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The Stone–Weierstrass theorem in certain Banach algebras of Fourier type

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1. Introduction

A subset S of a Banach algebra B is said to generate B if the smallest closed subalgebra of B which contains S is the whole of B. S is then said to be a system of generators for B. The object of this paper is to find systems of generators for $L^1(R)$ and a class of related algebras.

Let U be an arbitrary compact topological space, and $C(U)$ the algebra of all complex-valued continuous functions on U with the supremum norm. Then it is easy to see that the problem of finding finite systems of generators for $C(U)$ is equivalent to the difficult problem of uniform polynomial approximation on subsets of complex n-space, which is completely solved only for $n=1$. See Mergeljan [10] and Wermer [15]. For a positive result on the number of generators, see however Browder [4].

If the problem is limited to that of finding systems of *real-valued* generators of $C(U)$ it is much simpler and completely solved by the Stone–Weierstrass theorem (see e.g. Loomis [9], p. 9), which can be stated as follows.

Theorem A. *A subset S of $C(U)$, which consists of real-valued functions and contains unity, generates $C(U)$ if and only if it separates the points of U.*

Now let U be a compact manifold, differentiable of order m and $C^m(U)$ the algebra of all m times continuously differentiable, complex-valued functions on U, with the topology of uniform convergence of the functions together with their first m derivatives. For this algebra the problem of finding real-valued generators was solved by L. Nachbin [11].

Theorem B. *A subset S of $C^m(U)$, which consists of real-valued functions and contains unity, generates $C^m(U)$ if and only if it separates the points of U, and for every point $x \in U$ and every direction σ, tangent to U at x, there is an $f \in S$ such that $df(x)/\partial\sigma \neq 0$.*

We shall consider the cases when U is the unit circle, T, or the real line, R. That R is not compact does not make any important difference. We shall study functions, f, which have Fourier transforms, $\hat{f}$, defined on the group of integers,

Z, or on R, respectively. For $p \geqslant 0$ we define function spaces, $B_p(T)$ and $B_p(R)$, as the sets of all integrable functions such that the norms

$$\|f\|_p = \{\sum_{-\infty}^{\infty} (1+|n|^p)|\hat{f}(n)|^2\}^{\frac{1}{2}} \tag{1.1}$$

and

$$\|f\|_p = \left\{\int_{-\infty}^{\infty} (1+|u|^p)|\hat{f}(u)|^2 du\right\}^{\frac{1}{2}} \tag{1.2}$$

are finite, respectively. This property is equivalent to a certain continuity property of f or its derivatives, and therefore these spaces lie, in a sense, between C and C^m. We shall show that for $p>1$ $B_p(T)$ and $B_p(R)$ are Banach algebras, and then study the problem of finding systems of real-valued generators for these algebras.

If S is a point-separating subset of one of these algebras, and E_S is the set where the first derivatives of all functions in S exist and equal zero, the result is, generally speaking, that the size of E_S is decisive for whether or not S is a system of generators. Moreover, the larger one chooses p, the smaller E_S has to be, until for $p>3$ it must be empty, for S to generate the algebra. One can compare this to Theorem A where the size of E_S is unimportant, and Theorem B, where E_S must be empty. See Theorems 3. 4, 7, 9, 10.

It was proved by Katznelson and Rudin [8] that the Stone–Weierstrass property does not hold for the group algebra $L^1(G)$ of a locally compact abelian group whose dual group is not totally disconnected. As corollaries to Theorems 3 and 4 we obtain more precise results on systems of real-valued generators for $L^1(Z)$ and $L^1(R)$, or equivalently, for the dual algebras $A(T)$ and $A(R)$ of all functions with absolutely convergent Fourier transforms. See Theorems 5 and 6.

The above-mentioned material occupies Sections 3 and 4. Section 2 contains preliminary material, notably a previously unpublished uniqueness theorem of Carleson. See Theorem 1. The results in this section are not needed in the rest of the paper, but serve to characterize certain sets of uniqueness which will be needed.

I was introduced to the problems treated here by Professor Lennart Carleson, and his generous advice and criticism have been a continual source of inspiration, and have led to numerous improvements. For all this, and for permission to include the unpublished result mentioned above, I wish to express my deep gratitude.

2. Sets of uniqueness for $B_{1-p}(T)$ and $B_{1-p}(R)$, $0 \leqslant p < 1$

If a function $c(x)$ belongs to $B_{1-p}(T)$ or $B_{1-p}(R)$, $0 \leqslant p < 1$, it is known that if $c(x)$ is suitably modified on a set of measure zero

$$c(x) = \lim_{n\to\infty} \sum_{-n}^{n} \hat{c}(\nu)\, e^{i\nu x},$$

or
$$c(x)=\lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}\hat{c}(u)\,e^{iux}du,$$

except on a set of p-capacity zero (logarithmic capacity for $p=0$). This was proved for T by Beurling [2], Salem–Zygmund [14], and Broman [3], and the proof in [3] easily extends to R. In the following we assume that $c(x)$ is thus modified. We make the following definition.

Definition 1. *A set E in T or R is called a set of uniqueness for $B_{1-p}(T)$ or $B_{1-p}(R)$, if every function, $c(x)$, in $B_{1-p}(T)$ or $B_{1-p}(R)$, such that $c(x)=0$ for all x not in E, must be identically zero.*

These sets of uniqueness are characterized by the following theorem, which is due to Ahlfors and Beurling ([1], p. 124) for $p=0$, and to Carleson in the general case. As Carleson's result has not been published we include its proof here.

We denote p-capacity (capacity with respect to the kernel r^{-p}) by C_p and logarithmic capacity by C_0. For definitions and properties of capacities we refer to Frostman [7] and Carleson [5].

Theorem 1. *A Borel subset E of the unit circle T is a set of uniqueness for $B_{1-p}(T)$, $0\leqslant p<1$, if and only if $C_p(T-E)=C_p(T)$.*

A Borel subset E of the real line R is a set of uniqueness for $B_{1-p}(R)$ if and only if $C_p(I-E)=C_p(I)$ for every interval I.

Proof. We give the proof only for R and for $p>0$. The proof for T is similar.

Let E be a set such that $C_p(I-E)=C_p(I)$ for every I. (We use the notation $A-B$ for $A\cap C(B)$.) Choose an arbitrary interval I.

We can choose an increasing sequence $\{F_n\}_1^\infty$ of closed subsets of $I-E$, such that
$$\lim_{n\to\infty}C_p(F_n)=C_p(I-E)=C_p(I).$$

For each F_n there is a positive unit measure μ_n, the equilibrium measure, such that the corresponding potential,
$$U_n(x)=\int\frac{d\mu_n(t)}{|x-t|^p}=V_n=1/C_p(F_n),\ x\in F_n,$$

except possibly on a set of p-capacity zero (see Frostman [7], p. 56).

Put $\hat{\mu}_n(u)=\int e^{-iux}d\mu_n(x)$. Then, if Parseval's formula is applied to evaluate the energy integral of μ_n (see Carleson [5], p. 20), we find
$$\int_{-\infty}^{\infty}\frac{|\hat{\mu}_n(u)|^2}{|u|^{1-p}}\,du=\text{Const.}\int U_n(x)\,d\mu_n(x)=\text{Const.}\ V_n<\infty. \tag{2.1}$$

Now let $c(x)\in B_{1-p}(R)$ be such that
$$\lim_{A\to\infty}c_A(x)=\lim_{A\to\infty}\frac{1}{2\pi}\int_{-A}^{A}\hat{c}(u)\,e^{iux}du=0,\quad x\notin E.$$

By the Schwarz inequality

$$\left\{\int_{-\infty}^{\infty} |\overline{\hat{c}(u)}\,\hat{\mu}_n(u)|\,du\right\}^2 \leqslant \int_{-\infty}^{\infty} |u|^{1-p}|\hat{c}(u)|^2 du \cdot \int_{-\infty}^{\infty} \frac{|\hat{\mu}_n(u)|^2}{|u|^{1-p}}\,du < \infty,$$

because of (1.2) and (2.1). Thus, for all n, by Parseval's formula,

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{\hat{c}(u)}\,\hat{\mu}_n(u)\,du = \lim_{A\to\infty} \frac{1}{2\pi}\int_{-A}^{A} \overline{\hat{c}(u)}\,\hat{\mu}_n(u)\,du = \lim_{A\to\infty}\int \overline{c_A(x)}\,d\mu_n(x).$$

For each F_n we can choose a subset F'_n such that $\lim_{A\to 0} c_A(x) = 0$ uniformly on F'_n, and such that the restriction, μ'_n, of μ_n to $F_n - F'_n$ has arbitrarily small energy integral. In fact, this integral is less than $V_n \cdot \mu_n(F_n - F'_n)$, which can be made small by Egoroff's theorem. Using this and applying the Schwarz inequality again we obtain

$$\lim_{A\to\infty}\left|\int \overline{c_A(x)}\,d\mu_n(x)\right| = \lim_{A\to\infty}\left|\int \overline{c_A(x)}\,d\mu'_n(x)\right|$$

$$\leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} |\overline{\hat{c}(u)}\,\hat{\mu}'_n(u)|\,du \leqslant \text{Const.}\ \{\mu_n(F_n - F'_n)\}^{\frac{1}{2}},$$

which can be made as small as we please. Thus

$$\int_{-\infty}^{\infty} \overline{\hat{c}(u)}\,\hat{\mu}_n(u)\,du = 0, \quad \text{all } n.$$

If the equilibrium measure for I is μ, and the corresponding potential U, we then find by (2.1) and the Schwarz inequality

$$\left|\int_{-\infty}^{\infty} \overline{\hat{c}(u)}\,(\hat{\mu}_n(u) - \hat{\mu}(u))\,du\right|^2 \leqslant \text{Const.} \int (U_n(x) - U(x))\,(d\mu_n(x) - d\mu(x)) =$$

$$= \text{Const.}\left\{\int U_n\,d\mu_n + \int U\,d\mu - 2\int U\,d\mu_n\right\} = \text{Const.}\left\{\frac{1}{C_p(F_n)} - \frac{1}{C_p(I)}\right\}.$$

But $\lim_{n\to\infty} C_p(F_n) = C_p(I)$, and thus

$$\int_{-\infty}^{\infty} \overline{\hat{c}(u)}\,\hat{\mu}(u)\,du = 0.$$

As μ is the equilibrium measure for an interval,

$$d\mu(x) = m(x)\,dx,$$

and $\overline{c(x)}\,m(x) \in L^1$, since $m(x) \in L^2$. See Pólya–Szegö [12], p. 23. Thus, by Parseval's formula

$$\int_{-\infty}^{\infty} \bar{c}(x)\, m(x)\, dx = 0.$$

The same argument can be applied to the function $c(x)e^{ixt}$ for every t, so that

$$\int_{-\infty}^{\infty} \overline{c(x)}\, m(x)\, e^{-ixt}\, dx = 0, \text{ all } t.$$

Hence $\bar{c}(x)\, m(x) = 0$ almost everywhere, and as $m(x) > 0$ in I, $c(x) = 0$ almost everywhere in I. But I is arbitrary, and thus $c(x) \equiv 0$, which proves the sufficiency.

