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One-parameter semigroups of subsets of a real linear space

By Hans Rådström

CONTENTS

§ 1. Introduction

The main objective of this paper is to establish the structure theorem in § 5.

If K and M are subsets of a linear space we let $K+M$ denote the set of all sums $k+m$, where $k \in K$ and $m \in M$. If K consists of only one point k, we write $k+M$ instead of $\{k\}+M$. This operation of *addition* is obviously commutative and associative. Thus a family of subsets forms a commutative semigroup under addition if it is closed under this operation. Such a semigroup is called a *one-parameter semigroup* if there exists a one-to-one application: $\delta \to A(\delta)$, of the set of positive real numbers onto the semigroup, satisfying:

$$A(\delta_1+\delta_2)=A(\delta_1)+A(\delta_2).$$

In his paper (2) Gleason introduced the concept of a one-parameter semigroup of subsets of an arbitrary toplogical group. In my thesis (3, henceforth referred to as CN) I obtained results concerning the structure of such semigroups in Lie groups. In particular, it follows from results in CN that in a finite-dimensional linear space every one-parameter semigroup $A(\delta)$ of closed sets which come arbitrarily close to the origin as $\delta \to 0$ is of the form $A(\delta)=\delta \cdot M$ where M is a convex, compact set and δM denotes the set of all δm with $m \in M$. This is of course a consequence of the much more general results in § 5 below.

It should be observed that the definition of one-parameter semigroups given above is neither equivalent to the one in CN nor to Gleason's although there are no very essential differences.

We remark that:

1° In CN a one-parameter semigroup is the mapping $\delta \to A(\delta)$, and not a family of sets. Thus in CN $A(\delta)$ could be constant which is not possible with the present definition. (We require $\delta \to A(\delta)$ to be one-to-one.)

2° In CN the mapping is defined for $\delta \geqslant 0$ and not only for $\delta > 0$.

3° In CN the sets $A(\delta)$ are assumed to be closed for all small δ.

4° In CN the mapping is assumed to be continuous with respect to Hausdorff's topology for sets (see below, section 3.1).

If the semigroup is parametrized in two ways with parameters δ and ε, then there exists a one-to-one mapping, f, of the set of positive reals onto itself satisfying

$$\varepsilon = f(\delta) \text{ and } f(\delta_1 + \delta_2) = f(\delta_1) + f(\delta_2).$$

It is well known that the only positive solutions of this functional equation are of the form $f(\delta) = \text{const} \cdot \delta$. Therefore every possible reparametrization of a given semigroup consists simply in a change of unit for the parameter.

§ 2. Remarks and examples

2.1. We denote by δM the set of all δm where $m \in M$.

Example 1. If M is convex it is easy to prove that $(\delta_1 + \delta_2) M = \delta_1 M + \delta_2 M$. Thus, for a convex set M, which is not a cone, the sets δM constitute a one-parameter semigroup with δ as parameter. (We must require that M not be a cone in order to guarantee that the mapping $\delta \to \delta M$ be one-to-one.)

Conversely, suppose that M is a set such that the sets δM form a one-parameter semigroup with δ as parameter. *Then M is convex.* Namely, let x and y be elements of M. Then $\lambda x \in \lambda M$ and $(1-\lambda) y \in (1-\lambda) M$ for $0 < \lambda < 1$. It follows that $\lambda x + (1-\lambda) y \in \lambda M + (1-\lambda) M = M$.

2.2. *Definition: A one-parameter semigroup $A(\delta)$ is called linear if there is a set M such that*

$$A(\delta) = \delta M$$

for all $\delta > 0$.

We have seen that a linear semigroup consists of convex sets. The converse is not true, not even in the case of a one-dimensional space as the next example proves.

Example 2. Let $f(\delta)$ be a realvalued function of the positive real variable δ and satisfying the functional equation

$$f(\delta_1 + \delta_2) = f(\delta_1) + f(\delta_2).$$

Putting $A(\delta) = \{f(\delta)\}$ for each δ we obtain a one-parameter semigroup of points on the real line. It is known that under almost any (even very mild) regularity condition the function $f(\delta)$ must be of the form $\text{const} \cdot \delta$ and consequently $A(\delta)$ linear. On the other hand there also exist pathological solutions of the equation. Thus a semigroup of points need not be linear.

