem of giving upper bounds for the probabilities Prob $(S_n \in I_n)$ for some different types of interval families.

In Theorem 2 we give an inversion formula for characteristic functions.

We derive the following result in Theorem 3. If $E X_i = 0$, $E X_i^2 = \sigma^2 > 0$, then the series

$$\sum_1^\infty \frac{1}{n}\,(a_n - \tfrac{1}{2}) \tag{1.1}$$

is absolutely convergent. This strengthens the result, derived by F. Spitzer [2], that (1.1) is convergent.

2. Asymptotic properties of $F_n(x)$

Let $\varphi(t)$ be the characteristic function of the d.f. $F(x)$, i.e.

$$\varphi(t) = \int_{-\infty}^{\infty} e^{ixt}\, dF(x).$$

Lemma 1. *Let X be a nondegenerate r.v. with d.f. $F(x)$ and c.f. $\varphi(t)$. There exist two constants $\delta > 0$ and $C > 0$ such that*

$$|\varphi(t)| \leqslant 1 - Ct^2 \quad \textit{for} \quad |t| \leqslant \delta.$$

Proof. (1) The Lemma is true for a variable with mean zero and finite second moment $= \sigma^2$, because we then have

$$\varphi(t) = 1 - \tfrac{1}{2}\,\sigma^2 t^2 + t^2 o(1).$$

Thus $$|\varphi(t)| \leqslant 1 - t^2\left(\tfrac{1}{2}\sigma^2 - |o(1)|\right).$$

We now choose C and δ such that

$$\tfrac{1}{2}\sigma^2 - |o(1)| \geqslant C > 0 \quad \text{for} \quad |t| \leqslant \delta.$$

This is possible because $|o(1)| \to 0$ when $t \to 0$ and $\sigma^2 > 0$, as X is nondegenerate.

(2) The Lemma is valid for any distribution with a finite second moment, because the c.f. of such a distribution can be written

$$\varphi(t) = e^{i\mu t}\psi(t),$$

where $\mu = EX$ and $\psi(t)$ is the c.f. of a distribution with two finite moments and mean zero. In virtue of (1) we thus get

$$|\varphi(t)| = |\psi(t)| \leqslant 1 - Ct^2 \text{ for } |t| \leqslant \delta.$$

(3) The Lemma is true for any nondegenerate distribution

$$|\varphi(t)| = \left|\int_{-\infty}^{\infty} e^{itx}\,dF(x)\right| \leqslant \left|\int_{|x|\leqslant A} e^{itx}\,dF(x)\right| + \int_{|x|>A} dF(x).$$

We denote $$\int_{|x|\leqslant A} dF(x) = m. \quad \text{According to (2)}$$

$$\left|\int_{|x|\leqslant A} e^{itx}\,dF(x)\right| \leqslant m(1 - Ct^2) \quad \text{for} \quad |t| \leqslant \delta.$$

Thus $$|\varphi(t)| \leqslant m(1 - Ct^2) + 1 - m = 1 - mCt^2 \quad \text{for} \quad |t| \leqslant \delta$$

and m is positive if we choose A large enough.

Thus the Lemma is proved.

Theorem 1. *Let X_i $(i = 1, 2, \ldots)$ be independent random variables with the common d.f. $F(x)$, which is nondegenerate. Let $F_n(x)$ be the d.f. of the sum*

$$S_n = X_1 + X_2 + \cdots + X_n.$$

I_n is an interval on the x-axis and $1(I_n)$ its length. C is a constant which is independent of n and I_n.

(a) *If $1(I_n) \leqslant n^p$, $0 < p < \frac{1}{2}$, then*

$$\text{Prob}\,(S_n \in I_n) \leqslant C/n^{\frac{1}{2}-p}.$$

(b) *If $1(I_n) \leqslant \varepsilon\sqrt{n}$, $\varepsilon > 0$, then*

$$\text{Prob}\,(S_n \in I_n) \leqslant \varepsilon(C + \xi(\varepsilon, n))$$

where $\xi(\varepsilon, n) \to 0$ when $n \to \infty$ for every fixed $\varepsilon > 0$.

