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A connection between α-capacity and L^p-classes of differentiable functions

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1.

Let $x = (x^1, \ldots, x^m)$ be a point in the m-dimensional Euclidean space R^m. The following measure was recently used by Serrin [6] for investigating removable singularities of a class of quasi-linear partial differential equations:

Definition. Let E be a bounded set in R^m. $M_s(E)$, where $1 \leqslant s < \infty$, is defined by

$$M_s(E) = \inf \int |\operatorname{grad} \psi|^s \, dx, \tag{1.1}$$

where the infimum is taken over all continuously differentiable functions ψ which have compact supports and are $\geqslant 1$ on E. If $s \geqslant m$ we also require the support of ψ to belong to a certain fixed sphere $|x| < R_0 < \infty$ which is independent of E.

We intend to investigate the connection between $M_s(E)$ and the potential theoretic α-capacity of E. As $M_s(E) = M_s(\bar{E})$, where $\bar{E}$ is the closure of E, the only case of interest is to consider compact sets. The investigation has a close connection with [7], to which we shall refer concerning some details of the proofs.

Let us first introduce some notations. The support of a measure μ and of a function f is denoted by S_μ and S_f respectively. $S(r)$, $r > 0$, is the closed sphere $|x| \leqslant r$. The α-potential, $0 \leqslant \alpha < m$, of a measure μ is denoted by u_α^μ, where

$$u_\alpha^\mu(x) = \int \frac{d\mu(y)}{|x-y|^\alpha}, \quad \text{if} \quad 0 < \alpha < m,$$

and

$$u_0^\mu(x) = \int \log \frac{1}{|x-y|} \, d\mu(y).$$

Here and elsewhere, the integration is to be extended over the whole space, if no limits of integration are indicated. If μ is absolutely continuous and has a density f, $d\mu = f\,dx$, we also write u_α^f instead of u_α^μ.

If $I_\alpha(\mu)$ denotes the energy integral of μ,

$$I_\alpha(\mu) = \int u_\alpha^\mu \, d\mu(x),$$

we define the α-capacity of a bounded Borel set E, $C_\alpha(E)$, by

$$C_\alpha(E) = \{\inf I_\alpha(\mu)\}^{-1},$$

where the infimum is taken over all positive measures μ with total mass 1 and $S_\mu \subset E$. When $\alpha = 0$ we make this definition only if the diameter of E is less than 1. For an arbitrary Borel set E we put $C_0(E) = 0$ if and only if $C_0(E \cap S) = 0$ for every sphere S with diameter less than 1.

We shall use the well-known fact that if F is a compact set with $C_\alpha(F) > 0$ —we suppose the diameter of F less than 1 if $\alpha = 0$—then there exists a unique positive measure τ with total mass k, $k > 0$, and $S_\tau \subset F$ such that $\inf_\nu I_\alpha(\nu)$ is attained for $\nu = \tau$ where ν ranges over the class of all positive measures with total mass k and $S_\nu \subset F$. τ is called the *capacitary distribution* with total mass k of order α of F. u_α^τ has the following properties:

$$u_\alpha^\tau(x) \geq k\{C_\alpha(F)\}^{-1} \quad \text{for every } x \in F \text{ except when } x \text{ belongs to a set of } \alpha\text{-capacity zero.} \tag{1.2}$$

$$u_\alpha^\tau(x) \leq k\{C_\alpha(F)\}^{-1} \quad \text{for every } x \in S_\tau. \tag{1.3}$$

$$u_\alpha^\tau(x) \leq M \cdot k\{C_\alpha(F)\}^{-1} \quad \text{everywhere,} \tag{1.4}$$

where M is a constant which may be chosen only depending on m. If $\alpha > 0$ we may in fact choose $M = 2^\alpha < 2^m$. We shall also use the fact that if F is the union of a finite number of closed spheres, then

$$u_\alpha^\tau(x) \geq k\{C_\alpha(F)\}^{-1} \quad \text{for every} \quad x \in F. \tag{1.5}$$

The β-dimensional measure, $0 < \beta < m$, of a bounded set E, $L_\beta(E)$, is defined as

$$\inf \sum_\nu r_\nu^\beta,$$

where the infimum is taken over all the coverings of E by families of open spheres with radii $\{r_\nu\}$.