To prove the necessity we assume that the set E is such that for a certain interval I, $C_p(I-E) < C_p(I)$. Then, by the capacitability theorem (see Choquet [6] or Carleson [5]), there is a closed subset F of E such that we also have $C_p(I-F) < C_p(I)$. We let $\{F_n\}_1^\infty$ be an increasing sequence of closed sets, each consisting of finitely many closed intervals, such that $I - F = \bigcup_1^\infty F_n$. Again μ_n is the equilibrium measure for F_n, and $U_n(x)$ the corresponding potential. Then $U_n(x) = V_n = 1/C_p(F_n)$ for all $x \in F_n$. In the same way as in the proof of (2.1) we find

$$\int |u|^{1-p} \,|\hat{U}_n(u)|^2 du = \text{Const. } V_n < \infty.$$

The sequence $\{\mu_n\}_1^\infty$ has a subsequence which converges weakly to a measure μ with support in I. Then, if φ is an arbitrary differentiable function with support contained in $I-F$, we find

$$\int \varphi(x)\, dx \int \frac{d\mu(t)}{|x-t|^p} = \int d\mu(t) \int \frac{\varphi(x)\, dx}{|x-t|^p} =$$

$$= \lim_{i\to\infty} \int d\mu_{n_i}(t) \int \frac{\varphi(x)\, dx}{|x-t|^p} = \lim_{i\to\infty} \int \varphi(x)\, dx \int \frac{d\mu_{n_i}(t)}{|x-t|^p} =$$

$$= \lim_{i\to\infty} V_{n_i} \int \varphi(x)\, dx = \frac{1}{C_p(I-F)} \int \varphi(x)\, dx.$$

Hence
$$U(x) = \int \frac{d\mu(t)}{|x-t|^p} = \frac{1}{C_p(I-F)}$$

for almost all x in $I-F$, and because of the semicontinuity of the energy integral

$$\int |u|^{1-p} \,|\hat{U}(u)|^2 du < \infty.$$

But $U(x)$ is not constant on the whole of I, for in that case, if ν is the equilibrium distribution for I and $V(x)$ the corresponding potential, we would obtain

$$\frac{1}{C_p(I)} = \int V(x)\,d\mu(x) = \int U(x)\,d\nu(x) = \frac{1}{C_p(I-F)},$$

which is contrary to assumption. Thus $U(x) \not\equiv 1/C_p(I-F)$ on a set of positive measure.

Now let $a(x)$ be a differentiable function with support in I, and let

$$c(x) = a(x)\left(\frac{1}{C_p(I-F)} - U(x)\right).$$

It is easy to prove that $c(x) \in B_{1-p}(R)$ (cf. p. 85 below), and clearly $c(x) = 0$ almost everywhere outside F. But since the complement of F is open it follows that $\lim_{A\to\infty} c_A(x) = 0$ everywhere outside F, and hence also outside E.

If $a(x)$ is suitably chosen $c(x) \not\equiv 0$, and we have proved Theorem 1.

Remark. We note that the function $c(x)$ constructed in the second part of the proof has the additional property that $(1+|u|^{1-p})\hat{c}(u)$ is bounded. It is easy to prove that a similar construction can be made in the case $p=0$, and then $(1+|u|)\hat{c}(u)$ is bounded, a fact which will be of use later.

From now on we only consider functions on R, although all results and proofs hold, with only small changes, for functions on T as well.

We need a somewhat stronger uniqueness property, which we define as follows.

Definition 2. *A subset E of R is called a strong set of uniqueness for $B_{1-p}(R)$ if every function, $c(x)$, in $B_{1-p}(R)$, such that $c(x)=0$ for almost all x not in E, must be identically zero.*

If $E \subset R$ we denote the characteristic function of E by $\chi_E(x)$, and by E^* we mean the set of all density points of E, i.e. all $x \in R$ such that

$$\lim_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} \chi_E(t)\,dt = 1.$$

Theorem 2. *A subset E of R is a strong set of uniqueness for $B_{1-p}(R)$, $0<p<1$, if and only if for every interval I*

$$C_p(I-E^*) = C_p(I).$$

Proof. The necessity follows from Theorem 1, and the well-known fact that for every interval I $m(I-E^*) = m(I-E)$. (See Saks [13], p. 128.)

Now we let E be a set which satisfies the condition in the theorem. We denote the complement of E by F, and the complement of E^* by F^*, and we let $c(x) \in B_{1-p}(R)$ be such that $c(x) = \lim_{A\to\infty} c_A(x) = 0$ almost everywhere on F.

By assumption, for every point x in F^*

$$\limsup_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} \chi_F(t)\,dt > 0.$$

If we define, for an arbitrary positive integer k, a set F_k^* by

$$F_k^* = \left\{x;\ \limsup_{h\to 0} \frac{1}{2h}\int_{x-h}^{x+h} \chi_F(t)\,dt > \frac{1}{k}\right\},$$

it is clear that $F^* = \bigcup_1^\infty F_k^*$. Now let I be an arbitrary interval and K be any closed subset of $I \cap F_k^*$ for some k, and let μ be the equilibrium measure corresponding to K. If we can show that $\int c(x)\,d\mu(x) = 0$ the theorem will follow in the same way as Theorem 1.

Choose $\delta > 0$. For every x in K we can find a number $h_x \leqslant \delta$ such that $\int_{x-h_x}^{x+h_x} \chi_F(t)\,dt \geqslant 2h_x/k$. Hence, as K is compact, there is a finite covering of K by intervals, I_j, such that $m(I_j \cap F) \geqslant (1/k)\,mI_j$. It is easily seen that superfluous intervals can be removed so that no point is contained in more than two intervals. Let $\{I_j\}_1^n$ be the resulting covering, and assume that the left endpoints of the intervals form an increasing sequence.

We now define a measure ν by $\nu = \sum_{j=1}^n \nu_j$, where the measures ν_j are defined as follows. The intervals $I_1, I_3, I_5, \ldots$ do not intersect, and corresponding to these we put

$$d\nu_j(x) = \frac{\mu(I_j)}{m(I_j \cap F)}\chi_{I_j\cap F}(x)\,dx, \quad j \ \text{ odd}.$$

Each of the remaining intervals can intersect at most two intervals, and we put

$$d\nu_j(x) = \frac{\mu(I_j - I_{j-1} - I_{j+1})}{m(I_j \cap F)}\chi_{I_j\cap F}(x)\,dx, \quad j \ \text{ even}.$$

It follows from the assumptions that $\int c(x)\,d\nu(x) = 0$. One easily sees that the measures ν converge weakly to μ as δ tends to zero. We shall see that this implies $\int c(x)\,d\mu(x) = 0$.

Choose $\varepsilon > 0$. Then, if the number A is chosen so large that

$$\int_{|u|>A} |u|^{1-p}\,|\hat{c}(u)|^2\,du < \varepsilon,$$

and if the energy integral of a measure μ is denoted by $J(\mu)$, it follows as before that

$$\left|\int (c(x) - c_A(x))\,d\mu(x)\right|^2 \leqslant \text{Const. } \varepsilon\cdot J(\mu).$$

Similarly

$$\left|\int (c(x) - c_A(x))\,d\nu(x)\right|^2 \leqslant \text{Const. } \varepsilon\cdot J(\nu).$$

Now, because of the weak convergence, we can choose δ so small that

$$\left|\int c_A(x)\,d\nu(x) - \int c_A(x)\,d\mu(x)\right| < \varepsilon.$$

Thus, if we can prove that $J(\nu) \leqslant \text{Const.}\, J(\mu)$ independently of the choice of δ, it will follow that $\int c(x)\, d\mu(x) = 0$.

Let the length of the interval I_j be l_j, and let the greatest distance between a point in I_j and a point in I_i be a_{ij}. Then, clearly

$$J(\mu) = \iint \frac{d\mu(x)\, d\mu(y)}{|x-y|^p} \geqslant \frac{1}{4} \sum_{j=1}^{n} \sum_{i=1}^{n} \int_{I_i} \int_{I_j} \frac{d\mu(x)\, d\mu(y)}{|x-y|^p} \geqslant \frac{1}{4} \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{\mu(I_i)\,\mu(I_j)}{a_{ij}^p}.$$

On the other hand,

$$J(\nu) = \iint \frac{d\nu(x)\, d\nu(y)}{|x-y|^p} = \sum_{j=1}^{n} \sum_{i=1}^{n} \iint \frac{d\nu_i(x)\, d\nu_j(y)}{|x-y|^p} \leqslant \sum_{j=1}^{n} \sum_{i=1}^{n} \frac{k^2 \mu(I_i)\,\mu(I_j)}{l_i l_j} \int_{I_i} \int_{I_j} \frac{dx\,dy}{|x-y|^p}.$$

Thus, it suffices to show that for all i and j

$$\frac{1}{l_i l_j} \int_{I_i} \int_{I_j} \frac{dx\,dy}{|x-y|^p} \leqslant \text{Const.}\, \frac{1}{a_{ij}^p}.$$

Consider two intervals, I_i and I_j, $i \leqslant j$, and assume $l_i \leqslant l_j$. Then either $a_{ij} \leqslant 3 l_j$ or $a_{ij} > 3 l_j$. In the first case

$$\frac{1}{l_i l_j} \int_{I_i} dx \int_{I_j} \frac{dy}{|x-y|^p} \leqslant \frac{1}{l_i l_j} \int_{I_i} dx \int_{|t| < l_j/2} \frac{dt}{|t|^p} = \frac{2^p}{(1-p)\, l_j^p} \leqslant \frac{6^p}{(1-p)\, a_{ij}^p}.$$

If $a_{ij} > 3 l_j$ it follows that $|x-y| \geqslant a_{ij}/3$ for $x \in I_i$, $y \in I_j$, and we find that in this case

$$\frac{1}{l_i l_j} \int_{I_i} dx \int_{I_j} \frac{dy}{|x-y|^p} \leqslant \frac{3^p}{a_{ij}^p},$$

which proves Theorem 2.