2.3. The result in § 5 below shows that for one-parameter semigroups of compact sets in a locally convex space, the two examples given cover all possibilities. We shall see that if $A(\delta)$ is such a semigroup, it can be expressed in the form $A(\delta) = f(\delta) + \delta \cdot M$, where $f(\delta)$ is a semigroup of points and M a compact convex set. If the compactness condition is not made the conclusion $A(\delta) = f(\delta) + \delta M$ with convex M is no longer true.

Example 3. On the real line, let $A(\delta)$ be the set of all rational numbers r with

$$0 < r < \delta.$$

Then the sets $A(\delta)$ form a one-parameter semigroup of non-convex sets.

However, in this example the closure, $\overline{A(\delta)}$, of each set in the semigroup is convex. Examples 5, 6 and 7 below show, among other things, that this is not always the case.

2.4. We shall also be somewhat interested in finding conditions to ensure that $A(\delta)$ or $\overline{A}(\delta)$ is linear. If $A(\delta)$ is given and $\phi(\delta)$ is a one-parameter semigroup of points, then $\phi(\delta) + A(\delta)$ is a one-parameter semigroup. Can $\phi(\delta)$ be chosen so that $\phi(\delta) + A(\delta)$ is linear? Our next example shows that this need not be possible even if the sets $A(\delta)$ are convex.

Example 4. Let $f(\delta)$ be a pathological function in example 2. In the xy-plane, consider the set $A(\delta)$ of all points (x, y) with $0 < x < \delta$ together with the points $(0, 0)$ and $(\delta, f(\delta))$. Then the sets $A(\delta)$ form a non-linear, one-parameter semigroup of convex sets such that $\phi(\delta) + A(\delta)$ is not linear for any choice of ϕ.

2.5. The following statement is an application of Theorem 5.6 of CN (p. 133) to the case of a linear space: Let a translation invariant metric be given on a linear space and denote the closed sphere of radius δ around the origin by $S(\delta)$. A necessary and sufficient condition that $S(\delta)$ forms a one-parameter semigroup is that the metric be convex. (See CN p. 133.) A well known case when this occurs is when the metric is defined by a norm. It can, however, occur also in other cases.

Example 5. Consider for $0 < p \leqslant 1$ the space L^p of functions $f(t)$ (with the usual identifications) defined for $0 \leqslant t \leqslant 1$ and such that $\|f\| = \int_0^1 |f(t)|^p dt < \infty$. (Observe! Unusual notation. If $p \geqslant 1$ the norm is defined by $\|f\|^p = \int_0^1 |f(t)|^p dt$. Our $\|f\|$ is *not* a norm for $p < 1$ since it is not homogeneous.) If $\|f - g\|$ is taken to mean the distance between f and g this makes the space a complete metric space which is not locally convex for $p < 1$ but locally bounded (see Tychonoff 4, or Bourbaki 1 p. 224 ex 17). The triangle inequality follows from the elementary inequality

$$|a+b|^p \leqslant |a|^p + |b|^p$$

in which for $p < 1$ the sign of equality holds if and only if either a or b is zero. Thus the triangle inequality is strict unless the two functions involved are different from zero on disjoint sets in which case equality holds. Now the triangle inequality:

$$\|f+g\| \leqslant \|f\| + \|g\|$$

is obviously equivalent to the following inequality for the spheres:

$$S(\delta_1) + S(\delta_2) \subset S(\delta_1 + \delta_2).$$

We obtain equality if and only if to any h with $\|h\| \leqslant \delta_1 + \delta_2$ it is possible to find f and g with $f + g = h$, $\|f\| \leqslant \delta_1$ and $\|g\| \leqslant \delta_2$. It is obviously sufficient to prove that given h with $\|h\| = \delta_1 + \delta_2$, there exist f and g with $f + g = h$ and $\|f\| = \delta_1$ and $\|g\| = \delta_2$. This is done in the following way. Let λ be a number such that

$$\int_0^{\lambda} |h(t)|^p \, dt = \delta_1$$

which exists since $0 < \delta_1 < \delta_1 + \delta_2 = \int_0^1 |h|^p dt$.

