ARKIV FÖR MATEMATIK Band 5 nr 16

Communicated 12 February 1964 by L. GÅRDING and L. CARLESON

Real Banach algebras

By LARS INGELSTAM

CONTENTS

Introduction

Most of the general theory of Banach algebras has been concerned with algebras over the complex field. The reason for this is clear: the power of function theoretic methods and the Gelfand representation [11]. But the complex algebras can be regarded as a subclass of the real algebras and it is natural to ask what can be said about this larger class. In several respects the extension of results that are known for complex algebras is easy; many techniques valid for complex algebras will work in the real case as well (cf. sec. 2). Also since any real algebra can be embedded in a complex (sec. 3), some general results can be obtained in this way from the complex case. In quite a few areas, however, new approaches are needed or different ("typically real") phenomena occur.

In much of the literature on Banach algebras incidental remarks on the real scalar case can be found. Among more systematic contributions we can mention [1], [19], [20] and in particular the monograph [24], in large portions of which the theory is presented simultaneously for real and complex scalars. The author's papers [15], [16], [17] deal with special problems for real normed algebras.

This article intends to contribute to the theory of real Banach algebras in three central areas: (1) the structure of the (quasi-) regular group (Ch. III), (2) abstract characterization of real function algebras (Ch. IV), (3) the relation between real B^*- and C^*-algebras (Ch. V). The questions studied here and in [15] lead us to introduce and investigate a certain classification of real Banach algebras (Ch. II). An introductory chapter (Ch. I) gives some of the standard material from the general theory, modified for the real scalar case.

For a more detailed survey of the contents the reader is referred to the short summaries which are found at the beginning of each chapter.

Chapter I. General theory of real normed algebras

This chapter intends to give a brief survey of those parts of the general theory of real normed algebras that will be used in later chapters. Following the definitions (sec. 1), some standard notions and techniques (regular and quasi-regular group, adjunction of identity, natural norms, the function $\nu(x)$, quotient algebras, completion) are described (sec. 2). The spectrum of an element in a real algebra is defined and its relation to the complexification is discussed in section 3. For the Frobenius-Mazur Theorem 3.6 on real normed division algebras we give a complete proof whose algebraic part is self-contained and elementary. In the last part (sec. 4) of the chapter the real counterpart of the Gelfand representation theory [11] for commutative complex Banach algebras is presented.

All the material in the chapter is known in principle, although many things, most notably those of sec. 4, are rarely given explicitly for the real case.

1. *Definitions*

We let A be an associative algebra over the real numbers (R) or the complex numbers (C). A is called a *topological algebra* if it is also a Hausdorff topological space such that addition, multiplication and multiplication by scalars (from $A \times A$, $A \times A$ and $R \times A$ or $C \times A$ into A, respectively) are continuous functions.

A *norm* on A is a real-valued function $x \to \|x\|$ on A with the properties

$$\|x+y\| \leqslant \|x\| + \|y\|,$$

$$\|\alpha x\| = |\alpha| \|x\|,$$

$$\|x\| > 0 \text{ if } x \neq 0$$

for all $x, y \in A$ and scalars α. A norm defines a topology on A, making A a topological vector space. Two norms, $\|\cdot\|$ and $|||\cdot|||$, define the same topology if and only if there are two numbers c and C such that

$$0 < c \leqslant \frac{\|x\|}{|||x|||} \leqslant C$$

for all x. Two norms satisfying such a relation are called equivalent.

A topological algebra, the topology of which can be defined by a norm will be called a *normed algebra*. (This terminology, though slightly unusual, is quite practical; when a certain norm is replaced by an equivalent norm we can still speak of the same normed algebra.) A norm that defines the topology for A is called *admissible*. A *Banach algebra* is a complete normed algebra.

For a given norm $\|\cdot\|$ on A multiplication is continuous, i.e. A is a normed algebra under $\|\cdot\|$, if and only if there exists a number K such that

$$\|xy\| \leqslant K\|x\| \cdot \|y\|.$$

We will see that there are always admissible norms for which we can take $K=1$ (Proposition 2.3).

If A is an algebra over C we can make A an algebra over R simply by restricting scalar multiplication to R. Hence the class of complex algebras can be regarded as a subclass of the real algebras; later on we will devote some effort to the characterization of this subclass, the real algebras of complex type (Ch. II).

2. *Basic techniques*

Throughout this section A is a normed algebra. If A has an identity element e ($ex = xe = x$ for all x) the set A together with algebra multiplication forms a semigroup $(A, \cdot)$ with neutral element e. The elements with two-sided inverse in $(A, \cdot)$ are called *regular* and form a group G, *the regular group*.