Let C^∞ be the class of all infinitely differentiable functions in R^m and C_0^∞ those functions in C^∞ that have compact supports. L_{loc}^p, $p \geq 1$, is the class of all Lebesgue measurable functions f in R^m such that $\int_F |f(x)|^p \, dx < \infty$ for every compact set F and L^p, $p \geq 1$, is the class of all measurable functions f such that $\int |f(x)|^p \, dx < \infty$. We use the notation

$$\|f\|_{L^p(E)} = \left\{\int_E |f(x)|^p \, dx\right\}^{1/p},$$

and we write $\|f\|_{L^p}$ instead of $\|f\|_{L^p(R^m)}$.

2.

Theorem. (A). *Let F be a compact set in R^m with $C_\alpha(F) = 0$, where $0 \leq \alpha < m$. If $\alpha = 0$ we suppose that F is a subset of the sphere $|x| < R_0$, where R_0 is the constant occurring in the definition of $M_s(F)$. The following conclusions are true:*

If $0 \leqslant \alpha \leqslant m-2$, *then* $M_{m-\alpha}(F)=0$. (2.1)

If $0 \leqslant m-2 < \alpha < m-1$, *then* $M_{m-\alpha-\varepsilon}(F)=0$ *for every* $\varepsilon > 0$ *such that* $m-\alpha-\varepsilon \geqslant 1$. (2.2)

(B). *Let* F *be a compact set in* R^m *with* $M_p(F)=0$, *where* $1 \leqslant p \leqslant m$. *The following conclusions are true:*

If $1 \leqslant p \leqslant 2$, *then* $C_{m-p}(F)=0$. (2.3)

If $2 < p \leqslant m$, *then* $C_{m-p+\varepsilon}(F)=0$ *for every* $\varepsilon > 0$. (2.4)

For the proof we need the following lemma.

Lemma 1. *Let* $0 < \alpha < \beta < m$. *Let* μ *be a positive measure with* $\mu(R^m) < \infty$. *Then*

$$\| u_\beta^\mu \|_{L^p} \leqslant M_1 \cdot \{\mu(R^m)\}^{1/p} \cdot \{\sup_{x \in R^m} u_\alpha^\mu(x)\}^{(p-1)/p}, \quad \textit{provided} \quad 2 \leqslant p = \frac{m-\alpha}{\beta-\alpha}; \tag{2.5}$$

and for every sphere S *with radius* r *we have*

$$\| u_\beta^\mu \|_{L^p(S)} \leqslant M_2 \cdot \{\mu(R^m)\}^{1/p} \cdot \{\sup_{x \in R^m} u_\alpha^\mu(x)\}^{(p-1)/p}, \quad \textit{provided} \quad 1 \leqslant p < \frac{m-\alpha}{\beta-\alpha} < 2. \tag{2.6}$$

M_1 *is a constant depending on* m, p *and* α *and* M_2 *is a constant depending on* m, p, α, β *and* r.

References to papers where this lemma is proved can be found in [7], p. 70.