Put

$$f(t) = \begin{cases} h(t) & \text{for} \quad 0 \leqslant t < \lambda \\ 0 & \text{for} \quad \lambda \leqslant t \leqslant 1 \end{cases}$$

$$g(t) = \begin{cases} 0 & \text{for} \quad 0 \leqslant t < \lambda \\ h(t) & \text{for} \quad \lambda \leqslant t \leqslant 1 \end{cases}$$

Since the sets where f and g are different from zero are disjoint the result follows. Thus the spheres form a one-parameter semigroup and the metric is convex. Since the spheres are closed and it is known that they are bounded, this is also an example that *a one-parameter semigroup of closed bounded sets need not consist of convex sets.*

2.6. The spaces L^p for $p < 1$ are not locally convex and for $p = 1$ the semigroup of spheres in example 5 does in fact consist of convex sets. Thus we have not yet obtained an example of a pathological (= consisting of non-convex sets) semigroup of closed bounded sets in a locally convex space. Example 6, however, shows that such an example exists in L^1.

Example 6. In example 5 let $p = 1$ and put $A(\delta)$ = the set of those functions f which satisfy

1° $\int_0^1 f \, dt = \delta$

2° $f \geqslant 0$

3° The values of f are integers almost everywhere.

Then $\|f\| = \delta$ so that $A(\delta) \subset S(\delta)$. Thus $A(\delta)$ is bounded and it is obviously closed. Now let $f \in A(\delta_1)$ and $g \in A(\delta_2)$. Then $f + g$ is clearly in $A(\delta_1 + \delta_2)$ which shows that $A(\delta_1) + A(\delta_2) \subset A(\delta_1 + \delta_2)$. The reversed inequality is obtained by considering an element h in $A(\delta_1 + \delta_2)$ and applying the decomposition of example 5. This yields $f \in A(\delta_1)$ and $g \in A(\delta_2)$.

Now $A(\delta)$ is not a semigroup of convex sets. We prove that $A(\frac{1}{2})$ is not convex. Let f_1 and f_2 be defined as follows

$$f_1(x) = \begin{cases} 1 & \text{if} \quad 0 \leqslant x \leqslant \frac{1}{2} \\ 0 & \textit{if} \quad \frac{1}{2} < x \leqslant 1 \end{cases}$$

$$f_2(x) = \begin{cases} 0 & \text{if} \quad 0 \leqslant x \leqslant \frac{1}{2} \\ 1 & \text{if} \quad \frac{1}{2} < x \leqslant 1 \end{cases}$$

Thus f_1 and f_2 are elements of $A(\frac{1}{2})$ but $(f_1 + f_2)/2 = \frac{1}{2}$ for all x and is therefore not integervalued almost everywhere and so it is not an element of $A(\frac{1}{2})$.

2.7.

Example 7. Let $A(\delta)$ be the set of all integervalued functions in $L^1(0, 1)$ of norm $\leqslant \delta$. Then $A(\delta)$ is closed and bounded and $A(\frac{1}{2})$ contains the functions f_1 and f_2 of example 6, but it does not contain the function $(f_1 + f_2)/2$. Thus the function $(f_1 + f_2)/2$ has a positive distance to the closed set $A(\frac{1}{2})$ and therefore there is a positive number k so small that $A(\frac{1}{2}) + kS(\frac{1}{2})$ is not convex. ($S(\delta)$ denotes the sphere of radius δ in the norm.) Hence the semigroup $S_1(\delta) = A(\delta) + kS(\delta)$ does not consist of convex sets. On the other hand all these sets are symmetric neighborhoods of the origin and again by Theorem 5.6 of CN they are the spheres around the origin in a translation invariant *convex* metric for L^1. This shows therefore that such spheres need not be convex even in a locally convex space.

§ 3. Auxiliary material

3.1. *Hausdorff's topology for sets.*

The following is a well known method to define a topology on the family, F, of all subsets of a linear space. Let U be a neighborhood of the origin and A a given set, the neighborhoods in F of which we want to define. Let $U'(A)$ be the subset of F consisting of those sets B in the space for which the two inequalities hold:

$$A \subset B + U$$
$$B \subset A + U.$$

If U varies in the set of all neighborhoods of the origin, the corresponding sets $U'(A)$ run through a fundamental system of neighborhoods of A in a topology for F. This will be called *Hausdorff's topology.*

In general (i.e. if the space has more than one point) Hausdorff's topology does not satisfy Hausdorff's separation axiom, T_2, for it is clear that if A is not closed the sets (or points in F) A and $\bar{A}$ are two different elements of all sets $U'(A)$ or $U'(\bar{A})$. On the other hand, *the restriction of Hausdorff's topology to the family of closed subsets of the linear space satisfies T_2 and so does the restriction to the open convex subsets.* The first part of this proposition is standard and the second is an immediate consequence of the following theorem in the theory of convex sets: If A is open and convex then $A = \operatorname{Int} \bar{A}$.