If there is no identity we can still do something equivalent. Let $x \circ y = x + y - xy$; then $(A, \circ)$ is a semigroup with neutral element 0. The elements with two-sided inverse in $(A, \circ)$ are called *quasi-regular* and form a group G^q, *the quasi-regular group*.

G and G^q are topological groups with the (metric) topology of A [24, p. 19].

Proposition 2.1. *If A has identity e, G^q is homeomorphically isomorphic to G, under the map $x \to e - x$.*

We can also "adjoin an identity". Let A be real and $A_1 = R \oplus A$, direct sum as real normed vector spaces. Multiplication is defined by

$$(\alpha, x)\,(\beta, y) = (\alpha\beta, \alpha y + \beta x + xy)$$

and the topology for instance given through a norm

$$\|(\alpha, x)\| = |\alpha| + \|x\|.$$

Thus A_1 is a normed algebra with identity $(1, 0)$ and we have

Proposition 2.2. *Through $x \to (0,x)$ A is embedded (homeomorphically and isomorphically) as an ideal of codimension 1 in A_1. An element $x \in A$ is quasi-regular in A if and only if $(1, -x)$ is regular in A_1.*

A norm is called *natural* if

$$\|xy\| \leqslant \|x\| \; \|y\| \quad \text{for all} \quad x,y \text{ (submultiplicative)},$$

$$\|e\| = 1 \quad \text{if } A \text{ has identity.}$$

Proposition 2.3. *Every normed algebra has an admissible natural norm.*

Proof. If A has identity, let

$$|||x||| = \sup_{y \neq 0} \frac{\|xy\|}{\|y\|}$$

for some admissible norm $\|\cdot\|$. Then $|||\cdot|||$ is natural and equivalent to $\|\cdot\|$. If there is no identity put $|||x||| = |||(0,x)|||$, the latter taken in A_1.

Remark. Using this device, the left regular representation, Gelfand [11] proved that a Banach space that is also an algebra such that multiplication with any fixed element is a continuous operation, is actually a topological (Banach) algebra.

Definition 2.4.

$$\nu(x) = \inf_n \|x^n\|^{\frac{1}{n}}$$

for some natural norm $\|\cdot\|$.

Proposition 2.5.

$$\nu(x) = \lim_{n \to \infty} \|x^n\|^{\frac{1}{n}}$$

for any admissible norm $\|\cdot\|$.

For proof and further properties of ν, see [24, p. 10].

We notice that $\nu(x)$ is a topological invariant, independent of which admissible norm was chosen to define it.

Proposition 2.6. *If A is Banach and $\nu(x) < 1$ then x is quasi-regular. If A has identity and $\nu(e-x) < 1$ then x is regular.*

Proof. The series $-\sum_{n=1}^{\infty} x^n$ and $e + \sum_{n=1}^{\infty} (e-x)^n$ converge and are the desired inverses.

Consequences of this are, among others, that 0 has a whole neighbourhood consisting of quasi-regular elements and that G^q (or G) is an open subset of A.

Proposition 2.7. *If I is a closed two-sided ideal in a normed algebra A, A/I is also a normed algebra. A/I is Banach if A is Banach.*

Proof. For $[x]\in A/I$ and $\|\cdot\|$ a norm for A, take $\|[x]\|=\inf_{y\in I}\|x+y\|$. Verification that this is a norm, that multiplication is continuous and that completeness is preserved, is routine.

For normed algebras we also have a standard procedure of completion.

Proposition 2.8. *Any normed algebra A can be embedded (isomorphically and topologically) as a dense subalgebra of a Banach algebra A', called the completion of A.*

Proof. Let A_∞^c be the space of all Cauchy sequences $(x_n)_{n=1}^\infty$, $x_n\in A$. With a norm $\|(x_n)\|=\sup\|x_n\|$ A_∞^c becomes a normed algebra. $N=\{(x_n);\, x_n\to 0\}$ is a closed two-sided ideal. Then $A'=A_\infty^c/N$ is the desired Banach algebra and the embedding $x\to(x_n=x)_{n=1}^\infty+N$. The conclusion follows from Proposition 2.7 and routine calculations.

3. *Complexification and spectrum*

As was indicated in sec. 1 the notion of a real normed algebra is more general than that of a complex normed algebra. The complex theory is better known and in some respects more satisfactory and we describe a procedure by which we can draw some results (but far from all) for real algebras in general from the complex theory. The real algebra A can be embedded in a larger ("twice as big") complex algebra.