Proof of (*A*) *of the theorem.* We first treat the case $\alpha > 0$. Let F_n, for $n = 1, 2, 3, \ldots$, be the union of finitely many closed spheres such that $C_\alpha(F_n) < n^{-1}$ and $F_n \supset F$, where F is the given compact set with $C_\alpha(F)=0$. Let μ_n be the capacitary distribution of order α of F_n with total mass

$$\mu_n(F_n) = 2n \cdot C_\alpha(F_n).$$

This means that $0 < \mu_n(F_n) < 2$. According to (1.5) and (1.4) we have

$$u_\alpha^{\mu_n}(x) \geqslant 2n \quad \text{for every} \quad x \in F_n, \tag{2.7}$$

and

$$u_\alpha^{\mu_n}(x) < 2Mn \quad \text{everywhere}. \tag{2.8}$$

(2.7) and (1.3) give that $u_\alpha^{\mu_n}$ is constant on S_{μ_n}, i.e. the restriction of $u_\alpha^{\mu_n}$ to S_{μ_n} is continuous and consequently $u_\alpha^{\mu_n}$ is continuous everywhere according to the continuity principle.

We now form $\psi_n = \varphi_n \ast \mu_n$, i.e. $\psi_n(x) = \int \varphi_n(x-y)\, d\mu_n(y)$, where $\varphi_n \in C_0^\infty$ is a non-negative function with $\int \varphi_n dx = 1$. This means that $\int \psi_n dx < 2$. By choosing S_{φ_n} belonging to a sufficiently small neighborhood of the origin we can make

S_{ψ_n} a subset of a given neighborhood of F_n and accordingly also of F. As $u_\alpha^{\mu_n}$ is continuous we can, in this way, also make the difference $|u_\alpha^{\psi_n}(x)-u_\alpha^{\mu_n}(x)|$ less than any given positive number everywhere (cf. [7], p. 59). Due to (2.7) and (2.8) we can consequently choose φ_n such that

$$u_\alpha^{\psi_n}(x)\geqslant n \quad \text{for every} \quad x\in F, \tag{2.9}$$

and

$$u_\alpha^{\psi_n}(x)< M\cdot n \quad \text{everywhere}, \tag{2.10}$$

where M is a new constant which is independent of n. We also observe that $u_\alpha^{\psi_n}\in C^\infty$ as $\psi_n\in C_0^\infty$.

Now we choose r_0 such that $F\cup S_{\psi_n}\subset S(r_0/2)$ for every n. [According to the above we can make the construction of ψ_n so that this choice of r_0 is possible.] Let $\varphi\in C_0^\infty$ be a function, independent of n, which is identically equal to 1 in $S(r_0)$ and put $f_n(x)=u_\alpha^{\psi_n}(x)$ and

$$g_n(x)=n^{-1}\cdot f_n(x)\cdot\varphi(x).$$

We observe that

$$g_n\in C_0^\infty \quad \text{and} \quad g_n(x)\geqslant 1 \quad \text{for every} \quad x\in F. \tag{2.11}$$

For every $p\geqslant 1$ we obtain, with constants M which are independent of n:

$$\int|\operatorname{grad} g_n|^p\,dx\leqslant M\cdot n^{-p}\int|f_n\operatorname{grad}\varphi|^p\,dx+M\cdot n^{-p}\int|\varphi\operatorname{grad} f_n|^p\,dx=I_n+II_n.$$

As φ is identically equal to 1 in $S(r_0)$ and

$$|f_n(x)|\leqslant 2\cdot\left(\frac{r_0}{2}\right)^{-\alpha} \quad \text{if} \quad |x|\geqslant r_0,$$

we get

$$I_n\leqslant M\cdot n^{-p}\cdot\max_{|x|\geqslant r_0}|f_n(x)|\int|\operatorname{grad}\varphi|^p\,dx<\text{const}\cdot n^{-p}.$$

Let r_1 be independent of n and chosen so that $S(r_1)\supset S_\varphi$. As

$$II_n<\text{const}\cdot n^{-p}\int_{S(r_1)}|\operatorname{grad} f_n|^p\,dx,$$

we want to estimate $|\operatorname{grad} f_n|$. Due to the properties of ψ_n we obtain

$$|\operatorname{grad} f_n(x)|<\text{const}\cdot u_{\alpha+1}^{\psi_n}(x) \quad \text{for every } x. \tag{2.12}$$

We now choose p:

$$p=m-\alpha \quad \text{if} \quad 0<\alpha\leqslant m-2$$

and

$$p=m-\alpha-\varepsilon \quad \text{if} \quad m-2<\alpha<m-1,$$

where $\varepsilon>0$ is chosen satisfying $m-\alpha-\varepsilon\geqslant 1$.