Although limits are not unique in Hausdorff's topology we can thus nevertheless pick at most one limit among the closed sets and if there exists an open convex limit it is also unique.

3.2. *The convex hull.*

We denote the convex hull of a set A by HA. The closed convex hull will be denoted by $\bar{H}A$. In this section we prove the formula:

$$H(A + B) = HA + HB$$

which is important in the sequel and, rather surprisingly, does not seem to be explicitly mentioned in the literature on convex sets.

Although the formula follows rather easily from the barycentric representation of the points in the convex hull, we prefer to deduce it as a special case from the following consideration which is much to general for our immediate purpose but has also other applications.

By a *hull operation* we mean an operation, T, which is defined on a family of subsets of a given space, takes its values in the same family and which satisfies

$$1^\circ \ TA \supset A$$

$$2^\circ \ T^2 A = TA$$

$$3^\circ \ A \subset B \text{ implies } TA \subset TB.$$

If we call any set A satisfying $TA = A$ a hull, it is immediate that for an arbitrary set A the set TA is the intersection of all hulls containing A. We remark also that if T is not defined for all subsets of the space, we can extend it by defining TA as the intersection of all hulls containing A if there is such a hull and as the entire space otherwise. Now suppose that the space is a linear space and that our hull operation is *translation invariant*, that is

$$4^\circ \ x + TA = T(x + A) \text{ for all } x \text{ in the space.}$$

Then we have

$$TA + TB \subset T(A + B)$$

and equality holds if and only if the left member is a hull.

Proof. Let $a \in A$. Then by 4° and 3°

$$a + TB = T(a + B) \subset T(A + B).$$

It follows that

$$A + TB \subset T(A + B). \tag{1}$$

Substituting TA for A we obtain

$$TA + TB \subset T(TA + B)$$

and by use of (1)

$$TA + TB \subset TT(A + B) = T(A + B),$$

which proves the first part of the proposition.

Now observe that from $A \subset TA$ and $B \subset TB$ follows

$$A + B \subset TA + TB.$$

Thus $T(A + B) \subset T(TA + TB) \subset T^2(A + B) = T(A + B)$ so that $T(TA + TB) = T(A + B)$. In this equation the left member equals $TA + TB$ if and only if this set is a hull, which proves the second part of the proposition.

As special cases we take $T = H$ and obtain the formula which we set out to prove. Other specializations are obtained by letting TA denote the linear variety spanned by A or the additive variety generated by A. In these cases we have $TA + TB = T(A + B)$. An example where equality does not always hold is furnished by choosing T to be the operation of closure: $TA = \bar{A}$. Thus $\bar{A} + \bar{B} \subset \overline{A + B}$ and equality holds if and only if $\bar{A} + \bar{B}$ is a closed set. In particular equality holds if A or B has compact closure.

3.3. *The support function.*

Let T be a linear system. By T^* we denote the algebraic adjoint of T, that is the linear system consisting of all linear functionals on T. Let L be such a functional and K a set in T. By $L \circ K$ we denote the supremum of L on K. This supremum may of course assume the value $+\infty$, but if K is non-empty it does not assume the value $-\infty$. If K and M are two non-empty sets we have

3.31. $$L \circ (K + M) = L \circ K + L \circ M,$$

where in the case when the value $+\infty$ occurs the obvious conventions "∞ + real $= \infty$" and "$\infty + \infty = \infty$" should be made. If the set K is kept fixed and the functional L varies in T^* the mapping $L \to L \circ K$ is a (real- or $+\infty$-valued) function on T^* which is called the support function S_K of K. Formula 3.31 above shows that

3.32. $$S_{K+M} = S_K + S_M.$$

It is easy to set that the support function is *positively homogeneous:*

$$S_K(\lambda L) = \lambda S_K(L) \text{ if } \lambda \geqslant 0,$$

and *subadditive:*

$$S_K(L_1 + L_2) \leqslant S_K L_1 + S_K L_2.$$

It is therefore a convex function.