Remark. Theorems 1 and 2 imply that if a set E has the property that for all I $C_p(I - E^*) = C_p(I)$, the same holds for E. The converse statement is not true, however. In fact, let F_n be an increasing sequence of closed sets contained in an interval I, and assume that $mF_n = 0$ and $\lim_{n\to\infty} C_p(F_n) = C_p(I)$. Such sets can be constructed. Let $F = \bigcup_1^\infty F_n$ and $E = I - F$. Then $C_p(I - E) = C_p(I)$, but $E^* = I$ and hence $C_p(I - E^*) = 0$.

3. Generators in $B_{1+p}(R)$, $0 < p < 1$, and $A(R)$

As we have already remarked we formulate our results for R, although with minor changes they hold also for T. Also, those results which we formulate for sets of real-valued functions are easily seen to hold also for sets which are closed under complex conjugation.

It is well known (see e.g. Beurling [2]), and easy to prove by the Parseval formula, that for a function f in $B_{1+p}(R)$, $0 < p < 1$,

$$\int_{-\infty}^{\infty} |u|^{1+p} |\hat{f}(u)|^2 du = \text{Const.} \cdot \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x+t)-f(x)|^2 |t|^{-2-p} dx dt,$$

where the constant depends only on p. Thus we can renorm the space by putting

$$\|f\|_{1+p}^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x+t)-f(x)|^2 |t|^{-2-p} dx dt.$$

It is then clear that e.g. all square integrable functions which satisfy a Hölder condition of order $(1+p')/2$ for some $p'>p$ are in $B_{1+p}(R)$.

By the Schwarz inequality we find

$$\operatorname*{Max}_x |f(x)| \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(u)| du$$
$$\leqslant \frac{1}{2\pi}\left\{\int_{-\infty}^{\infty} \frac{1}{1+|u|^{1+p}} du\right\}^{\frac{1}{2}} \left\{\int_{-\infty}^{\infty} (1+|u|^{1+p}) |\hat{f}(u)|^2 du\right\}^{\frac{1}{2}} \leqslant \text{Const. } \|f\|_{1+p}. \tag{3.1}$$

By means of this inequality it is easy to verify that the norm satisfies the relation

$$\|fg\|_{1+p} \leqslant \text{Const. } \|f\|_{1+p} \|g\|_{1+p},$$

and hence $B_{1+p}(R)$ is a Banach algebra for $0<p<1$.

For a subset S of such an algebra to generate the algebra it is clearly necessary that the following two conditions hold.

(3.2) *For every couple* (x, y) *with* $x \neq y$ *there is an* f *in* S *such that* $f(x) \neq f(y)$.

(3.3) *For every* x *there is an* f *in* S *such that* $f(x) \neq 0$.

Theorem 3. *Let* $S=\{f_i\}_1^\infty$ *be a subset of* $B_{1+p}(R)$, $0<p<1$, *satisfying* (3.2) *and* (3.3). *For* S *to generate* $B_{1+p}(R)$ *it is necessary that every set of points* $x \in R$, *where for each* i $\lim_{h\to 0}(f_i(x+h)-f_i(x))/h)=0$ *uniformly, be a strong set of uniqueness for* $B_{1-p}(R)$. (*See Def.* 2).

Proof. We assume that there is a set E where $f_i'(x)=0$ uniformly for every i, and that E is not a strong set of uniqueness for $B_{1-p}(R)$. Then there is a non-zero function, $c(x)$, in $B_{1-p}(R)$, such that $c(x)=0$ for almost all x not in E.

We observe that the space of bounded linear functionals on $B_{1+p}(R)$ can be identified with the space of those tempered distributions, C, on R, whose Fourier transforms, $\hat{C}$, are functions which satisfy

$$\int_{-\infty}^{\infty} \frac{|\hat{C}(u)|^2}{1+|u|^{1+p}} du < \infty,$$

and then

$$\langle C, f\rangle = \int_{-\infty}^{\infty} \overline{\hat{C}(u)} \hat{f}(u) du.$$

This is a consequence of the Schwarz inequality.

As the function $c(x)$ above belongs to $B_{1-p}(R)$, we have

$$\int_{-\infty}^{\infty} \frac{|iu\hat{c}(u)|^2}{1+|u|^{1+p}}\,du \leqslant \int_{-\infty}^{\infty} |u|^{1-p}|\hat{c}(u)|^2 du < \infty.$$

Hence the function $iu\hat{c}(u)$ is the Fourier transform of a bounded linear functional on $B_{1+p}(R)$, which we denote by C_0.

If f is a polynomial in elements of S it is now easy to see that $\langle C_0, f\rangle = 0$. In fact,

$$\langle C_0, f\rangle = \int_{-\infty}^{\infty} \overline{iu\hat{c}(u)}\,\hat{f}(u)\,du = -\lim_{h\to 0}\int_{-\infty}^{\infty} \overline{\hat{c}(u)}\,\hat{f}(u)\frac{e^{iuh}-1}{h}\,du,$$

because $|e^{iuh}-1|/|h| = 2|\sin \frac{1}{2}uh|/|h| \leqslant |u|$, and the first integral is absolutely convergent. But both $\hat{c}$ and $\hat{f}$ are in L^2, and thus by the Parseval formula

$$\langle C_0, f\rangle = -\lim_{h\to 0} 2\pi\int_{-\infty}^{\infty} \overline{c(x)}\,\frac{f(x+h)-f(x)}{h}\,dx,$$

which is equal to zero, because apart from a set of measure zero $c(x)$ is different from zero only in E, where $f'(x)=0$ uniformly, since f depends on only finitely many f_i.

As $C_0 \neq 0$ this proves that S cannot generate $B_{1+p}(R)$.

Remark. It is seen from the proof that it is enough to assume about E that there is a sequence $\{h_\nu\}_1^\infty$, $\lim_{\nu\to\infty} h_\nu = 0$, such that for all i

$$\lim_{h_\nu\to 0} (f_i(x+h_\nu) - f_i(x))/h_\nu = 0$$

uniformly on E, or even

$$\lim_{h_\nu\to 0} (1/h_\nu^2)\int_E (f_i(x+h_\nu) - f_i(x))^2 dx = 0.$$

If this is compared to Theorem 4 below, it follows that the size of such sets is not independent of the size of the set where $f_i'(x)=0$ in the ordinary sense.

For a given set S of functions on R we denote by E_S the set of all $x\in R$ such that for every f in S $f'(x)$ exists and equals zero. We denote the complement of E_S by F_S.

Theorem 4. *Let $S=\{f_i\}_1^\infty$ be a set of real-valued functions in $B_{1+p}(R)$, $0<p<1$, satisfying* (3.2) *and* (3.3). *Then S generates $B_{1-p}(R)$ if E_S is a strong set of uniqueness for $B_{1-p}(R)$ (see Def. 2), and if the moduli of continuity, $\omega_i(\delta)$, of the functions f_i satisfy $\omega_i(\delta) = o(\delta^{(1-p)/2})$.*

Proof. We assume that S satisfies the conditions in the theorem. It is clearly no restriction to assume that

$$\sum_1^\infty \|f_i\|_{1+p}^2 < \infty, \tag{3.4}$$

and that

$$\sum_1^\infty \omega_i(\delta)^2 \leqslant K_\delta\cdot\delta^{1+p}, \quad \text{where} \quad \lim_{\delta\to 0} K_\delta = 0. \tag{3.5}$$

We denote by B_S the closed subalgebra of $B_{1+p}(R)$ generated by S.

By the Riemann–Lebesgue lemma $\lim_{x\to\pm\infty} f_i(x)=0$. Thus, as x describes the extended line, $\bar{R}$, for every N the N-tuple $F_N(x)=\{f_i(x)\}_1^N$ describes a closed curve, Γ_N, in N-dimensional space, R^N. We denote the Euclidean norm of $F_N(x)$ by $|F_N(x)|$.

If $\varphi(\xi)$ is a function on R^N which is defined and has continuous partial derivatives in a neighbourhood of Γ_N, and if $\varphi(0)=0$, we shall see that the function $\varphi(F_N(x))$ is in B_S. In fact, there is a compact neighbourhood, W, of Γ_N where $|\text{grad } \varphi(\xi)|$ is bounded, and hence there is a $\delta>0$ such that for every $\xi_0\in\Gamma_N$ the ball $|\xi-\xi_0|\leqslant\delta$ is contained in W, and then if $|\xi_1-\xi_0|\leqslant\delta$

$$|\varphi(\xi_1)-\varphi(\xi_0)|\leqslant \operatorname*{Max}_{\xi\in W}|\text{grad } \varphi(\xi)|\,|\xi_1-\xi_0|.$$

It follows by uniform continuity that there is an $A>0$ such that for $|x|\geqslant A$

$$|\varphi(F_N(x))|\leqslant \operatorname*{Max}_{\xi\in W}|\text{grad } \varphi(\xi)|\,|F_N(x)|,$$

and that there is an $\eta>0$ such that for all x and $|t|\leqslant\eta$

$$|\varphi(F_N(x+t))-\varphi(F_N(x))|\leqslant \operatorname*{Max}_{\xi\in W}|\text{grad } \varphi(\xi)|\,|F_N(x+t)-F_N(x)|.$$

This implies that $\varphi(F_N)\in B_{1+p}(R)$. Furthermore, by a version of Weierstrass' approximation theorem (see Whitney [16], p. 74), there is a sequence of polynomials, $\{P_\nu\}$, converging to φ in $C^1(W)$, and if the above inequalities are applied to the functions $\varphi-P_\nu$ it follows that $\varphi\in B_S$.

For every positive integer, n, we define a set $F_{S,n}$, by

$$F_{S,n}=\{x;\ \limsup_{h\to 0}|f_i(x+h)-f_i(x)|/|h|>1/n,\ \text{ some } f_i\in S\}.$$

Then $F_S=\bigcup_{n=1}^{\infty}F_{S,n}$.