(c) *If* $l(I_n) \leqslant M$ *(constant) then*

$$\text{Prob}\ (S_n \in I_n) \leqslant C/\sqrt{n}.$$

(d) $\max_a \text{Prob}\ (S_n = a) \leqslant C/\sqrt{n}$.

These results cannot be generally improved.

Proof. We use two auxiliary functions $\psi_n(t)$ and $\hat{\psi}_n(x)$ with the properties

$$\int_{-\infty}^{\infty} |\psi_n(t)|\, dt < \infty,\ |\psi_n(t)| \leqslant 1, \tag{1}$$

$$\hat{\psi}_n(x) = \int_{-\infty}^{\infty} e^{ixt} \psi_n(t)\, dt, \tag{2}$$

$$\hat{\psi}_n(x) \geqslant 0. \tag{3}$$

The c.f. of $F_n(x)$ is $\varphi(t)^n$. Thus

$$\int_{-\infty}^{\infty} \hat{\psi}_n(x)\, dF_n(x) = \int_{-\infty}^{\infty} \psi_n(t)\, \varphi(t)^n\, dt. \tag{2.1}$$

As $\hat{\psi}_n(x) \geqslant 0$, we can estimate

$$\int_{-\infty}^{\infty} \hat{\psi}_n(x)\, dF_n(x) \geqslant \int_{I_n} \hat{\psi}_n(x)\, dF_n(x) \geqslant \min_{x \in I_n} \hat{\psi}_n(x) \int_{I_n} dF_n(x),$$

which combined with (2.1) gives

$$\int_{I_n} dF_n(x) \leqslant \left\{\min_{x \in I_n} \hat{\psi}_n(x)\right\}^{-1} \int_{-\infty}^{\infty} |\varphi(t)|^n\, |\psi_n(t)|\, dt.$$

As $|\varphi(t)| \leqslant 1$ and $|\psi_n(t)| \leqslant 1$, we get

$$\int_{I_n} dF_n(x) \leqslant \left\{\min_{x \in I_n} \hat{\psi}_n(x)\right\}^{-1} \left\{\int_{|t| \leqslant \delta} |\varphi(t)|^n\, dt + \int_{|t| > \delta} |\psi_n(t)|\, dt\right\},$$

where δ is the δ in Lemma 1.

In virtue of this Lemma we have

$$\int_{|t| \leqslant \delta} |\varphi(t)|^n\, dt \leqslant \int_{|t| \leqslant \delta} (1 - Ct^2)^n\, dt \leqslant \int_{|t| \leqslant \delta} e^{-Cnt^2}\, dt \leqslant \frac{C_1}{\sqrt{n}},$$

where C_1 is independent of n. Thus

$$\int_{I_n} dF_n(x) \leqslant \left\{\min_{x \in I_n} \hat{\psi}_n(x)\right\}^{-1} \left\{\frac{C_1}{\sqrt{n}} + \int_{|t| > \delta} |\psi_n(t)|\, dt\right\}. \tag{2.2}$$

We now choose the functions $\psi_n(t)$ and $\hat{\psi}_n(x)$ conveniently.

To prove (a) we choose

$$\hat{\psi}_n(x) = (\sqrt{2\pi}/n^p)\exp\{-(x-\mu_n)^2/2n^{2p}\}$$

and

$$\psi_n(t) = \exp\left(-\tfrac{1}{2}\cdot t^2 n^{2p} - i\mu_n t\right),$$

where μ_n is the midpoint of I_n. It is easily verified that $\hat{\psi}_n(x)$ and $\psi_n(t)$ are functions with the desired properties.

As $|x-\mu_n| \leqslant \tfrac{1}{2}n^p$, for $x \in I_n$ we get

$$\min_{x\in I_n} \hat{\psi}_n(x) \geqslant \frac{e^{-\frac{1}{8}}\sqrt{2\pi}}{n^p}$$

and (2.2) gives

$$\int_{I_n} dF_n(x) \leqslant \frac{n^p}{e^{-\frac{1}{8}}\sqrt{2\pi}}\left\{\frac{C_1}{\sqrt{n}} + \int_\delta^\infty \exp\left(-\tfrac{1}{2}\cdot t^2\cdot n^{2p}\right)dt\right\}.$$

For the last integral we have

$$\int_\delta^\infty \exp\left(-\tfrac{1}{2}\cdot t^2\cdot n^{2p}\right)dt \leqslant \frac{C_2}{\sqrt{n}},$$

where C_2 is independent of n and I_n. Thus

$$\int_{I_n} dF_n(x) \leqslant \frac{C}{n^{\frac{1}{2}-p}}$$

and (a) is proved.