Due to (2.12) and this choice of p we can use (2.5) or (2.6) in Lemma 1, with β equal to $\alpha+1$, to estimate II_n. This gives, as $\int \psi_n dx < 2$,

$$II_n < \text{const} \cdot n^{-p} \{ \sup_{x \in R^m} u_\alpha^{\psi_n}(x) \}^{p-1},$$

and, according to (2.10),

$$II_n < \text{const} \cdot n^{-1},$$

with constants that are independent of n.

The estimates of I_n and II_n show that

$$\lim_{n\to\infty} \int |\operatorname{grad} g_n|^p \, dx = 0$$

with our choice of p. Combined with (2.11) this gives $M_p(F)=0$, which means that (A) of the theorem is proved in the case when $\alpha > 0$.

We now prove (A) when $\alpha = 0$. We can cover F by finitely many closed spheres $\{S_i\}_1^N$ with diameters less than 1 such that $\bigcup_1^N S_i$ is a subset of $|x| < R_0$. It is clearly enough to prove that $M_m(F \cap S_i) = 0$ for $i = 1, 2, \ldots, N$, and it is consequently enough to consider the case when F itself has diameter less than 1. We can then repeat the construction which we used when $\alpha > 0$ but with obvious modifications. For instance, the choices of φ_n and φ are made so that S_{φ_n} and S_φ are subsets of $|x| < R_0$ and we use the following lemma [cf. Fuglede 3, p. 301] instead of Lemma 1:

Lemma 2. *Let $0 < \beta < m$ and $2 \leqslant p = m/\beta$. Let μ be a positive measure with compact support, $S_\mu \subset S(r_1)$. If S is a sphere of radius r_2 and ω_m denotes the surface of the unit sphere in R^m, then*

$$\int_S \{u_\beta^\mu(x)\}^p \, dx \leqslant \{\mu(R^m)\}^{p-2} \cdot \omega_m I_0(\mu) + M \cdot \{\mu(R^m)\}^p,$$

where M is a constant depending on m, r_1 and r_2.

Remark. The following result and its proof has been communicated to me by Professor Lennart Carleson:

If F is a compact set with $L_\alpha(E)=0$, $0 < \alpha \leqslant m-1$, then $M_{m-\alpha}(F)=0$.

As $C_\alpha(F)=0$ implies $L_{\alpha+\varepsilon}(F)=0$ for every $\varepsilon > 0$, this gives a better result than (A) of the theorem when $0 \leqslant m-2 < \alpha < m-1$.

The proof that $L_\alpha(F)=0$ implies $M_{m-\alpha}(F)=0$ proceeds in the following way. Let $\{x_\nu\}_1^n$ be given points and $S_\nu(r)$, $r > 0$, the open sphere $|x - x_\nu| < r$, $\nu = 1, 2, \ldots, n$, and suppose that $\{r_\nu\}_1^n$ are chosen so that $\bigcup_1^n S_\nu(r_\nu) \supset F$.

We define linear functions l_ν by

$$l_\nu(r) = \frac{2r_\nu - r}{r_\nu}, \quad r_\nu \leqslant r \leqslant 2r_\nu$$

and put

$$\varphi_\nu(x) = \begin{cases} 1 \quad \text{when } x \in S_\nu(r_\nu), \\ l_\nu(|x - x_\nu|) \quad \text{when } x \in S_\nu(2r_\nu) - S_\nu(r_\nu), \\ 0 \quad \text{when } x \text{ belongs to the complement of } S_\nu(2r_\nu). \end{cases}$$