Now we choose a set $F_{S,n}$, a compact interval I, and a function

$$g(x)=\int_{-\infty}^{x}k(t)\,dt,$$

where $k(t)=0$ for $t\notin I\cap F_{S,n}$, $|k(t)|\leqslant 1$, and $\int_{-\infty}^{\infty}k(t)dt=0$. It is clear that $g\in B_{1+p}(R)$. We shall show that $g\in B_S$.

In virtue of the above argument it suffices to show that for a given $\varepsilon>0$, we can choose an integer N and construct a differentiable φ so that

$$\|g-\varphi(F_N)\|_{1+p}<\varepsilon.$$

For every point x in $I\cap F_{S,n}$ there is a function f_i and an increasing (or decreasing) sequence, $\{h_\nu\}_1^\infty$, such that $\lim_{\nu\to\infty}h_\nu=0$, and

$$|f_i(x+h_\nu)-f_i(x)|\geqslant|h_\nu|/n,\quad \nu=1,2,\dots.$$

We fix a number $\delta>0$. By Vitali's covering theorem ([13], p. 109), applied to the set $I\cap F_{S,n}$, there is a finite sequence of non-intersecting intervals, $I_\nu=(a_\nu,b_\nu)$, $\nu=1,2,\dots,q$, contained in the interior of $I=(a,b)$, with the property that

$$|f_i(b_\nu)-f_i(a_\nu)|\geqslant(b_\nu-a_\nu)/n,\quad i=i(\nu),\quad \nu=1,2,\dots q,$$

$$m(I\cap F_{S,n}-\bigcup_1^q I_\nu)<\delta, \tag{3.6}$$

and $$\operatorname{Max}_\nu\,(b_\nu-a_\nu)\leqslant\delta.$$

We can assume $$b_0=a<a_1<b_1<a_2<\dots<b_q<b=a_{q+1}.$$

We shall now choose N in a way suitable for the sequel. First, N has to be greater than $\operatorname{Max}_{1\leqslant\nu\leqslant q} i(\nu)$. For every point (x,y) in $\bar{R}\times\bar{R}$ which is not on the diagonal, $x=y$, there is an N such that $F_N(x)\neq F_N(y)$, by assumption, and because of continuity this also holds in a neighbourhood of the point (x,y). We choose an interval, $I'=(a,b')$, where $b'>b$ and is independent of δ. Then, by the Heine–Borel property, for any $\eta>0$ the compact subset, V, of $\bar{R}\times\bar{R}$, defined by $V=\{(x,y);\ |x-y|\geqslant\eta,\ x\in I'\}$, can be covered by a finite number of such neighbourhoods. It follows that there is an integer, N_η, such that for $N\geqslant N_\eta$ we have $F_N(x)\neq F_N(y)$ for all (x,y) in V.

We shall now choose the number η. $\sum_{i=1}^\infty \operatorname{Max}_x f_i(x)^2<\infty$ by (3.1) and (3.4), and by assumption $\sup_N|F_N(x)-F_N(y)|>0$ for all $x,y\in\bar{R}$ with $x\neq y$. We put

$$\operatorname*{Min}_{1\leqslant\nu\leqslant q+1}\ \sup_N\ |F_N(b_\nu)-F_N(a_\nu)|=\gamma,$$

where b_{q+1} denotes b'.

It is easily seen that we can choose m so that $\{\sum_{m+1}^\infty f_i(x)^2\}^{\frac{1}{2}}\leqslant\gamma/36$ for all x. We choose $\eta>0$ so small that $\{\sum_1^m(f_i(x)-f_i(y))^2\}^{\frac{1}{2}}\leqslant\gamma/36$ for all (x,y) with $|x-y|\leqslant\eta$, which is possible because of uniform continuity. It follows that $|F_N(x)-F_N(y)|\leqslant\gamma/12$ for all N, as soon as $|x-y|\leqslant\eta$.

Now we fix the value of N so that for this choice of η

$$N\geqslant N_\eta,\quad N\geqslant\operatorname*{Max}_{1\leqslant\nu\leqslant q} i(\nu),$$

and so that $$\operatorname*{Min}_{1\leqslant\nu\leqslant q+1}|F_N(b_\nu)-F_N(a_\nu)|\geqslant\gamma/2.$$

Consider the curve Γ_N. We introduce the notation $F_N(a_\nu)=\alpha_\nu$, $F_N(b_\nu)=\beta_\nu$, etc. An interval $a_\nu\leqslant x\leqslant b_\nu$ corresponds to an arc $A(\alpha_\nu,\beta_\nu)$ on Γ_N, and the sets $x\leqslant a$ and $x\geqslant b'$ correspond to arcs $A(0,\alpha)$ and $A(\beta',0)$. Because of our choice of N we have

$$|\beta_\nu-\alpha_\nu|\geqslant(b_\nu-a_\nu)/n,\quad \nu=1,2,\dots,q, \tag{3.7}$$

$$F_N(x)\neq F_N(y),\quad |x-y|\geqslant\eta,\quad x\in I', \tag{3.8}$$

$$|F_N(x)-F_N(y)|\leqslant\operatorname*{Min}_{1\leqslant\nu\leqslant q+1}|\beta_\nu-\alpha_\nu|/6,\quad |x-y|\leqslant\eta. \tag{3.9}$$

Let h be an increasing function in $C^\infty(0,\infty)$ such that

$$h(r)=\begin{cases}0, & r\leqslant\frac{1}{3}\\ 1, & r\geqslant\frac{2}{3}.\end{cases}$$

For every ν, $1 \leqslant \nu \leqslant q$, we now define a function φ_ν on Γ_N as follows:

$$\varphi_\nu(\xi) = \begin{cases} 0, & \xi \in A(0, \alpha_\nu) \\ (g(b_\nu) - g(a_\nu))\, h\left(\dfrac{|\xi - \alpha_\nu|}{|\beta_\nu - \alpha_\nu|}\right), & \xi \in A(\alpha_\nu, \beta_\nu) \\ g(b_\nu) - g(a_\nu), & \xi \in A(\beta_\nu, \beta) \\ (g(b_\nu) - g(a_\nu))\left(1 - h\left(\dfrac{|\xi - \beta|}{|\beta' - \beta|}\right)\right), & \xi \in A(\beta, \beta') \\ 0, & \xi \in A(\beta', 0). \end{cases} \tag{3.10}$$

Then we define φ by $\varphi = \sum_{\nu=1}^{q} \varphi_\nu$.

We shall see that this definition of φ_ν makes sense, i.e if Γ_N intersects itself (3.10) gives only one value for φ_ν at that point. It is clear that

$$\eta < \operatorname{Min}_{1 \leqslant \nu \leqslant q+1} (b_\nu - a_\nu),$$

and hence the arcs $A(\alpha, \alpha_\nu)$ and $A(\beta_\nu, \beta')$ do not intersect. Let $a_\nu < x < b_\nu$ and assume that $\xi = F_N(x)$ is such that $\varphi_\nu(\xi) \neq 0$, i.e. $|\xi - \alpha_\nu| > \frac{1}{3}|\beta_\nu - \alpha_\nu|$. It follows from (3.9) that we must have $x - a_\nu > \eta$, and this implies that for any $y \leqslant a_\nu$ we have $x - y > \eta$, and thus $F_N(x) \neq F_N(y)$, by (3,8). Similarly, if we choose an x such that $\xi = F_N(x)$ satisfies $|\xi - \alpha_\nu| < \frac{2}{3}|\beta_\nu - \alpha_\nu|$, it follows that $b_\nu - x > \eta$, and thus $F_N(x) \neq F_N(y)$ for all $y \geqslant b_\nu$. The case when $b < x < b'$ can be handled in the same way, and it follows that the definition (3.10) is consistent. It also readily follows that the definition of φ can be extended consistently by the same formulas to a small neighbourhood of Γ_N, and that φ is differentiable to all orders there. Thus $\varphi(F_N(x)) \in B_S$.

We shall show that $\lim_{\delta \to 0} \|g - \varphi(F_N)\|_{1+p} = 0$. Put $\lambda(x) = g(x) - \varphi(F_N(x))$. Then we observe that for all x

$$|\lambda(x)| \leqslant 3\delta. \tag{3.11}$$

In fact, for $\nu = 1, 2, \ldots, q+1$,

$$|\lambda(a_\nu)| = |g(a_\nu) - \sum_{i=1}^{\nu-1} (g(b_i) - g(a_i))| = |\sum_{i=1}^{\nu} (g(a_i) - g(b_{i-1}))| \leqslant \delta,$$

by (3.6). Furthermore, if $b_{\nu-1} < x < b_\nu$ we have

$$|g(x) - g(a_\nu)| \leqslant \delta, \quad \text{and} \quad |\varphi(F_N(x)) - \varphi(F_N(a_\nu))| \leqslant \delta,$$

which proves (3.11).