For a one-point set K the support function is a linear and everywhere finite-valued function on T^*, i.e. S_K is an element of T^{**}. The mapping of T into T^{**} defined in this way: $a \to S_{\{a\}}$; is called the canonical embedding of T in T^{**}. In the following considerations we shall assume that T is embedded in T^{**} in this way.

If $K \subset T$ is a translate of $M \subset T$, say $K = M + a$, then 3.32 gives

3.33. $$S_K = S_M + S_{\{a\}}$$

where the second term in the right member is a linear functional on T^*. Conversely, suppose we know that $S_K = S_M +$ (linear functional on T^*). Does it follow that K is a translate of M? The answer is yes if both sets are convex and compact in a locally convex, Hausdorff topology for T. In fact

3.34. *Let a locally convex, Hausdorff topology be given on T. Then there exists a projection P of T^{**} onto a subspace containing T such that if*

1° *K and M are compact, convex subsets of T and $s \in T^{**}$,*

2° *$S_K(L) = S_M(L) + s(L)$ for those functionals $L \in T^*$ which are continuous on T, then $K = M + Ps$.*

Proof. Any linear functional L on T has a natural extension to T^{**}, for given $s \in T^{**}$ defined by $L(s) = s(L)$. We assume henceforth that the elements of T^* are extended in this way.

Now we consider the weak topology on T^{**} defined by the set T' of functionals L which are continuous on T. This topology need not be Hausdorff but the topology which it induces on the subset T of T^{**} is Hausdorff since it is the weakening of the original topology on T. Therefore K and M are compact also in the new topology. The closure of the origin in T^{**} is the intersection R of all nullspaces of functionals in T'.

We define P in the following way. Let T_1 be a supplement of $T + R$ in T^{**}. Thus $T^{**} = T + R + T_1$ and this is a decomposition of T^{**} in a direct sum. We define P to be the projection on $T + T_1$.

Now the relation $S_K = S_M + s$ says that for each $L \in T'$:

$$L \circ K = L \circ M + L(s) = L \circ (M + s).$$

Thus any closed halfspace in T^{**} which contains one of the sets K or $M + s$ also contains the other set. This means that the closed convex hulls of the sets coincide. Since the sets are convex it follows that

$$\bar{K} = \bar{M} + s. \tag{1}$$

Since R is the closure of the origin in T^{**} the set $\bar{K}$ will contain $K + R$. On the other hand this set, being the sum of a compact set and a closed set, is closed, so that we have $\bar{K} = K + R$. The same argument applies to M. Substituting in (1) we obtain

$$K + R = M + R + s. \tag{2}$$

We now use the fact that P is a linear transformation so that for any two sets M and N the formula $P(M + N) = P(M) + P(N)$ is valid. Applying P to both members of (2) and observing that $P(K) = K$, $P(M) = M$ and $P(R) = 0$ we obtain

$$K = M + Ps$$

which was to be proved.

Incidentally, it follows that $Ps \in T$ so that $s \in T + R$. This shows that condition 2° is a rather strong restriction on s. For in general, of course, $T + R$ is very far from being all of T^{**}.

§ 4. One-parameter semigroups of convex sets

4.1. We suppose in this paragraph that *the sets $A(\delta)$ are convex and form a one-parameter semigroup. Then for arbitrary $\lambda > 0$ and rational $\delta > 0$:*

$$A(\delta\lambda) = \delta A(\lambda).$$

Proof. We have $A(\lambda) = A\left(\frac{\lambda}{n}\right) + A\left(\frac{\lambda}{n}\right) + \cdots + A\left(\frac{\lambda}{n}\right)$, ($n$ terms), which is $= n A\left(\frac{\lambda}{n}\right)$ since $A\left(\frac{\lambda}{n}\right)$ is convex. Thus $A\left(\frac{\lambda}{n}\right) = \frac{1}{n} A(\lambda)$. By a change of notation we have also $A(m\lambda) = m a(\lambda)$. Combining these two formulas we obtain

$$A\left(\left(\frac{m}{n}\right)\lambda\right) = m A\left(\frac{\lambda}{n}\right) = \frac{m}{n} \cdot A(\lambda).$$

4.2. Example 2 (section 2.2) shows that the above proposition need not be true for all $\delta > 0$, i.e. a one-parameter semigroup of convex sets need not be linear. *If we make the following topological assumptions,* however, *the linearity follows* almost trivially.