Then we have

$$|\operatorname{grad} \varphi_\nu| = r_\nu^{-1} \quad \text{in the interior of} \quad S_\nu(2r_\nu) - S_\nu(r_\nu).$$

If we put

$$\psi(x) = \max_{1 \leqslant i \leqslant n} \varphi_i(x),$$

then $\psi(x) \geqslant 1$ on F and

$$\int |\operatorname{grad} \psi|^{m-\alpha} dx \leqslant \sum_{\nu=1}^{n} \int_{S_\nu(2r_\nu)} r_\nu^{-(m-\alpha)} dx \leqslant \text{const.} \sum_{\nu=1}^{n} r_\nu^\alpha,$$

where the constant only depends on m. If $L_\alpha(F) = 0$ we can make $\sum r_\nu^\alpha$ arbitrarily small, which means that

$$\int |\operatorname{grad} \psi|^{m-\alpha} dx$$

will be arbitrarily small. By using standard methods to approximate ψ it is possible to prove the existence of a continuously differentiable function f with compact support and $f(x) \geqslant 1$ on F so that

$$\int |\operatorname{grad} f|^{m-\alpha} dx$$

is less than any given positive number. This means that $L_\alpha(F) = 0$ implies $M_{m-\alpha}(F) = 0$.

3.

Proof of (*B*) *of the theorem.* Let F be a compact set with $M_p(F) = 0$ where $1 \leqslant p \leqslant m$. We may assume $m > 1$ because if $m = p = 1$, then $M_p(F) > 0$ for every F. We define α by

$$\alpha = m - p \quad \text{if} \quad 1 \leqslant p \leqslant 2,$$

and

$$\alpha = m - p + \varepsilon \quad \text{if} \quad 2 < p \leqslant m, \quad \text{where } \varepsilon > 0, \tag{3.1}$$

and we shall prove that $C_\alpha(F) = 0$.

As $M_p(F) = 0$ there exists a sequence $\{f_n\}_1^\infty$ of continuously differentiable functions with compact supports and

$$f_n(x) > n \quad \text{for every} \quad x \in F,$$

and

$$\int |\operatorname{grad} f_n|^p \, dx < \text{const.}, \quad n = 1, 2, \ldots . \tag{3.2}$$

In the case when $m = p$ we suppose furthermore, as we may, that S_{f_n} is a subset of $|x| < R_0$ for every n, where R_0 is the constant occurring in the definition of $M_p(F)$.

Considered as a distribution, f_n belongs to the class $BL_1(L^p_{\text{loc}})$ of distributions in R^m such that all the partial derivatives (in the distribution sense) of the first order are functions in L^p_{loc}. This fact and the fact that S_{f_n} is compact mean [see for instance 7, p. 71] that there exist constants b_i and d_i, not depending on n, such that

$$f_n(x) = \sum_{i=1}^{m} b_i \int \frac{\partial}{\partial y^i} |x-y|^{2-m} \frac{\partial}{\partial y^i} f_n(y) \, dy \text{ a.e. for } n = 1, 2, \ldots, \text{ if } m > 2, \tag{3.3}$$

and

$$f_n(x) = \sum_{i=1}^{2} d_i \int \frac{\partial}{\partial y^i} \log |x-y| \cdot \frac{\partial}{\partial y^i} f_n(y) \, dy \text{ a.e. for } n = 1, 2, \ldots, \text{ if } m = 2. \tag{3.4}$$

However, since all the partial derivatives of the first order of f_n are continuous, we conclude that also the integrals in (3.3) and (3.4) are continuous and, consequently, that the relations (3.3) and (3.4) are true everywhere in R^m.