As $\lambda(x) = 0$ outside I' it follows that $\int_{-\infty}^{\infty} |\lambda(x)|^2 dx \to 0$, as $\delta \to 0$. It is thus enough to prove that

$$J = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\lambda(x+t) - \lambda(x)|^2 |t|^{-2-p} dx dt \to 0, \quad \delta \to 0.$$

We find
$$J \leqslant 2\int_{-\infty}^{\infty} dx \int_{|t|<\delta} |\varphi(F_N(x+t)) - \varphi(F_N(x))|^2 |t|^{-2-p} dt$$
$$+2\int_{-\infty}^{\infty} dx \int_{|t|<\delta} |g(x+t)-g(x)|^2 |t|^{-2-p} dt$$
$$+\int_{-\infty}^{\infty} dx \int_{|t|\geqslant\delta} |\lambda(x+t)-\lambda(x)|^2 |t|^{-2-p} dt = J_1 + J_2 + J_3.$$

By (3.11)
$$J_3 \leqslant \int_a^{b'} dx \int_{|t|\geqslant\delta} 36\delta^2 |t|^{-2-p} dt \leqslant \text{Const. } \delta^{1-p}.$$

By the definition of g
$$J_2 \leqslant 2\int_{a-\delta}^{b+\delta} \int_{|t|<\delta} |t|^{-p} dt \leqslant \text{Const. } \delta^{1-p}.$$

Hence it suffices to prove that $\lim_{\delta\to 0} J_1 = 0$, or which is the same thing, that
$$\lim_{\delta\to 0} \int_{-\infty}^{\infty} dx \int_x^{x+\delta} |\varphi(F_N(x)) - \varphi(F_N(y))|^2 (y-x)^{-2-p} dy = 0. \tag{3.12}$$

If $a_\nu \leqslant x < y \leqslant b_\nu$, $1 \leqslant \nu \leqslant q+1$, we have
$$|\varphi(F_N(x)) - \varphi(F_N(y))| \leqslant \text{Const. } |F_N(x) - F_N(y)|. \tag{3.13}$$

For in the case when $1 \leqslant \nu \leqslant q$, we find by (3.10) and (3.7)
$$|\varphi(F_N(x)) - \varphi(F_N(y))| = |\varphi_\nu(F_N(x)) - \varphi_\nu(F_N(y))|$$
$$\leqslant |g(b_\nu) - g(a_\nu)| \cdot \operatorname{Max}_r h'(r) \cdot \frac{|F_N(x) - F_N(y)|}{|\beta_\nu - \alpha_\nu|} \leqslant n \cdot \operatorname{Max}_r h'(r) \cdot |F_N(x) - F_N(y)|.$$

Similarly, for $\nu = q+1$
$$|\varphi(F_N(x)) - \varphi(F_N(y))| \leqslant \frac{|\sum_1^q (g(b_\nu) - g(a_\nu))|}{|\beta - \beta'|} |F_N(x) - F_N(y)| \leqslant \frac{\delta}{|\beta-\beta'|} |F_N(x) - F_N(y)|.$$

Now, if $a_\nu < x < b_\nu$, and $a_\mu < y < b_\mu$, where $1 \leqslant \nu < \mu \leqslant q+1$, we find, using (3.10) and (3.13)
$$|\varphi(F_N(x)) - \varphi(F_N(y))| \leqslant |\varphi(F_N(x)) - \varphi(F_N(b_\nu))|$$
$$+\sum_{i=\nu+1}^{\mu-1} |\varphi(F_N(a_i)) - \varphi(F_N(b_i))| + |\varphi(F_N(a_\mu)) - \varphi(F_N(y))|$$
$$\leqslant \text{Const. } |F_N(x) - F_N(b_\nu)| + \sum_{i=\nu+1}^{\mu-1} |g(a_i) - g(b_i)| + \text{Const. } |F_N(a_\mu) - F_N(y)|$$
$$\leqslant \text{Const. } \{|F_N(x) - F_N(b_\nu)| + y - x + |F_N(a_\mu) - F_N(y)|\}. \tag{3.14}$$

In the same way we find that if $x \leqslant a_1$ or $b_\nu \leqslant x \leqslant a_{\nu+1}$ for some $\nu < \mu$, the term $|F_N(x) - F_N(b_\nu)|$ in (3.14) should be replaced by zero, and if $y \geqslant b'$ or $b_{\mu-1} \leqslant y \leqslant a_\mu$ for some $\mu > \nu$, the term $|F_N(a_\mu) - F_N(y)|$ should be replaced by zero.

Substituting (3.13) and (3.14) in (3.12) we thus find

$$J_1 \leqslant \text{Const.} \left\{ \int_{a-\delta}^{b'} dx \int_x^{x+\delta} (y-x)^{-p} dy + \sum_{\nu=1}^{q+1} \int_{a_\nu}^{b_\nu} dx \int_x^{b_\nu} |F_N(x) - F_N(y)|^2 (y-x)^{-2-p} dy \right.$$
$$+ \sum_{\nu=1}^{q+1} \int_{a_\nu}^{b_\nu} |F_N(x) - F_N(b_\nu)|^2 dx \int_{b_\nu}^{\infty} (y-x)^{-2-p} dy$$
$$\left. + \sum_{\mu=1}^{q+1} \int_{a_\mu}^{b_\mu} |F_N(a_\mu) - F_N(y)|^2 dy \int_{-\infty}^{a_\mu} (y-x)^{-2-p} dx \right\}$$
$$= \text{Const.} \{J_{11} + J_{12} + J_{13} + J_{14}\}.$$

Here
$$J_{11} \leqslant \text{Const.}\ \delta^{1-p},$$
and
$$J_{12} \leqslant \int_{-\infty}^{\infty} dx \int_0^{\delta} \sum_{i=1}^{\infty} |f_i(x+t) - f_i(x)|^2 t^{-2-p} dt,$$

which tends to zero with δ because of (3.4). If we apply (3.5) to J_{13} and J_{14} we find respectively

$$J_{13} \leqslant \text{Const.}\ K_\delta \sum_{\nu=1}^{q+1} \int_{a_\nu}^{b_\nu} dx \leqslant \text{Const.}\ K_\delta (b'-a);$$

$$J_{14} \leqslant \text{Const.}\ K_\delta \sum_{\mu=1}^{q+1} \int_{a_\mu}^{b_\mu} dy \leqslant \text{Const.}\ K_\delta (b'-a).$$

These inequalities prove that $\lim_{\delta \to 0} J_1 = 0$, and hence that $g \in B_S$.

We shall now show that this implies that $B_S = B_{1+p}(R)$. We assume that C is a bounded linear functional on $B_{1+p}(R)$ which annihilates B_S, and we shall prove that $C = 0$.

From the preceding it follows that for every interval $(-A, A)$ there is a function $a(x)$ in B_S such that $a(x) = 1$ on $(-A, A)$ and $a(x) = 0$ outside $(-A-1, A+1)$. In fact, for every interval I $m(I - E_S) > 0$, for otherwise I would be a strong set of uniqueness for $B_{1-p}(R)$, which is impossible. We can thus construct the function a with a bounded derivative.

We now define a new bounded linear functional, C_1, by

$$\langle C_1, g \rangle = \langle C, ag \rangle, \quad g \in B_{1+p}(R).$$

Obviously, C_1 also annihilates B_S, and if $g(x) = 0$ outside $(-A, A)$ we have

$$\langle C_1, g \rangle = \langle C, g \rangle.$$

C_1 is a distribution with compact support, and thus $\hat{C}_1(u)$ is differentiable. Moreover, if $g \in B_S$ and $g(x) = 1$ on $(-A-1, A+1)$, we find

$$\hat{C}_1(0) = \langle C_1, 1 \rangle = \langle C_1, g \rangle = 0.$$

It follows that the function $\hat{C}_1(u)/iu$ is continuous everywhere. Also

$$\int_{|u|\geq 1}(1+|u|^{1-p})\left|\frac{\hat{C}_1(u)}{iu}\right|^2 du \leq 2\int_{|u|\geq 1}|u|^{-1-p}|\hat{C}_1(u)|^2 du < \infty,$$

and thus
$$\int_{-\infty}^{\infty}(1+|u|^{1-p})\left|\frac{\hat{C}_1(u)}{iu}\right|^2 du < \infty.$$

Hence there is a function $c(x)$ in $B_{1-p}(R)$ whose Fourier transform is $\hat{C}_1(u)/iu$, and one sees easily that $c(x)=0$ outside $(-A-1,\ A+1)$.

Let $k(x)$ be an arbitrary bounded function, such that $k(x)=0$ outside $F_{S,\,n}$ for some n and outside a compact interval. Then choose a similar function, $k_1(x)$, such that $k_1(x)=0$ outside some $F_{S,\,n}$ and on $(-A-1,\ A+1)$ and

$$\int_{-\infty}^{\infty}(k(x)+k_1(x))\,dx = 0.$$

It follows from the first part of the proof that $g(x)=\int_{-\infty}^{x}(k(t)+k_1(t))\,dt$ belongs to B_S, and thus $\langle C_1, g\rangle = 0$. On the other hand, the Parseval formula yields

$$\langle C_1, g\rangle = \int_{-\infty}^{\infty}\overline{\hat{C}_1(u)}\,\hat{g}(u)\,du = \int_{-\infty}^{\infty}\overline{\hat{C}_1(u)}\,(\hat{k}(u)+\hat{k}_1(u))/iu\,du$$
$$= -2\pi\int_{-\infty}^{\infty}\overline{c(x)}\,(k(x)+k_1(x))\,dx = -2\pi\int_{-\infty}^{\infty}\overline{c(x)}\,k(x)\,dx.$$

As n and k are arbitrary this implies that $c(x)=0$ almost everywhere outside E_S. But E_S is a strong set of uniqueness for $B_{1-p}(R)$, and thus $c(x)\equiv 0$, and hence also $C_1=0$. Moreover, A is arbitrary, and thus $\langle C, \varphi\rangle = 0$ for every differentiable φ with compact support. Hence $C=0$, which proves Theorem 4.

Before applying these results to $A(R)$ we shall generalize Theorem 4 slightly. We define a class of weight functions as follows.

Definition 3. *A function $h(x)$ on R is said to belong to the class W if it is continuous, even, non-negative, increasing for $x>0$,*

$$\int_0^{\infty}(1+xh(x))^{-1}dx < \infty,$$

and
$$\int_0^1 h\left(\frac{1}{x}\right)dx < \infty.$$

For every $h\in W$ we define a space $B_h(R)$ as the set of functions $f(x)$ on R for which

$$\|f\|_h^2 = \int_{-\infty}^{\infty}|f(x)|^2 dx + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f(x+t)-f(x)|^2\,|t|^{-2}h(t^{-1})\,dx\,dt < \infty.$$

Then $B_h(R)=B_{1+p}(R)$ for $h(x)=|x|^p$, $0<p<1$.

It is easily seen that

$$\int_{-\infty}^{\infty} (1+|u|h(u))\,|\hat{f}(u)|^2 du \leqslant \text{Const. } \|f\|_h^2, \tag{3.15}$$

where the constant is independent of h, and hence also

$$\operatorname*{Max}_x |f(x)| \leqslant \frac{1}{2\pi}\int_{-\infty}^{\infty} |\hat{f}(u)|\,du \leqslant \text{Const. } \|f\|_h.$$

It follows as before that $B_h(R)$ is a Banach algebra, and Theorem 4 easily extends to this case. It also follows that $B_h(R) \subset A(R)$ for all h in W.

The bounded linear functionals on $A(R)$ are those distributions on R whose Fourier transforms are bounded. We denote by $D(R)$ the set of all functions $c(x)$ on R such that $(1+|u|)\hat{c}(u)$ is bounded. Then we can define sets of uniqueness and strong sets of uniqueness for $D(R)$ exactly as in Definitions 1 and 2.