1. *The convex sets $A(\delta)$ are bounded and either all closed or all open.*
2. *The mapping A is continuous.*

It is easy to see that for a bounded set K the sets δK vary continuously with δ. (Not true for all unbounded sets. Counterexample see CN p. 106.) From 4.1 we have

$$A(\delta) = \delta A(1) \text{ for rational } \delta > 0.$$

Since both members vary continuously with δ the result would follow if Hausdorff limits were unique. Now the result follows from proposition 3.1. Example 2 shows that the assumption 1 alone does not imply linearity. Example 4 shows that neither does assumption 2.

4.3. The following result is basic in the proof of the main theorem in § 5.

Let $A(\delta)$ be a one-parameter semigroup of compact convex sets in a Hausdorff, locally convex linear space, and let $a \in A(1)$ be given. Then there exists a function f defined for all $\delta > 0$ taking (point) values in the space and satisfying

1. $f(\delta_1 + \delta_2) = f(\delta_1) + f(\delta_2)$
2. $f(\delta) \in A(\delta)$
3. $f(1) = a$.

Proof. Let the space be T and its topological adjoint T'. For each $\delta > 0$ we denote by $\varphi(\delta, L)$ the restriction to T' of the support function of $A(\delta)$. Then φ is a realvalued function. From the semigroup property of $A(\delta)$ it follows that φ is *additive* as a function of δ for each L. Since φ is a support function it is *subadditive* as a function of L for each $\delta > 0$.

Now let L_1 and L_2 be two elements of T'. By the subadditivity in L of $\varphi(\delta, L)$ we have

$$\varphi(\delta, L_1) + \varphi(\delta, L_2) - \varphi(\delta, L_1 + L_2) \geqslant 0.$$

The left member of this inequality is an additive real-valued function $\psi(\delta)$ for $\delta > 0$. It is wellknown that such a function can be non-negative only if $\psi(\delta) = \delta\psi(1)$. Thus $\psi(\delta) - \delta\psi(1) = 0$ or in other words $\varphi(\delta, L) - \delta\varphi(1, L)$ is additive in L on T' for each $\delta > 0$. Moreover $\varphi(\delta, L)$ is positively homogenous in L. Thus we can state the preliminary result: For $\delta > 0$ and $L \in T'$: $\varphi(\delta, L) = \delta\varphi(1, L) + \lambda(\delta, L)$ where $\lambda(\delta, L)$ is a realvalued function with $\lambda(1, L) = 0$, additive in δ and linear in L.

By use of 3.34 we see that there is a function $g(\delta)$ taking values from T and such that

$$A(\delta) = \delta A(1) + g(\delta).$$

Here $g(\delta)$ is the element of T on which $\lambda(\delta, \cdot) \in T^{**}$ is projected. Since the projection is linear and λ is additive in δ it follows that g is additive and $g(1) = 0$. Now put $f(\delta) = g(\delta) + \delta a$. Then f is additive so that the statement 1 of the theorem is proved. The result 3 is immediate and 2 follows since $a \in A(1)$ implies

$$f(\delta) = \delta a + g(\delta) \in \delta A(1) + g(\delta) = A(\delta).$$

4.4. If all the sets in a one-parameter semigroup contain the origin the semigroup is monotone in the sense that $\delta_1 < \delta_2$ implies $A(\delta_1) \subset A(\delta_2)$. For $A(\delta_2) = A(\delta_1) + A(\delta_2 - \delta_1)$ and since $A(\delta_2 - \delta_1)$ contains the origin $A(\delta_2) \subset A(\delta_1)$.

One-parameter semigroups of convex sets containing the origin are "almost" linear. The following proposition gives the exact formulation.

If $B(\delta)$ is a one-parameter semigroup of convex sets containing the origin, there exist two convex sets K_1 and K_2 containing the origin and such that $\delta K_1 \subset B(\delta) \subset \delta K_2$. The set K_2 is obtained from K_1 by adjoining to K_1 the endpoints of all radii of K_1.