To finish the proof of (B) of the theorem we need an estimate of the α-capacity of the set $H_a^{(n)}$ where the right members of (3.3) and (3.4) are larger than a, $a > 0$. By majorizing the integrals in (3.3) and (3.4) we obtain that there exists a sequence $\{g_n\}$ of non-negative continuous functions such that $H_a^{(n)}$ is a subset of the set $G_a^{(n)}$ where

$$u_{m-1}^{g_n}(x) = \int \frac{g_n(y)}{|x-y|^{m-1}} \, dy$$

is larger than a. We may also assume that $S_{g_n} \subset S_{f_n}$ and, due to (3.2), that

$$\int g_n^p \, dx < \text{const}, \quad n = 1, 2, \ldots, \tag{3.5}$$

where the constant is independent of n.

$C_\alpha(G_a^{(n)})$ is estimated by standard methods. The two cases $p \leqslant 2$ and $p > 2$ give different calculations. For the sake of completeness we treat one of them, the case $p > 2$, in detail.

The estimate of $C_\alpha(G_a^{(n)})$ is somewhat facilitated for certain values of m and p if $\bigcup_n S_{f_n}$ is a bounded set. $\bigcup_n S_{f_n}$ is bounded if $m = p$ as S_{f_n} is a subset of $|x| < R_0$, $n = 1, 2, 3, \ldots$, in this case. But even when $m > p \geqslant 1$ we may choose $\{f_n\}$ so that $\bigcup_n S_{f_n}$ is a bounded set. Because if $\bigcup_n S_{f_n}$ is not bounded for the sequence $\{f_n\}$ which was chosen originally, we replace $\{f_n\}$ by a sequence $\{f_n^*\}$ defined by $f_n^*(x) = f_n(x) \cdot \psi(x)$, where $\psi(x)$ is a function in C_0^∞ which is identically equal to 1 in a neighborhood of F. Then f_n^* is continuously differentiable, $f_n^*(x) > n$ on F and the set $\bigcup_n S_{f_n^*}$ is bounded. From the estimate

$$\int |\operatorname{grad} f_n^*|^p \, dx \leqslant \text{const} \int |f_n \operatorname{grad} \psi|^p \, dx + \text{const} \int |\psi \operatorname{grad} f_n|^p \, dx, \tag{3.6}$$

we may, finally, prove that (3.2) is true with f_n replaced by f_n^*: according to (3.2) the second term of the right member of (3.6) is less than a constant which is independent on n. In order to realize that the whole expression (3.6) is less than a constant it is hence enough to prove that

$$\sup_n \|f_n\|_{L^p(E)} < \infty, \quad m > p \geqslant 1,$$

for every bounded set E. But this is an immediate consequence of (3.2), (3.3) and (3.4) if $p = 1$ and of (3.2) and the following inequality by Sobolev if $m > p > 1$,

$$\|f_n\|_{L^r} \leqslant \text{const} \, \|\operatorname{grad} f_n\|_{L^p}, \quad r = \frac{mp}{m-p},$$

where the constant is independent of n.

In the calculations below, we assume, as we accordingly may, that $\bigcup_n S_{f_n}$ is bounded. This means that also $\bigcup_n S_{g_n}$ is bounded.

Now let $m \geqslant p > 2$. Since $\bigcup_n S_{g_n}$ is bounded, we can choose a finite number r_0 such that $\bigcup_n S_{g_n} \subset S(r_0) = S_0$. Let μ be a positive measure with $S_\mu \subset G_a^{(n)}$ and $\mu(R^m) = 1$. We obtain by means of Hölder's inequality, if $p' = p/(p-1)$,

$$a < \int u_{m-1}^{g_n}(x) \, d\mu(x) = \int_{S_0} u_{m-1}^{\mu}(x) \, g_n(x) \, dx \leqslant \|g_n\|_{L^p} \cdot \|u_{m-1}^{\mu}\|_{L^{p'}(S_0)}. \tag{3.7}$$

To estimate the last norm we use formula (2.6) of Lemma 1 with $\beta = m - 1$. An easy calculation shows that the conditions of the lemma are satisfied if we choose ε small enough in (3.1), a choice which we may obviously make without limitation. (2.6) gives then

$$\|u_{m-1}^{\mu}\|_{L^{p'}(S_0)} \leqslant \text{const} \, \{\sup_{x \in R^m} u_\alpha^\mu(x)\}^{(p'-1)/p'}.$$