If E^* is defined as on p. 82, it follows from the remark following the proof of Theorem 1 that a set E is not a strong set of uniqueness for $D(R)$ if $C_0(I-E^*) < C_0(I)$ for some interval I, and it follows from Theorem 2 that E is a strong set of uniquenesss for $D(R)$ if for some p, $0<p<1$, $C_p(I-E^*) = C_p(I)$, for all intervals I.

Theorem 3 has the following counterpart for $A(R)$.

Theorem 5. *Let $S=\{f_i\}_1^\infty$ be a subset of $A(R)$ satisfying* (3.2) *and* (3.3). *For S to generate $A(R)$ it is necessary that every set of points $x \in R$ where for each i $\lim_{h\to 0}(f_i(x+h)-f_i(x))/h = 0$ uniformly, is a strong set of uniqueness for $D(R)$.*

The proof is the same as that of Theorem 3.

In the other direction Theorem 4 gives the following corollary.

Theorem 6. *Let $S=\{f_i\}_1^\infty$ be a set of real-valued functions in $A(R)$ satisfying* (3.2) *and* (3.3). *Then S generates $A(R)$ if the following three conditions are satisfied:*

(a) *E_S is a strong set of uniqueness for $D(R)$;*
(b) *for each i there is an $h_i \in W$ such that $f_i \in B_{h_i}(R)$;*
(c) *for each i the modulus of continuity, ω_i, of f_i satisfies $\omega_i(\delta)^2 = o(\delta/h_i(1/\delta))$.*

Proof. We assume that S satisfies the conditions stated. Let $H_N(u) = \operatorname{Min}_{i\leqslant N} h_i(u)$. Then H_N is also in W, for

$$\int_0^\infty (1+uH_N(u))^{-1}du \leqslant \sum_{i=1}^N \int_0^\infty (1+uh_i(u))^{-1}du.$$

It is clearly no restriction to assume that the series on the right is bounded. If $\|f\| = \int_{-\infty}^\infty |\hat{f}(u)|\,du$ it follows from the Schwarz inequality and (3.15) that

$$\|f\| \leqslant \text{Const. } \|f\|_{H_N}, \tag{3.16}$$

where the constant is independent of N.

We can also assume that

$$\sum_{i=1}^{\infty} \|f_i\|_{h_i}^2 < \infty,$$

and that

$$\sum_{1}^{N} \omega_i(\delta)^2 \leq K_\delta \cdot \delta / H_N(1/\delta),$$

where K_δ is independent of N and $\lim_{\delta\to 0} K_\delta = 0$. It is now easy to see that the proof of Theorem 4 applies to this case almost without change. Thus, with the notations used there, we can make $\|g-\varphi(F_N)\|_{H_N}$ arbitrarily small, and then the theorem follows by (3.16).

An amusing consequence of Theorem 6 is that a function f in $A(R)$ together with its complex conjugate generates $A(R)$ if f separates points on $\bar{R}$, f is Hölder continuous of order p for some $p > \frac{1}{2}$, and f is nowhere differentiable. We give an example of such a function (in $A(T)$ for simplicity).

Let $r(x) = \sum_{n=1}^{\infty} a^n \cos b^n x + C$, where b is an integer, $b^{-1} < a < b^{-p}$, $p > \frac{1}{2}$, and C is so large that $r(x) > 0$ for all x. It is well known that r is nowhere differentiable, and one can easily prove that r satisfies a Hölder condition of order p. If we put $f(x) = r(x)e^{ix}$, f satisfies all conditions, and thus f and $\bar{f}$ generate $A(T)$.

On the other hand, there are C^∞ functions, f, which separate points on T and are never zero although f and $\bar{f}$ do not generate $A(T)$. (Cf. Katznelson–Rudin [8], where the existence of such functions is proved in a somewhat different way.)

Let E be any closed, totally disconnected set on T such that $C_0(T-E) < C_0(T)$. Such sets exist, for one can construct $T-E$ as the union of sufficiently small intervals containing the rational points. It is easy to construct a strictly monotonic C^∞ function $a(x)$ such that $a'(x) = 0$ for $x \in E$, $a(0) = 0$, and $a(2\pi) = 2\pi$. Then if we put $f(x) = \exp(i a(x))$, f and $\bar{f}$ do not generate $A(T)$ by Theorem 5.

4. Generators in $B_p(R)$, $p \geq 2$

We shall first study $B_2(R)$. If $f \in B_2(R)$, f' exists and belongs to $L^2(R)$. We can thus renorm the space by putting

$$\|f\|_2^2 = \int_{-\infty}^{\infty} |f(x)|^2 dx + \int_{-\infty}^{\infty} |f'(x)|^2 dx,$$

and it is then easily seen that $B_2(R)$ is a Banach algebra.

For the sake of simplicity we only study finite systems of generators.

Theorem 7. *Let $S = \{f_i\}_1^N$ be a set of real-valued functions in $B_2(R)$, satisfying* (3.2) *and* (3.3). *A necessary and sufficient condition for S to generate $B_2(R)$ is that the set E_S, where $f_i'(x) = 0$ for $i = 1, 2, \ldots, N$, has Lebesgue measure zero.*

Proof. The necessity is obvious. Here we do not need the hypothesis that the functions f_i are real-valued, nor that S is finite.

Suppose now $mE_S = 0$. Then, if E'_S is the union of E_S and the set where for some i $f'_i(x)$ does not exist, E'_S is also of measure zero. Choose a $\delta > 0$, and let K be a compact set contained in the complement of E'_S, such that for every $x \in K$ there is an $f_i \in S$ with $|f'_i(x)| \geq \delta$.

Let $g(x) = \int_{-\infty}^{x} k(t)\,dt$, where k is a bounded function which is zero outside K and has the property that $\int_{-\infty}^{\infty} k(t)\,dt = 0$. Clearly $g \in B_2(R)$. We shall show that $g \in B_S$ (the subalgebra generated by S).

As x describes the extended line, the N-tuple $F(x) = \{f_i(x)\}_1^N$ describes a closed Jordan curve, Γ, in R^N. As on p. 87 it is easy to prove by means of the Weierstrass approximation theorem that any function $\varphi(F(x))$, where $\varphi(\xi) = \varphi(\xi^{(1)}, \ldots, \xi^{(N)})$ belongs to C^1 in a neighbourhood of Γ, is in B_S. Thus it is enough if we can find a sequence, $\{\varphi_n\}_1^\infty$, of C^1 functions, such that as $n \to \infty$

$$\int_{-\infty}^{\infty} |g(x) - \varphi_n(F(x))|^2\,dx + \int_{-\infty}^{\infty} \Big| k(x) - \sum_{i=1}^{n} \frac{\partial \varphi_n(F(x))}{\partial \xi^{(i)}} f'_i(x) \Big|^2 dx \to 0.$$

We can determine bounded functions, γ_i, such that

$$k(x) = \sum_{i=1}^{N} \gamma_i(x) f'_i(x),$$

and $\gamma_i(x) = 0$ outside K. By Lebesgue's theorem on dominated convergence it suffices if we can construct the functions φ_n so that for $i = 1, \ldots, N$

$$\lim_{n\to\infty} \frac{\partial \varphi_n(F(x))}{\partial \xi^{(i)}} = \gamma_i(x)$$

boundedly, for almost all x.

For each γ_i there is a sequence of step functions which converges boundedly to γ_i almost everywhere, and thus it is no restriction to assume that the functions γ_i are step functions such that

$$\int_{-\infty}^{\infty} \sum_{i=1}^{N} \gamma_i(x) f'_i(x)\,dx = 0.$$

We let $\{a_\nu\}_1^m$, $a_\nu < a_{\nu+1}$, be the set of all points of discontinuity of the γ_i, $i = 1, \ldots, N$. Then, for all i, $\gamma_i(x) = 0$ for $x < a_1$ and $x > a_m$. We put $\gamma_i(x) = b_{i\nu}$, $a_\nu < x < a_{\nu+1}$, and we denote the point $F(a_\nu)$ on Γ by $\alpha_\nu = (\alpha_\nu^{(1)}, \ldots, \alpha_\nu^{(N)})$. We then define a function $\varphi(\xi) = \varphi(\xi^{(1)}, \ldots, \xi^{(N)})$ on Γ by

$$\varphi(\xi) = \sum_{i=1}^{N} b_{i\nu}(\xi^{(i)} - \alpha_\nu^{(i)}) + c_\nu, \text{ for } \xi = F_N(x),\ a_\nu \leq x < a_{\nu+1},\ \nu = 1, \ldots, m-1,$$

and
$$\varphi(\xi) = 0 \text{ for } \xi = F_N(x),\ x < a_1 \text{ or } x \geq a_m.$$

Since
$$\varphi(\xi) - \varphi(\alpha_\nu) = \sum_{i=1}^{N} b_{i\nu}(\xi^{(i)} - \alpha_\nu^{(i)}) = \sum_{i=1}^{N} b_{i\nu}(f_i(x) - f_i(a_\nu)) = \int_{a_\nu}^{x} \sum_{i=1}^{N} \gamma_i(t) f'_i(t)\,dt$$

for $a_\nu \leq x < a_{\nu+1}$, it follows that the constants c_ν can be chosen so that φ is continuous on Γ.

Now let h be an increasing function in $C^\infty(0, \infty)$ such that

$$h(r)=\begin{cases}0, & r\leqslant \frac{1}{3}\\ 1, & r\geqslant \frac{2}{3}.\end{cases}$$

For every positive integer n such that $1/n\leqslant \frac{1}{3}\operatorname{Min}_{\nu\neq\mu}|\alpha_\nu-\alpha_\mu|$ we modify the function φ by putting

$$\varphi_n(\xi)=(\varphi(\xi)-\varphi(\alpha_\nu))\cdot h(n|\xi-\alpha_\nu|)+\varphi(\alpha_\nu)$$

on the intersection of the arcs $(\alpha_{\nu-1}, \alpha_\nu)$ and $(\alpha_\nu, \alpha_{\nu+1})$ with the ball $|\xi-\alpha_\nu|\leqslant 1/n$, for $\nu=1,\ldots,m$ (we put $\alpha_0=0$, $\alpha_{m+1}=0$), and $\varphi_n(\xi)=\varphi(\xi)$ elsewhere on Γ. It is then easily seen that $\varphi_n(\xi)$ is extendable by the same definition to a C^∞ function defined in a neighbourhood of Γ, and thus $\varphi_n(F)\in B_S$.