In case (b) we choose

$$\hat{\psi}_n(x) = \frac{\sqrt{2\pi}}{\varepsilon\sqrt{n}}\exp\{-(x-\mu_n)^2/2\varepsilon^2 n\}$$

and

$$\psi_n(t) = \exp\left(-\tfrac{1}{2}\cdot t^2\cdot\varepsilon^2\cdot n - i\mu_n t\right).$$

Then

$$\min_{x\in I_n} \hat{\psi}_n(x) \geqslant e^{-\frac{1}{8}}\sqrt{2\pi}\cdot(\varepsilon\sqrt{n})^{-1}$$

and (2.2) gives

$$\int_{I_n} dF_n(x) \leqslant \varepsilon\left\{C_1 + \frac{\sqrt{n}}{e^{-\frac{1}{8}}\sqrt{2\pi}}\int_\delta^\infty \exp\left(-\tfrac{1}{2}\cdot t^2\cdot\varepsilon^2\cdot n\right)dt\right\},$$

where the function

$$\xi(n,\varepsilon) = \frac{\sqrt{n}}{e^{-\frac{1}{8}}\sqrt{2\pi}}\int_\delta^\infty \exp\left(-\tfrac{1}{2}\cdot t^2\cdot\varepsilon^2\cdot n\right)dt$$

satisfies $\lim_{n\to\infty} \xi(n,\varepsilon) = 0$ for every fixed $\varepsilon > 0$.

This proves (b).

Choose
$$\hat{\psi}_n(x)=\delta_1\left(\frac{\sin \frac{1}{2}\delta_1(x-\mu_n)}{\frac{1}{2}\delta_1(x-\mu_n)}\right)^2$$

and
$$\psi_n(t)=\begin{cases}(1-|t/\delta_1|)\exp(i\mu_n t) & \text{for } |t|\leqslant\delta_1,\\ 0 & \text{for } |t|>\delta_1,\end{cases}$$

where μ_n is the midpoint of the interval I_n. δ_1 is chosen so that $\delta_1\leqslant\delta$ and $M\leqslant 2\pi/\delta_1$ which assures that

$$\min_{x\in I_n}\hat{\psi}_n(x)\geqslant\varrho>0.$$

Thus (2.2) gives
$$\int_{I_n} dF_n(x)\leqslant\frac{C}{\sqrt{n}},$$

which proves (c).

We can choose I_n so that it covers the maximal jump of $F_n(x)$. Then (*d*) immediately follows from (c).

Let $F(x)$ be the normal distribution with mean zero and variance 1. Then

$$\frac{\text{Prob}\,(|S_n|\leqslant n^p)}{n^{\frac{1}{2}-p}}\to\sqrt{\frac{2}{\pi}},\quad 0<p<\tfrac{1}{2}.$$

This shows that the result in (a) cannot be generally improved. The same is true for the result in (b). The proof is not difficult but somewhat laborious and we omit it. An example which shows that the results in (c) and (d) cannot be improved is given in, e.g. [1], p. 53.

3. An inversion formula

Theorem 2. *Let $F(x)$ be a d.f. and $\varphi(t)$ its c.f. If* $\int_{-\infty}^{\infty}\log(1+|x|)\,dF(x)<\infty$ *then the following inversion formula holds:*

$$\tfrac{1}{2}[F(x-0)+F(x+0)]=\frac{1}{2}+\lim_{N\to\infty}\frac{1}{2\pi i}\int_0^N\frac{1}{t}\{e^{ixt}\varphi(-t)-e^{-ixt}\varphi(t)\}\,dt.$$

Proof. We define

$$\psi(y,x)=\frac{1}{2}+\frac{1}{\pi}\int_0^\infty\frac{\sin(x-y)t}{t}\,dt=\begin{cases}1 & (y<x),\\ \frac{1}{2} & (y=x),\\ 0 & (y>x).\end{cases}$$