Remembering (3.5), we obtain, after simplification, from (3.7) and the estimate above,

$$a^p < \text{const} \, \{\sup_x u_\alpha^\mu(x)\}.$$

Now let μ be the capacitary distribution with total mass 1 of order α of an arbitrarily chosen closed subset $F_a^{(n)}$ of $G_a^{(n)}$. This and (1.4) give

$$a^p < \text{const} \, \{C_\alpha(F_a^{(n)})\}^{-1}.$$

Hence the same inequality is true with $F_a^{(n)}$ replaced by $G_a^{(n)}$ and we have proved the following inequality when $m \geqslant p > 2$:

If $m > p \geqslant 1$ or $m = p > 2$, then there exists a positive constant M, not depending on a and n, such that

$$C_\alpha(G_a^{(n)}) < Ma^{-p}, \quad a > 0, \quad n = 1, 2, \ldots . \tag{3.8}$$

The proof of (3.8) when $1 \leqslant p \leqslant 2$, $m > p$—which may be completed even without the assumption that $\bigcup_n S_{f_n}$ is bounded—is first carried through when $p = 2$ or 1, after which the case $1 < p < 2$ is reduced to the case $p = 2$ by an application of Hölder's inequality. Compare for instance [7] formulas (8.11) and (8.15) where, however, the presence of a function q complicates the proof.

When $m = p = 2$ we have the following inequality instead of (3.8): If S is an arbitrary sphere with diameter less than 1, then there exist constants M and a_0, such that

$$C_0(G_a^{(n)} \cap S) < M \cdot a^{-2} \quad \text{if} \quad a > a_0, \quad m = p = 2, \quad n = 1, 2, \ldots . \tag{3.9}$$

Remembering that $f_n(x) > n$ on F, that $H_a^{(n)} \subset G_a^{(n)}$, and that (3.3) and (3.4) are true everywhere, we obtain

$$C_\alpha(F) \leqslant C_\alpha(G_n^{(n)}), \quad n = 1, 2, \ldots . \tag{3.10}$$

(When $\alpha = 0$, i.e. when $m = p = 2$, F is to be replaced by $F \cap S$ and $G_n^{(n)}$ by $G_n^{(n)} \cap S$ where S is a sphere having diameter less than 1.) (3.10) combined with (3.8) or (3.9) give that $C_\alpha(F) = 0$, and (B) of the theorem is proved.

Remark 1. The same methods of proofs also give an analogous theorem if we introduce derivatives of higher orders in (1.1).

Remark 2. Restricting ourselves to the case $m > p$ we observe that the result (2.4) of (B) of the theorem is best possible in the following sense:

If $m > p > 2$ there exists a compact set F satisfying

$$M_p(F) = 0, \; C_{m-p}(F) > 0. \tag{3.11}$$

To prove this we shall use the following result by du Plessis [5, Theorem 4 and p. 131 ff.]:

Let α and q be given numbers, $0 < \alpha < m$, $2 < q < \infty$. There exists a compact set E with $C_{m-\alpha}(\mathrm{E}) > 0$ and a function $f \in L^q$ with compact support such that, if $\gamma = m - \alpha/q$, then $u_\gamma^f(x) = \infty$ everywhere on E.