If $\xi\in(\alpha_\nu, \alpha_{\nu+1})$ and $\varphi_n(\xi)=(\varphi(\xi)-\varphi(\alpha_\nu))\cdot h(n|\xi-\alpha_\nu|)+\varphi(\alpha_\nu)$ we find for all i

$$\frac{\partial\varphi_n(\xi)}{\partial\xi^{(i)}}=b_{i\nu}\cdot h(n|\xi-\alpha_\nu|)+(\varphi(\xi)-\varphi(\alpha_\nu))\cdot h'(n|\xi-\alpha_\nu|)\cdot n\cdot\frac{\xi^{(i)}-\alpha_\nu^{(i)}}{|\xi-\alpha_\nu|}.$$

Here $h'(n|\xi-\alpha_\nu|)=0$ for $|\xi-\alpha_\nu|\geqslant 2/3n$, and since $|\varphi(\xi)-\varphi(\alpha_\nu)|\leqslant\text{Const.}\ |\xi-\alpha_\nu|$, we have $|\varphi(\xi)-\varphi(\alpha_\nu)|\cdot h'(n|\xi-\alpha_\nu|)\cdot n\leqslant\text{Const.}$ for all n. Hence $|\partial\varphi_n(\xi)/\partial\xi^{(i)}|$ is uniformly bounded as $n\to\infty$, and $\lim_{n\to\infty}\partial\varphi_n(\xi)/\partial\xi^{(i)}=b_{i\nu}$. This proves that the function $g\in B_S$.

Since the number δ is arbitrary, and $mE_S'=0$, it is now obvious that every function in $B_2(R)$ can be approximated by functions in B_S, i.e. $B_S=B_2(R)$, which proves the theorem.

We now turn to the spaces $B_{2+p}(R)$, $0<p\leqslant 1$, and we renorm them by putting

$$\|f\|_{2+p}^2=\int_{-\infty}^{\infty}|f(x)|^2dx+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f'(x+t)-f'(x)|^2|t|^{-1-p}dxdt.$$

To prove that these spaces are Banach algebras we need the following lemma.

Lemma 1. *If* $f\in B_{2+p}(R)$, $0<p\leqslant 1$, *the modulus of continuity,* ω, *of* f *satisfies*

$$\omega(\delta)\leqslant\text{Const.}\cdot\|f\|_{2+p}\cdot\delta^{(1+p)/2}\quad \textit{for } 0<p<1,$$

and

$$\omega(\delta)\leqslant\text{Const.}\cdot\|f\|_3\cdot\delta\log 1/\delta\quad\textit{for } p=1.$$

Proof.

$$f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{f}(u)\,e^{ixu}du.$$

Hence, for $\delta>0$,

$$|f(x+\delta)-f(x)|\leqslant\frac{1}{2\pi}\int_{-\infty}^{\infty}|\hat{f}(u)|\,|e^{i\delta u}-1|\,du$$

$$\leqslant\frac{\delta}{2\pi}\int_{-1/\delta}^{1/\delta}|u|\,|\hat{f}(u)|\,du+\frac{1}{\pi}\int_{|u|>1/\delta}|\hat{f}(u)|\,du.$$

For all $p \geq 0$

$$\int_{|u|>1/\delta} |\hat{f}(u)|\,du \leq \left\{\int_{|u|>1/\delta} |u|^{2+p}|\hat{f}(u)|^2du\right\}^{\frac{1}{2}} \left\{\int_{|u|>1/\delta} |u|^{-2-p}du\right\}^{\frac{1}{2}}$$
$$\leq \text{Const.} \cdot \|f\|_{2+p} \cdot \delta^{(1+p)/2}.$$

For $p<1$

$$\delta \int_{-1/\delta}^{1/\delta} |u|\,|\hat{f}(u)|\,du \leq \delta \left\{\int_{-1/\delta}^{1/\delta} |u|^{2+p}|\hat{f}(u)|^2du\right\}^{\frac{1}{2}} \left\{\int_{-1/\delta}^{1/\delta} |u|^{-p}du\right\}^{\frac{1}{2}}$$
$$\leq \text{Const.} \cdot \|f\|_{2+p} \cdot \delta^{(1+p)/2},$$

which proves the lemma in this case, and for $p=1$

$$\delta \int_{-1/\delta}^{1/\delta} |u|\,|\hat{f}(u)|\,du \leq \delta \left\{\int_{-1/\delta}^{1/\delta} (1+|u|)\,|u|^2|\hat{f}(u)|^2du\right\}^{\frac{1}{2}} \left\{\int_{-1/\delta}^{1/\delta} (1+|u|)^{-1}du\right\}^{\frac{1}{2}}$$
$$\leq \text{Const.} \cdot \|f\|_3 \cdot \delta \log (1+1/\delta).$$

This proves the lemma.

Theorem 8. *$B_{2+p}(R)$, $0<p\leq 1$, is a Banach algebra.*

Proof. We introduce the notation

$$I(f) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |f(x+t)-f(x)|^2|t|^{-1-p}dxdt.$$

Let f and g belong to $B_{2+p}(R)$. As before $\text{Max}_x|f(x)| \leq \text{Const.}\ \|f\|_{2+p}$, so it is enough to prove that e.g.

$$I(f'g) \leq \text{Const.}\ \|f\|_{2+p}^2\|g\|_{2+p}^2.$$

But
$$I(f'g) \leq 2\int_{-\infty}^{\infty}|f'(x)|^2dx \int_{-\infty}^{\infty}|g(x+t)-g(x)|^2|t|^{-1-p}dt$$
$$+2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|g(x+t)|^2|f'(x+t)-f'(x)|^2|t|^{-1-p}dxdt,$$

and by Lemma 1 we find

$$I(f'g) \leq \text{Const.}\ \|g\|_{2+p}^2 \int_{-\infty}^{\infty}|f'(x)|^2dx + 2\,\text{Max}_x|g(x)|^2 I(f') \leq \text{Const.}\ \|f\|_{2+p}^2\|g\|_{2+p}^2,$$

which proves the theorem.

It was proved by Beurling [2] and Broman [3] that if a function f belongs to $B_p(T)$, $0<p\leq 1$, the Fourier series of f converges, except possibly on a set of $(1-p)$-capacity zero, and the sum equals the derivative of the primitive function of f. The proof in [3] extends easily to $B_p(R)$.

If $f \in B_{2+p}(R)$, its derivative f' is in $B_p(R)$, and it follows that $f'(x)$ exists and equals its Fourier integral except on a set of $(1-p)$-capacity zero.

Theorem 9. *Let $S = \{f_i\}_1^N$ be a set of real-valued functions in $B_{2+p}(R)$, $0 < p \leq 1$, satisfying* (3.2) *and* (3.3). *For S to generate $B_{2+p}(R)$ it is necessary that $C_{1-p}(E_S) = 0$, and if, in addition, the derivatives f_i', $i = 1, \ldots, N$, are continuous, this condition is also sufficient.*

Proof. If $C_{1-p}(E_S) > 0$, $0 < p < 1$, there is a measure μ with support contained in E_S and with finite energy integral, i.e.

$$\int_{-\infty}^{\infty} |u|^{-p} |\hat{\mu}(u)|^2 du < \infty.$$

For $p = 1$ one finds similarly, using the kernel $\log^+ 1/r$, that there is a measure μ with

$$\int_{-\infty}^{\infty} (1 + |u|)^{-1} |\hat{\mu}(u)|^2 du < \infty.$$

Then, as in Theorem 3, the function $iu\hat{\mu}(u)$ is the Fourier transform of a non-zero bounded linear functional, C, on $B_{2+p}(R)$. If f is a polynomial in elements of S, it follows, after an application of Egoroff's theorem, as in Theorem 3 that $\langle C, f \rangle = 0$, which proves that S does not generate $B_{2+p}(R)$. Here we have neither used the reality, nor the finiteness of S.

Now we assume that $C_{1-p}(E_S) = 0$, and that f_i', $i = 1, \ldots, N$, are continuous. Let g be an arbitrary C^∞ function with compact support. Such functions are dense in $B_{2+p}(R)$ and it is thus enough to show that $g \in B_S$.

We let I be an open interval which contains the support of g, and for every positive integer n we denote by K_n the subset of I where $\sum_{i=1}^N |f_i'(x)| \leq 1/n$. Then $I \cap E_S = \bigcap_{n=1}^\infty K_n$. Since E_S is closed, it is easy to see that we can find open sets O_n, $K_n \subset O_n \subset I$, such that $\lim_{n\to\infty} C_{1-p}(O_n) = C_{1-p}(I \cap E_S) = 0$. (See Carleson [5], p. 22). We can also assume that O_n consists of a finite number of intervals, since K_n is compact.

We denote by μ_n the equilibrium distribution of $\bar{O}_n$ with mass $C_{1-p}(O_n)$, and by U_n the corresponding $(1-p)$-potential. Then $1 - U_n(x)$ vanishes everywhere on $\bar{O}_n$, and is positive outside $\bar{O}_n$.

We can then choose a C^∞ function, η_n, with support in $I - O_n$, so that

$$\int_{-\infty}^{\infty} \eta_n(x)\, dx = \int_{-\infty}^{\infty} g'(x) U_n(x)\, dx,$$

and define a function

$$g_n(x) = \int_{-\infty}^{x} k_n(t)\, dt,$$

by

$$k_n(x) = g'(x)(1 - U_n(x)) + \eta_n(x).$$

It is easily seen that $k_n \in B_p(R)$, and since $g_n(x)=0$ outside I it follows that $g_n \in B_{2+p}(R)$. We shall show that $\lim_{n\to\infty} \|g-g_n\|_{2+p}=0$. It suffices to show that $\lim_{n\to\infty} \|g'-k_n\|_p=0$.

