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An extremal property of the Riemann zeta-function

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Introduction

When it is known *a priori* that a function φ is the solution of a certain extremal problem, this very fact usually enables us to derive without difficulty the characteristic properties of the function. If, however, the same function is explicitly defined while the extremal problem is unknown, we find ourselves in a quite different situation where it may be extremely difficult to recognize these same properties of φ.

In the special case

$$\varphi(s) = \zeta(s) = \sum_1^\infty \frac{1}{n^s} \qquad (\sigma > 1)$$

we have an explicit definition of the Riemann zeta-function from which the "elementary" properties of ζ are easily derived, whereas this definition gives very poor information as regards the truth of the Riemann hypothesis and other "deeper" properties of ζ. It should also be recalled that all this properties which we believe to be true but cannot verify, indicate that ζ in a certain sense is of minimal order of magnitude.

It therefore seems worth while to ask whether ζ is the solution of an hitherto unknown extremal problem, for example, whether in a certain class C of functions, ζ minimizes an integral of the form[1]

$$\int_{-\infty}^{\infty} |\varphi(\tfrac{1}{2} + it)|^2 p(t)\, dt. \qquad (p > 0)$$

Suppose this is so, and suppose futhermore the class C has the following property: For any $\varphi \in C$ such that $\varphi(\alpha \pm i\beta) = \varphi(1 - \alpha \pm i\beta) = 0$, $\alpha < \frac{1}{2}$, this quadruple of zeros may be displaced in such a way that φ remains in C while its modulus decreses on $\sigma = \frac{1}{2}$. Then the Riemann hypothesis is obviously true.

However this may be, it certainly is an interesting problem to investigate the extremal properties of the zeta-function. The main purpose of this paper

[1] Except for the question of the existence of classes C and weight functions p of this kind, which will not be considered here, this paper is a summary of a lecture given at the Harvard Mathematical Colloquium in 1949.

is to show that in a certain class of Dirichlet series ζ actually is of minimal growth and that it has a remarkable position in this class even in other respects.

On a class of Dirichlet series

We shall consider sequences of positive numbers $\{\lambda_n\}$,

$$0 < \lambda_1 \leq \lambda_2 \leq \ldots \leq \lambda_n \leq \ldots,$$

such that the Dirichlet series

$$\varphi(s) = \sum_1^\infty \frac{1}{\lambda_n^s} \tag{1}$$

converges for $\sigma > 1$. For any positive number k, C_k shall denote the class of series (1) with the properties

$$(\alpha) \qquad \varphi(s) - \frac{1}{s-1} \qquad \textit{is entire}$$

$$(\beta) \qquad \varphi(-2n) = 0, \qquad n = 1, 2, \ldots$$

$$(\gamma) \qquad \operatorname*{Max}_{|s|=r} |\varphi(s)| < \text{const.} \frac{\Gamma(r)}{(2\pi k)^r}, \qquad (r \geq 2).$$

The class C_k thus defined is obviously decreasing for increasing k. By D_k we shall denote the set of functions φ which for $\varepsilon > 0$ are contained in $C_{k-\varepsilon}$ but not in $C_{k+\varepsilon}$.

From the functional equation of the Riemann zeta-function,

$$\zeta(1-s) = 2 \cdot (2\pi)^{-s} \cos\frac{\pi}{2} s \cdot \Gamma(s) \cdot \zeta(s),$$

we conclude that ζ belongs to the class D_1.

Theorem: *For $0 < k \leq \frac{1}{2}$ each class D_k contains an infinity of functions, while D_k is empty for $k > \frac{1}{2}$ except for the class D_1 that contains as only elements the two functions $\zeta(s)$ and $(2^s - 1)\zeta(s)$, corresponding to the sequences $\{n\}_1^\infty$, and $\{n - \frac{1}{2}\}_1^\infty$, respectively.*

The essence of this theorem could be expressed by saying that the Riemann zeta-function is uniquely determined by the property $\lambda_1 = 1$ added to the conditions (α), (β) and (γ), if $\frac{1}{2} < k \leq 1$.