It should be noted that the proof of this fact which is illustrated for the case $m = 2$ in [5] is incomplete. The set E which is constructed in [5], p. 132, (where it is denoted by M) can not be used if $1 < \alpha < 2 = m$. However, for E it is possible to use the m-dimensional Cantor set which is the Cartesian product of m equal 1-dimensional Cantor sets, G, where G is the usual Cantor set which is obtained starting from an interval of length 1 and a sequence $\{\xi_n\}$ such that $0 < \xi_n < 1/2$; i.e. $G = \bigcap G_n$, where G_n consists of 2^n closed intervals each of length $\xi_1 \xi_2 \cdots \xi_n$. It is well known that if $0 < \beta < m$, then $C_\beta(E) > 0$ if and only if

$$\sum_{n=1}^{\infty} 2^{-nm} (\xi_1 \xi_2 \dots \xi_n)^{-\beta} < \infty.$$

By using this it is possible to construct the function f and to carry through the proof by obvious modifications of the proof given by du Plessis for the case $m=1$ [4, p. 896 ff.].

We now turn to the proof of the existence of a compact set F satisfying (3.11), where p is given, $m>p>2$. According to the above there exists a compact set F with $C_{m-p}(F)>0$ and a non-negative function $g \in L^p$ with compact support such that $u^g_{m-1}(x)=\infty$ on F. We shall prove that $M_p(F)=0$. Let, for $n=1, 2, \dots, \varphi_n \in C_0^\infty$ be a non-negative function with $\int \varphi_n \, dx = 1$ such that $\bigcup_n S_{\varphi_n}$ is a bounded set. As $u^g_{m-1}(x)>n$ on an open set containing F we have $u^g_{m-1} * \varphi_n(x) > n$ on F if we choose S_{φ_n} in a sufficiently small neighborhood of the origin. Putting $g_n = g * \varphi_n$ this means that $u^{g_n}_{m-1}(x) > n$ on F since

$$u^{g_n}_{m-1} = \frac{1}{r^{m-1}} * g_n = \frac{1}{r^{m-1}} * g * \varphi_n = u^g_{m-1} * \varphi_n.$$

Furthermore, we have $u^{g_n}_{m-1} \in C^\infty$. We now choose a function $\varphi \in C_0^\infty$ which is identically equal to 1 on a set, the interior of which contains F and $\bigcup_n S_{g_n}$, and put

$$f_n(x) = n^{-1} u^{g_n}_{m-1}(x) \cdot \varphi(x).$$

Hence $$f_n \in C_0^\infty, \; f_n(x) \geq 1 \text{ on } F. \tag{3.12}$$

There exists a constant M such that

$$\int |\operatorname{grad} f_n|^p \, dx \leq M n^{-p} \int |u^{g_n}_{m-1} \operatorname{grad} \varphi|^p \, dx + M n^{-p} \int |\varphi \operatorname{grad} u^{g_n}_{m-1}|^p \, dx. \tag{3.13}$$

In the same way as in the proof of (A) of the theorem we realize that the first term of the right member tends to zero when $n \to \infty$. The second term of the right member may, for instance, be estimated by means of the theory of singular integrals. We have [1, p. 129]

$$\frac{\partial u^{g_n}_{m-1}(x)}{\partial x^i} = \lim_{\varepsilon \to 0} (1-m) \int_{|x-y| \geq \varepsilon} \frac{x^i - y^i}{|x-y|^{m+1}} g_n(y) \, dy \text{ a.e.,}$$

and from this we infer [1, p. 116],

$$\| \operatorname{grad} u^{g_n}_{m-1} \|_{L^p} \leq \text{const} \, \| g_n \|_{L^p},$$

where the constant is independent of n. But an application of Hölder's inequality shows that (see for instance [2, p. 192]),

$$\| g_n \|_{L^p} = \| g * \varphi_n \|_{L^p} \leq \| g \|_{L^p} \cdot \| \varphi_n \|_{L^1} = \| g \|_{L^p},$$

and consequently we have proved that also the second term of the right member of (3.13) tends to zero when $n \to \infty$. Hence

$$\lim_{n\to\infty} \int |\operatorname{grad} f_n|^p \, dx = 0.$$

This combined with (3.12) finally gives that $M_p(F) = 0$ and so we have proved the existence of a compact set F satisfying (3.11) if $m > p > 2$.

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