We first observe that for any $M>0$ $\lim_{n\to\infty} \int_{-M}^{M} U_n(x)\,dx=0$. Indeed,

$$\int_{-\infty}^{\infty} U_n(x)\,dx = \int d\mu_n(t) \int_{-M}^{M} \frac{dx}{|x-t|^{1-p}} \leqslant \text{Const.}\, C_{1-p}(O_n),$$

which can be made arbitrarily small. It follows that we can choose η_n so that $\lim_{n\to\infty} \|\eta_n\|_p=0$, and thus it is enough to show that $\lim_{n\to\infty} \|g'U_n\|_p=0$. As

$$\int_{-\infty}^{\infty} g'(x)^2 U_n(x)^2\,dx \leqslant \text{Const.} \int_I U_n(x)\,dx,$$

it suffices to prove that $\lim_{n\to\infty} I(g'U_n)=0$. We choose $\varepsilon>0$, and then $M>0$ so large that for all n $U_n(x)\leqslant\varepsilon$ for $|x|\geqslant M$. We find

$$I(g'U_n) \leqslant 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |g'(x+t)|^2\, |U_n(x+t)-U_n(x)|^2\, |t|^{-1-p}\,dx\,dt$$

$$+\,2\int_{-\infty}^{\infty} |U_n(x)|^2\,dx \int_{-\infty}^{\infty} |g'(x+t)-g'(x)|^2\,|t|^{-1-p}\,dt$$

$$\leqslant \text{Const.}\; I(U_n) + \text{Const.} \int_{-M}^{M} U_n(x)^2\,dx + 2\varepsilon^2 I(g')$$

$$\leqslant \text{Const.} \int U_n(x)\,d\mu_n(x) + \text{Const.} \int_{-M}^{M} U_n(x)\,dx + \text{Const.}\; \varepsilon^2 \leqslant \text{Const.}\; (C_{1-p}(O_n)+\varepsilon^2),$$

which proves the assertion. Hence $g\in B_S$ if $g_n \in B_S$ for all n.

Thus, from now on we can assume that g is a function in $B_{2+p}(R)$ such that g' is continuous and $g'(x)=0$ outside a set $I-O_n$.

Every point in the support of g' has an open neighbourhood where for some i $|f_i'(x)|>\delta>0$ for some δ, and thus the support can be covered by a finite number of such intervals, $\{\omega_j\}_1^m$.

We can multiply g by any C^∞ function which is constant on O_n, and still remain in the class considered. Hence, by a partition of unity, we can reduce the problem to the cases when $g(x)\neq 0$ on only one interval ω_j, or when the support of g' is contained in two non-intersecting intervals, ω_1 and ω_2, say. It is enough to consider the second case, and we assume that $f_1'(x)>\delta>0$ on ω_1 and $f_2'(x)>\delta>0$ on ω_2.

As before, $F(x)=\{f_i(x)\}_1^N$ maps $\bar{R}$ onto a closed Jordan curve Γ in R^N. We shall show that, given $\varepsilon>0$, there is a function φ which belongs to C^2 in a neighbourhood of Γ, such that $\|g-\varphi(F)\|_{2+p}<\varepsilon$.

We start by defining functions h_i, $i=1, 2$, by $h_i(x)=g'(x)/f_i'(x)$, for $x\in\omega_i$, and $h_i(x)=0$ elsewhere. We show that $h_i\in B_p(R)$. In fact, $|h_i(x)|\leqslant \text{Const.}/\delta$, and, if x and y belong to ω_i,

$$|h_i(x)-h_i(y)|\leqslant \frac{1}{\delta^2}\{\operatorname*{Max}_{x\in\omega_i}|f_i'(x)|\,|g'(x)-g'(y)|+\operatorname*{Max}_{y\in\omega_i}|g'(y)|\,|f_i'(x)-f_i'(y)|\},$$

and if x is in ω_i and y belongs to neither ω_1 nor ω_2

$$|h_i(x)-h_i(y)|\leqslant \frac{1}{\delta}|g'(x)|=\frac{1}{\delta}|g'(x)-g'(y)|,$$

which proves the assertion, since either of these alternatives holds for $|x-y|$ sufficiently small.

It is easy to see, e.g. by means of convolutions with positive kernels having compact support, that we can find continuously differentiable functions, γ_i, such that

$$\|h_i-\gamma_i\|_p<\varepsilon,$$

and

$$\operatorname*{Max}_{x}|h_i(x)-\gamma_i(x)|<\varepsilon,$$

and such that the support of γ_i is contained in ω_i. We can also assume that

$$\int_{\omega_i}\gamma_i(x)f_i'(x)\,dx=\int_{\omega_i}g'(x)\,dx.$$

We denote $h_i-\gamma_i$ by λ_i, and we shall show that

$$\|g'-\gamma_1 f_1'-\gamma_2 f_2'\|_p\leqslant\|\lambda_1 f_1'\|_p+\|\lambda_2 f_2'\|_p\leqslant \text{Const. }\varepsilon.$$

Since

$$\int_{-\infty}^{\infty}|\lambda_i f_i'|^2\,dx\leqslant\varepsilon^2\cdot\operatorname*{Max}_{x\in\omega_i}f_i'(x)^2\cdot m\omega_i,$$

it suffices to prove that $I(\lambda_i f_i')\leqslant\text{Const.}\,\varepsilon^2$. We choose $M>0$ so large that $\omega_i\subset(-M,M)$, $i=1, 2$. As $\lambda_i(x)=0$ outside ω_i, we find that

$$\begin{aligned}I(\lambda_i f_i')\leqslant&\int_{-2M}^{2M}f_i'(x)^2\,dx\int_{-\infty}^{\infty}|\lambda_i(x+t)-\lambda_i(x)|^2|t|^{-1-p}\,dt\\&+\int_{|x|>2M}f_i'(x)^2\,dx\int_{|t|>M}|\lambda_i(x+t)-\lambda_i(x)|^2|t|^{-1-p}\,dt\\&+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|\lambda_i(x+t)|^2|f_i'(x+t)-f_i'(x)|^2|t|^{-1-p}\,dx\,dt\\&\leqslant\operatorname*{Max}_{|x|\leqslant 2M}f_i'(x)^2\cdot I(\lambda_i)+4\varepsilon^2\cdot\int_{-\infty}^{\infty}f_i'(x)^2\,dx\cdot\int_{|t|>M}|t|^{-1-p}\,dt+\varepsilon^2 I(f_i')\leqslant\text{Const. }\varepsilon^2.\end{aligned}$$

Now we define a function φ on Γ by

$$\varphi(F(x))=\int_{-\infty}^{x}\{\gamma_1(t)f_1'(t)+\gamma_2(t)f_2'(t)\}\,dt.$$

Because of the choice of γ_1 and γ_2, $\varphi(F(x))=0$ outside a compact interval, and thus $\varphi(\xi)$ is continuous on Γ.

Since f_i is monotonic on ω_i, we can extend φ to small neighbourhoods of the images of ω_1 and ω_2 by defining φ to be constant sufficiently near to Γ on the hyperplanes $\xi^{(i)}=f_i(x)$, $x\in\omega_i'\supset\omega_i$, $i=1,2$. Everywhere outside the intervals ω_i, $\varphi(F(x))$ is constant, and hence we can extend φ continuously to a neighbourhood of Γ by putting it equal to a constant in a small neighbourhood of each of the two remaining arcs of Γ.

On the image of ω_i we clearly have

$$\frac{\partial\varphi(F(x))}{\partial\xi^{(i)}}=\gamma_i(x),$$

and all other derivatives are zero. Since $\gamma_i(x)$ is continuously differentiable, and $1/f_i'(x)$ is continuous on ω_i, it follows that φ has continuous second derivatives on Γ, and because of the way we extended φ, also in a neighbourhood of Γ.

Now it is easy to see that $\|g-\varphi(F)\|_{2+p}<\text{Const.}\,\varepsilon$. Indeed, we have

$$\frac{d\varphi(F(x))}{dx}=\gamma_1(x)f_1'(x)+\gamma_2(x)f_2'(x),$$

and it follows from the construction that

$$\int_{-\infty}^{\infty}|g(x)-\varphi(F(x))|^2\,dx\leqslant\text{Const. }\varepsilon^2,$$

and that

$$I\left(g'-\frac{d\varphi(F)}{dx}\right)\leqslant 2I(\lambda_1f_1')+2I(\lambda_2f_2')\leqslant\text{Const. }\varepsilon^2.$$

To complete the proof we need only observe that for all C^2 functions φ, $\varphi(F)\in B_S$, which is easily proved as before by the Weierstrass approximation theorem (Whitney [16], p. 74).

For the sake of completeness we also treat the spaces $B_p(R)$, $p>3$. We put $p=2k+q$, where k is an integer and $0\leqslant q<2$. If $q=0$ we renorm the space by

$$\|f\|_p^2=\int_{-\infty}^{\infty}|f(x)|^2\,dx+\int_{-\infty}^{\infty}|f^{(k)}(x)|^2\,dx,$$

and if $q>0$ we put

$$\|f\|_p^2=\int_{-\infty}^{\infty}|f(x)|^2\,dx+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}|f^{(k)}(x+t)-f^{(k)}(x)|^2|t|^{-1-q}\,dx\,dt.$$

If $f\in B_p(R)$, $p=2k+q$, the derivatives up to order $k-1$ have absolutely integrable Fourier transforms, and thus these derivatives are continuous and bounded. It follows easily, by means of the Leibniz formula for the derivatives of a product, that these spaces are also Banach algebras.

Theorem 10. *Let S be a set of real-valued functions in $B_p(R)$, $p>3$, satisfying* (3.2) *and* (3.3). *For S to generate $B_p(R)$ it is necessary and sufficient that the set E_S be empty.*

Sketch of proof: Since the proof does not involve any new ideas we only give a brief sketch. (Cf. also Theorem B.)

The necessity of the condition is obvious, since convergence in $B_p(R)$, $p>3$, implies uniform convergence of the first derivatives.

To prove the sufficiency, we assume that $E_S=\phi$, and we let g be a C^∞ function whose support is contained in an interval, ω, such that for some $f\in S$, $f'(x)\geqslant\delta>0$ in ω.

We then write $g(x)=\gamma(f)$, prove that $\gamma^{(k)}(f)\in B_q(R)$, and approximate $\gamma^{(k)}(f)$ in $B_q(R)$ with a continuously differentiable function $a(f)$, which can be modified so that it is the k-th derivative with respect to f of a function $A(f)$ in $B_p(R)$. Then we prove that $A(f)$ approximates g, and finish by showing that $A(f)\in B_S$.

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