Remark. By a radius of K_1 we mean, of course, the intersection between K_1 and a ray from the origin. Such a radius is either the entire ray, the origin or a segment, one endpoint of which is the origin and belongs to K_1, while the other endpoint may or may not belong to K_1.

Proof. By 4.1 we have for rational $r > 0$

$$B(r) = rB(1).$$

Now let K_1 be the set obtained from $B(1)$ by deleting from $B(1)$ all points which are endpoints of a radius and different from the origin. Let K_2 be obtained by adjoining these points to $B(1)$. Thus

$$rK_1 \subset B(r) \subset rK_2.$$

Since $B(\delta)$ is monotone it follows that for δ arbitrary

$$\bigcup_{r<\delta} rK_1 \subset B(\delta) \subset \bigcap_{r>\delta} rK_2.$$

It follows from the properties of K_1 and K_2 that

$$\bigcup_{r<\delta} rK_1 = \delta K_1$$

and

$$\bigcap_{r>\delta} rK_2 = \delta K_2$$

which proves the proposition.

4.5. By combination of 4.3 and 4.4 it is possible to complement the result of the discussion in 2.2.

If $A(\delta)$ is a semigroup of compact, convex sets, in a locally convex, Hausdorff, linear space, it is of the form

$$A(\delta) = f(\delta) + \delta K$$

where K is a compact convex set and $f(\delta)$ is a one-parameter semigroup of points.

Proof. Let f be the function of proposition 4.3, put $B(\delta) = A(\delta) - f(\delta)$ and apply 4.4.

§ 5. One-parameter semigroups of compact sets

5.1. *Theorem. In a locally convex space any one-parameter semigroup of compact sets consists of convex sets.*

Proof. Let the semigroup be $A(\delta)$. We fix δ and consider the sets

$$B_n = 2^n A\left(\frac{\delta}{2^n}\right), \qquad n = 0, 1, 2, \ldots.$$

Then

1° B_n is compact

2° $B_{n+1} \subset B_n$

3° $HB_n = HB_0$

1° is obvious. 2° follows from the formula $2M \subset M + M$. 3° follows by induction from 3.2 in the following way

$$HB_n = H\left(2^n A\left(\frac{\delta}{2^n}\right)\right) = 2^n H\left(A\left(\frac{\delta}{2^n}\right)\right) =$$

$$= 2^{n-1}\left(HA\left(\frac{\delta}{2^n}\right) + HA\left(\frac{\delta}{2^n}\right)\right) = 2^{n-1} HA\left(\frac{\delta}{2^{n-1}}\right) = H(B_{n-1}).$$

Consider now the compact set $B = \bigcap_{n \geqslant 0} B_n$. We observe first that B is convex. Let namely $x, y \in B$. Then $x, y \in B_n$ for all n. It follows that if $n \geqslant 1$ we have $\frac{x+y}{2} \in \frac{1}{2}(B_n + B_n) = B_{n-1}$. Thus $\frac{x+y}{2} \in B$. Since B is closed this implies that B is convex.

Next we observe that $B_n \to B$ in Hausdorff's topology for sets. That this follows from 1° and 2° is a standard topological theorem.

It follows that $HB_n \to HB$, for the operation H is a continuous operation in a locally convex space. (We can also prove the result directly in the following way. Let U be a convex neighborhood of the origin. We need the inequalities $HB_n \subset HB + U$ for all sufficiently large n. Simply choose n so large that $B_n \subset B + U$. Then by 3.2 we obtain $HB_n \subset HB + HU$. But $U = HU$ since U is convex.)

Combining this with 3° above we obtain $\bar{H}B_0 = \bar{H}B = B$ since B is convex and closed. On the other hand $B_0 \supset B = \bar{H}B_0 \supset B_0$ so that equality holds everywhere and it follows that $B_0 = A(\delta)$ is convex. The theorem is proved.

5.2. *Structure Theorem:*

Any one-parameter semigroup $A(\delta)$ of compact sets in a locally convex, Hausdorff, linear space is of the form

$$A(\delta) = f(\delta) + \delta K$$

where $f(\delta)$ is a one-parameter semigroup of points and K a compact convex set.

Proof. Theorem 5.1 and proposition 4.5.

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