Then
$$\tfrac{1}{2}[F(x-0)+F(x+0)]=\int_{-\infty}^{\infty}\psi(y,x)\,dF(y)=$$

where the right-hand side is an $L-S-$integral

$$=\frac{1}{2}+\int_{y\neq x}\{\psi(y,x)-\tfrac{1}{2}\}\,dF(y)=\frac{1}{2}+\frac{1}{\pi}\int_{y\neq x}dF(y)\int_0^\infty\frac{\sin(x-y)t}{t}\,dt$$

$$=\frac{1}{2}+\frac{1}{\pi}\int_{-\infty}^{\infty}dF(y)\int_0^N\frac{\sin(x-y)t}{t}\,dt+\frac{1}{\pi}\int_{y\neq x}dF(y)\int_N^\infty\frac{\sin(x-y)t}{t}\,dt$$

$$=\tfrac{1}{2}+I_1(N)+I_2(N).$$

For $I_1(N)$ we have

$$\pi\,|I_1(N)|\leqslant\int_{-\infty}^{\infty}dF(y)\int_0^N\left|\frac{\sin(x-y)t}{t}\right|dt=\int_{-\infty}^{\infty}dF(y)\int_0^{N|x-y|}\left|\frac{\sin s}{s}\right|ds$$

$$\leqslant\int_{-\infty}^{\infty}dF(y)\,\{1+\log(1+N|x-y|)\}.$$

In virtue of the assumption $\int_{-\infty}^{\infty}\log(1+|x|)\,dF(x)<\infty$ we thus have that $I_1(N)$ is absolutely convergent. Therefore we can change the order of integration in $I_1(N)$. This gives

$$I_1(N)=\frac{1}{2\pi i}\int_0^N\frac{dt}{t}\int_{-\infty}^{\infty}\{e^{i(x-y)t}-e^{-i(x-y)t}\}\,dF(y)$$

$$=\frac{1}{2\pi i}\int_0^N\frac{1}{t}\,\{e^{ixt}\varphi(-t)-e^{-ixt}\varphi(t)\}\,dt.$$

The following estimations are well known. For $N>0$

$$\left|\int_N^\infty\frac{\sin xt}{t}\,dt\right|\leqslant\frac{C_1}{|x|N}\tag{3.1}$$

$$\left|\int_N^\infty\frac{\sin xt}{t}\,dt\right|\leqslant C_2,\tag{3.2}$$

where C_1 and C_2 are absolute constants.

Estimate $I_2(N)$ as follows

$$\pi\,|I_2(N)|\leqslant\int_{|x-y|\geq 1/N}dF(y)\left|\int_N^\infty\frac{\sin(x-y)t}{t}\,dt\right|+\int_{0<|x-y|<1/N}dF(y)\left|\int_N^\infty\frac{\sin(x-y)t}{t}\,dt\right|.$$

According to (3.1) and (3.2) we get

$$\pi\,|I_2(N)|\leqslant\frac{C_1}{N}\int_{|x-y|\geq 1/N}\frac{dF(y)}{|x-y|}+C_2\int_{0<|x-y|<1/N}dF(y).$$

We have
$$\int_{0<|x-y|<1/N} dF(y) \to 0 \text{ for } N \to \infty$$

and
$$\int_{|x-y|\geq 1/N} \frac{dF(y)}{N|x-y|} \to 0 \text{ for } N \to \infty$$

by Lebesgue's theorem on dominated convergence, as

$$(N|x-y|)^{-1} \leqslant 1 \text{ for } |x-y| \geqslant 1/N$$

and
$$(N|x-y|)^{-1} \to 0 \text{ when } N \to \infty.$$

Thus $I_2(N) \to 0$ when $N \to \infty$.

Summing up

$$\tfrac{1}{2}[F(x-0)+F(x+0)] = \frac{1}{2} + \frac{1}{2\pi i}\int_0^N \frac{1}{t}\{e^{ixt}\varphi(-t) - e^{-ixt}\varphi(t)\}\,dt + I_2(N).$$

$N \to \infty$ gives the theorem.