The proof will be divided into two parts, both dealing with entire functions of exponential type. To the Dirichlet series (1) we associate the entire function

$$f(z) = \prod_1^\infty \left(1 + \frac{z^2}{\lambda_n^2}\right). \tag{2}$$

It is an elementary consequence of the condition (α) that $\lambda_n/n \geq \delta > 0$ and $f(z)$ is therefore always of exponential type at most. Starting from the formula

$$\int_0^\infty \log(1+x^2)\frac{dx}{x^{1+s}} = \frac{\pi}{s \sin\frac{\pi s}{2}}, \qquad (0 < \sigma < 2)$$

we obtain

$$\int_0^\infty \log\left(1+\frac{x^2}{\lambda_n^2}\right)\frac{dx}{x^{1+s}} = \frac{1}{\lambda_n^s}\frac{\pi}{s \sin\frac{\pi s}{2}}, \qquad (0 < \sigma < 2)$$

and by summing

$$\int_0^\infty \log f(x)\frac{dx}{x^{1+s}} = \frac{\pi\varphi(s)}{s \sin\frac{\pi s}{2}} \equiv \psi(s) \qquad (1 < \sigma < 2).$$

According to our assumptions (α) and (β), $\psi(s)$ is meromorphic in $\sigma < 2$ and has poles only at $s = 1$ and $s = 0$ with the principal parts

$$\frac{\pi}{s-1}, \qquad \frac{2\varphi(0)}{s^2} + \frac{2\varphi'(0)}{s},$$

respectively.

Let us now assume that $\varphi(s)$ belongs to a certain class C_k. Comparing the functions $\psi(s)$ and

$$\psi_0(s) = (2\pi k)^s\,\Gamma(2-s),$$

we find by (γ) and the inequalities

$$|\Gamma(\tfrac{1}{2}+it)| > \sqrt{\pi}\,e^{-\frac{\pi}{2}|t|}, \qquad |\varphi(\tfrac{3}{2}+it)| = O(1),$$

that $\psi/\psi_0 = O(1)$ on the boundary of the region $\sigma < \frac{3}{2}$, $t > 0$, $|s| > 2$. On applying the Phragmén-Lindelöf principle, we find that

$$|\psi(s)| \leq \text{const}\,|\Gamma(2-s)(2\pi k)^s|$$

in the region considered, as well as in the region symmetric with respect to the real axis. Using the inequality

$$\int_{-\infty}^\infty |\Gamma(\sigma+2+it)|\,dt \leq \text{const.}\,\sigma^3\,\Gamma(\sigma), \qquad (\sigma \geq 2)$$

we obtain in particular

$$\int_{-\infty}^\infty |\psi(-\sigma+it)|\,dt \leq \text{const.}\frac{\sigma^3\,\Gamma(\sigma)}{(2\pi k)^\sigma}, \qquad (\sigma \geq 2). \tag{3}$$

Now the Mellin inversion formula yields

$$\log f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \psi(s)\, x^s\, ds, \qquad (1 < c < 2);$$

we next move the part of integration to the left beyond the poles at $s = 1$ and $s = 0$ into the left half-plane and obtain

$$\log f(x) - \pi x - 2\varphi(0)\log x - 2\varphi'(0) = \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} \psi(s)\, x^s\, ds, \qquad (c > 0).$$

Setting

$$p = 2\varphi(0), \quad a = e^{2\varphi'(0)},$$

we get by (3)

$$|\log f(x) - \log\{a x^p e^{\pi x}\}| < \text{const}\,\frac{c^3\,\Gamma(c)}{(2\pi k x)^c}, \qquad (c \geq 2).$$

Taking $c = 2\pi k x$ and letting $x \to \infty$, we find from Stirling's formula that

(4) $$|\log f(x) - \log\{a x^p e^{\pi x}\}| = O(e^{-2\pi(k-\varepsilon)x})$$

and consequently, since $f(x)$ is even,

(5) $$f(x) = a|x|^p e^{\pi|x|} + O(e^{\pi(1+\varepsilon-2k)|x|})$$

which holds for any $\varepsilon > 0$.

Let us now assume that $\{\lambda_n\}_1^\infty$, is a sequence such that the entire function (2) satisfies (5) for some real constants $a > 0$ and p. Obviously (5) implies (4) and from this relation we conclude that $\psi(s)$ is meromorphic in the half-plane $\sigma < 2$ with poles only at $s = 1$ and $s = 0$ with the principal parts

$$\frac{\pi}{s-1}, \qquad \frac{p}{s^2} + \frac{\log a}{s}.$$

In the left halfplane $\sigma < 0$, $\psi(s)$ will have the representation

$$\psi(s) = \int_0^\infty \{\log f(x) - \pi x - p\log x - \log a\}\frac{dx}{x^{1+s}}$$

which yields

$$\operatorname*{Max}_t |\psi(-\sigma + it)| = O\left(\frac{\Gamma(\sigma)}{(2\pi(k-\varepsilon))^\sigma}\right), \qquad (\sigma \to \infty).$$

By an application of the Phragmén-Lindelöf principle in the same way as above, we finally see that φ satisfies (γ) with k replaced by $k - \varepsilon$. Thus $\varphi \in C_{k-\varepsilon}$ for any $\varepsilon > 0$. We summarize our results so far in

Lemma I: *A Dirichlet series* (1) *belongs to the class* D_k *if and only if there are real constants* $a > 0$ *and* p *such that the entire function* (2) *satisfies* (5) *for* $\varepsilon > 0$ *but not for* $\varepsilon < 0$.