We now apply the inversion formula to the d.f. $F_n(x)$ with c.f. $\varphi^n(t)$.

$$\tfrac{1}{2}[F_n(x-0)+F_n(x+0)] = \frac{1}{2} + \frac{1}{2\pi i}\int_0^\delta \frac{1}{t}\{e^{ixt}\varphi(-t)^n - e^{-ixt}\varphi(t)^n\}\,dt + R(n,x,\delta), \quad (3.3)$$

where δ is a positive number and

$$R(n,x,\delta) = \frac{1}{\pi}\int_{-\infty}^{\infty} dF_n(x)\int_\delta^\infty \frac{\sin(x-y)t}{t}\,dt.$$

Lemma 2. *$R(n,x,\delta)$ satisfies for fixed $\delta>0$ the inequality*

$$|R(n,x,\delta)| \leqslant C n^{-\frac{1}{4}}$$

where C is independent of n and x.

Proof.
$$\pi|R(n,x,\delta)| \leqslant \int_{-\infty}^{\infty} dF_n(y)\left|\int_\delta^\infty \frac{\sin(x-y)t}{t}\,dt\right|$$
$$\leqslant C_2\int_{|x-y|\leq \sqrt[4]{n}} dF_n(y) + C_1\int_{|x-y|>\sqrt[4]{n}} \frac{dF_n(x)}{|x-y|}$$

according to (3.1) and (3.2). Theorem 1 (a) gives

$$|R(n,x,\delta)| \leqslant C n^{-\frac{1}{4}}.$$

4. On the series $\sum_1^\infty \frac{1}{n}(a_n - \frac{1}{2})$

We introduce the notation $a_n = \text{Prob}\,(S_n < 0)$.

Theorem 3. *If* $E X_i = 0$ *and* $E X_i^2 = \sigma^2$, $0 < \sigma^2 < \infty$, *then the series* $\sum_1^\infty \frac{1}{n}(a_n - \frac{1}{2})$ *converges absolutely.*

We first need a lemma. Let $\varphi(t)$ be the c.f. of the variables X_i. $\varphi(t)$ has the Taylor expansion

$$\varphi(t) = 1 - \tfrac{1}{2}\sigma^2 t^2 + t^2 (R(t) + i I(t)),$$

where $R(t)$ and $I(t)$ are real functions such that $R(t) \to 0$, $I(t) \to 0$ when $t \to 0$.

Lemma 3. *For every* $\delta > 0$ *the integral* $\int_0^\delta \frac{|I(t)|}{t}\,dt$ *is convergent.*

Proof. We put $\delta = 1$, which is no loss of generality. As $\int_{-\infty}^{\infty} x\,dF(x) = 0$, we have

$$I(t) = \frac{1}{t^2}\int_{-\infty}^{\infty} (\sin tx - tx)\,dF(x)$$

and we get

$$\int_\varepsilon^1 \frac{|I(t)|}{t}\,dt \leqslant \int_{-\infty}^{\infty} dF(x) \int_\varepsilon^1 \frac{|\sin xt - xt|}{t^3}\,dt,$$

the inversion of integration being justified by absolute convergence. Put

$$\psi(x, \varepsilon) = \int_\varepsilon^1 \frac{|\sin xt - xt|}{t^3}\,dt.$$

By using the inequalities

$$|\sin xt - xt| \leqslant \tfrac{1}{6}|xt|^3 \text{ for } |xt| \leqslant 1$$

and

$$|\sin xt - xt| \leqslant 2|xt| \text{ for } |xt| > 1,$$

we get for $1 \leqslant |x| \leqslant 1/\varepsilon$

$$\psi(x, \varepsilon) \leqslant \int_\varepsilon^{1/|x|} |x^3|\,dt + \int_{1/|x|}^1 2|x| \cdot \frac{1}{t^2}\,dt = |x|^3\left\{\frac{1}{|x|} - \varepsilon\right\} + 2|x|\{|x| - 1\} \leqslant 3|x|^2.$$

For $|x| < 1$ we get
$$\psi(x, \varepsilon) \leqslant \int_\varepsilon^1 \tfrac{1}{6}|x|^3\,dt \leqslant \tfrac{1}{6}|x|^3.$$

For $|x| > 1/\varepsilon$ we have

$$\psi(x, \varepsilon) \leqslant \int_\varepsilon^1 2|x| \cdot \frac{1}{t^2}\,dt = 2|x|\left(\frac{1}{\varepsilon} - 1\right) \leqslant \frac{2|x|}{\varepsilon} \leqslant 2|x|^2.$$

Thus $0 \leqslant \psi(x, \varepsilon) \leqslant 3|x|^2$, which gives

$$\int_\varepsilon^1 \frac{|I(t)|}{t}\, dt \leqslant \int_{-\infty}^{\infty} \psi(x, \varepsilon)\, dF(x) \leqslant 3 \int_{-\infty}^{\infty} |x|^2 dF(x) = 3\sigma^2.$$

We now let $\varepsilon \to 0$ and get $\displaystyle\int_0^\delta \frac{|I(t)|}{t}\, dt < \infty$

and the Lemma is proved.

Proof of Theorem 3.

We put $x=0$ in (3.3)

$$\tfrac{1}{2}[F_n(-0)+F_n(+0)] = a_n + \tfrac{1}{2} \operatorname{Prob}\{S_n = 0\} =$$

$$= \frac{1}{2} + \frac{1}{2\pi i}\int_0^\delta \frac{1}{t}\{\varphi^n(-t) - \varphi^n(t)\}\, dt + R(n), \qquad (4.1)$$

where δ is a positive constant to be determined later.

For any c.f. $\varphi(-t) = \overline{\varphi(t)}$ holds. Thus we can write

$$a_n - \tfrac{1}{2} = \frac{1}{\pi}\int_0^\delta \frac{|\varphi(t)|^n}{t} \sin\{n \arg \varphi(t)\}\, dt + R(n) - \tfrac{1}{2}\operatorname{Prob}\{S_n = 0\},$$

which gives

$$|a_n - \tfrac{1}{2}| \leqslant \frac{n}{\pi}\int_0^\delta \frac{|\varphi(t)|^n}{t}|\arg\varphi(t)|\, dt + |R(n)| + \tfrac{1}{2}\operatorname{Prob}\{S_n = 0\}.$$

Thus

$$\sum_1^\infty \frac{1}{n}|a_n - \tfrac{1}{2}| \leqslant \frac{1}{\pi}\int_0^\delta \frac{|\arg\varphi(t)|}{t}\cdot\frac{|\varphi(t)|}{1-|\varphi(t)|}\, dt + \sum_1^\infty \frac{1}{n}|R(n)| + \tfrac{1}{2}\sum_1^\infty \frac{1}{n}\operatorname{Prob}\{S_n = 0\}.$$

From Lemma 2 we conclude

$$\sum_1^\infty \frac{1}{n}|R(n)| = D_1 < \infty$$

and Theorem 1 (*d*) gives

$$\tfrac{1}{2}\sum_1^\infty \frac{1}{n}\operatorname{Prob}\{S_n = 0\} = D_2 < \infty.$$

$$\arg\varphi(t) = \operatorname{arctg}\frac{t^2\cdot I(t)}{1-\frac{1}{2}\sigma^2 t^2 + t^2 R(t)},$$

and for δ_1 sufficiently small, there exists a constant $C_1 > 0$ so that

$$|\arg\varphi(t)| \leqslant C_1\cdot t^2|I(t)| \quad \text{for} \quad |t| \leqslant \delta_1.$$

Let δ_2 be the δ in Lemma 1. We choose δ in (4.1) to be $\delta = \min(\delta_1, \delta_2)$. By Lemma 1, $1 - |\varphi(t)| \geqslant C_2 t^2$ for $|t| \leqslant \delta$. Thus

$$\sum_1^\infty \frac{1}{n} \left| a_n - \tfrac{1}{2} \right| \leqslant \frac{C_1}{\pi C_2} \int_0^\delta \frac{|I(t)|}{t} \, dt + D_1 + D_2$$

and Lemma 3 gives

$$\sum_1^\infty \frac{1}{n} \left| a_n - \tfrac{1}{2} \right| < \infty$$

and Theorem 3 is proved.

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