The rest of the proof will be concerned with the existence of entire functions of this kind. We shall prove

Lemma II: *There are only two entire functions* f *of exponential type with* $f(0) = 1$ *which on the real axis satisfy a relation of the form*

$$h(x) = f(x) - a|x|^p e^{\pi |x|} = O(e^{-\delta |x|}) \tag{6}$$

where $a > 0$, $\delta > 0$ *and* p *are real constants, viz.:*

$$f(z) = \frac{e^{\pi z} - e^{-\pi z}}{2\pi z} = \prod_1^\infty \left(1 + \left(\frac{z}{n}\right)^2\right)$$

and

$$f(z) = \frac{e^{\pi z} + e^{-\pi z}}{2} = \prod_1^\infty \left(1 + \left(\frac{z}{n - \frac{1}{2}}\right)^2\right).$$

According to a well-known property of functions holomorphic in a half-plane, an entire function f of exponential type vanishes identically if $f(x) = O(e^{-\delta|x|})$ on the real axis. From this we first conclude that $f(z) - f(-z) \equiv 0$. Our next step is to prove that the relation (6) is differentiable, i. e.

$$h^{(n)}(x) = O(e^{-\delta|x|}), \qquad (n = 1, 2, \ldots). \tag{7}$$

From the inequality $\log|f(z)| < A|z|$, $A > \pi$, $|z| > 1$, if follows on applying the Phragmén-Lindelöf principle to

$$h(z) = f(z) - a z^p e^{\pi z}, \quad (z^p > 0 \text{ for } z = x > 0)$$

in the region $(x > 1,\ y > 0)$ that

$$|h(z)| < \text{const. } e^{Ay - \delta x}, \qquad (x > 1,\ y > 0).$$

Making the same argument for $(x > 1,\ y < 0)$, we conclude in particular that $h(z) = O(e^{-\delta x})$ in the strip $(|y| < 1,\ x > 1)$ and (7) follows from the formula

$$h^{(n)}(x) = \frac{\underline{|n}}{2\pi i} \int_{|z-x|=1} \frac{h(z)}{(z-x)^{n+1}}\, dz$$

for positive as well as negative x since $h(x)$ is even.

The functions $|x|^p e^{\pi x}$ and $|x|^p e^{-\pi x}$ are both for real $x \neq 0$ solutions of the linear differential equation

$$L(u) \equiv x^2 u'' - 2pxu' - (\pi^2 x^2 - p(p+1))u = 0.$$

Thus for real $x \neq 0$

$$L(f) = L(h) = O(x^2 e^{-\delta |x|}).$$

Since, however, $L(f)$ is itself an entire function of exponential type, $L(f)$ must vanish identically and $f(x)$ is hence for $x > 0$ of the form

$$f(x) = x^p (c_1 e^{\pi x} + c_2 e^{-\pi x}).$$

As readily seen, this function can be entire and even and have the property $f(0) = 1$ if and only if either

$$p = -1, \; c_1 = -c_2 = \frac{1}{2\pi}$$

or

$$p = 0, \; c_1 = c_2 = \frac{1}{2}$$

which establishes Lemma II.

Combining the two lemmas, we obtain the latter part of the theorem and it remains only to show that each class D_k, $0 < k \leq \frac{1}{2}$, contains an infinity of Dirichlet series. Because of Lemma I the truth of this statement is rather obvious and we shall only exhibit some elementary examples proving that none of these classes is empty. Defining f by the relation

$$f(iy) = \cos \pi k y \cdot \cos \pi (1-k) y, \qquad (0 < k \leq \tfrac{1}{2})$$

we will have for real x

$$f(x) = \tfrac{1}{4} e^{\pi |x|} + \tfrac{1}{4} e^{\pi (1-2k)|x|} + O(1)$$

and hence $\varphi \in D_k$.

An interesting problem is whether or not $\zeta(s)$ will still be of minimal growth if we extend the classes of Direchlet series C_k by omitting the condition (β).

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