Let $A_C=A\oplus A$, direct sum as real normed vector spaces, and define multiplication

$$(a,b)\,(c,d)=(ac-bd,\, ad+bc)$$

and multiplication by complex scalars

$$(\alpha+i\beta)\,(a,b)=(\alpha a-\beta b,\alpha b+\beta a).$$

Then A_C is a complex algebra and a real normed algebra, an admissible norm is for instance

$$\|(a,b)\|=\|a\|+\|b\|.$$

To show that A_C is a complex normed algebra we construct an admissible complex-homogeneous norm, following Kaplansky [20, p. 400],

$$|||x|||=\max_\varphi\|(\cos\varphi+i\sin\varphi)x\|.$$

A is embedded in A_C by the real-algebra monomorphism $x\to(x,0)$. (A more detailed discussion [24, p. 8] shows that, given a natural norm on A, there exists an admissible natural norm on A_C such that the embedding is an isometry with respect to these norms.)

The spectrum $\sigma_A^C(x)$ for an element x of a complex algebra A is defined as the set of complex numbers ζ such that $\zeta^{-1}x$ is not quasi-regular, together with 0 if x^{-1} does not exist or A lacks identity. If the corresponding definition, with real numbers, is used for real algebras the spectrum would frequently be empty and give no information at all about the element. We therefore adopt the following definition, due to Kaplansky [20].

Definition 3.1. *The spectrum of an element in a real algebra A is the set of complex numbers*

$$\sigma_A(x) = \{\alpha + i\beta \neq 0;\ (\alpha^2+\beta^2)^{-1}\,(x^2 - 2\alpha x) \text{ not quasi-regular}\}$$

plus 0 if x^{-1} does not exist or A does not have identity.

We notice in particular that for a real $\alpha \neq 0, \alpha \in \sigma_A(x)$ if and only if $\alpha^{-1}x$ is not quasi-regular.

Proposition 3.2. *The spectrum of an element of a real algebra is equal to the spectrum of the corresponding element in the complexification*

$$\sigma_A(x) = \sigma^C_{A_C}((x,0)).$$

Proof. 0 is easily checked out to belong to neither or both of the sets. For $\alpha + i\beta \neq 0$ the element $x' = (\alpha+i\beta)^{-1}(x,0) = (\alpha^2+\beta^2)^{-1}(\alpha x, -\beta x) \in G^q$ if and only if $x'' = (\alpha^2+\beta^2)^{-1}(\alpha x, \beta x) \in G^q$, hence if and only if $x' \circ x'' = (\alpha^2+\beta^2)^{-1}(x^2 - 2\alpha x, 0) \in G^q$.

Given a complex algebra A, "regarding" it as a real algebra may distort the notion of spectrum but in a non-essential way:

Proposition 3.3. *If A is a complex algebra*

$$\sigma_A(x) = \sigma^C_A(x) \cup \overline{\sigma^C_A(x)}.$$

Proposition 3.2 enables us to quote, directly from the well-known theory of complex algebras, the following two important results:

Proposition 3.4. *For any element x in a normed algebra A, $\sigma_A(x)$ contains at least one number ζ with $|\zeta| \geqslant \nu(x)$. In particular $\sigma_A(x)$ is never empty.*

Proposition 3.5. *In a Banach algebra A, $\sigma_A(x)$ is a compact set and*

$$\max_{\zeta \in \sigma_A(x)} |\zeta| = \nu(x)$$

for all $x \in A$.

A *division algebra* is an algebra with identity in which every non-zero element has a two-sided inverse.

Theorem 3.6. *A normed real division algebra A is isomorphic to the real numbers (R), the complex numbers (C) or the quaternions (Q).*

Proof. From Definition 3.1 and Proposition 3.4 follows that every element that is not a scalar multiple of e satisfies an irreducible quadratic equation. Such an algebra is called *quadratic* and we prove that every quadratic real (associative) algebra is isomorphic to R, C or Q.

If A has dimension 1 or 2 it is clearly isomorphic to R or C. Assume therefore that there are $x, y \in A$ so that $\{e, x, y\}$ is linearly independent. Since any x_0 which is not a multiple of e satisfies an equation $(x_0 - \xi e)^2 = -\eta^2 e, \eta \neq 0$, we can assume $x^2 = y^2 = -e$. With $a = x+y$, $b = x-y$ we have $ab + ba = 0$ and

$$xy + yz = a^2 - x^2 - y^2 = \alpha a + \gamma e,$$

$$xy + yx = x^2 + y^2 - b^2 = \beta b + \delta e$$

for scalars $\alpha,\beta,\gamma,\delta$. Since $\{e, a, b\}$ is also linearly independent $\alpha=\beta=0$ and $\gamma=\delta$. We now put $e_1=(2-\gamma)^{-\frac{1}{2}}a$, $e_2=(2+\gamma)^{-\frac{1}{2}}b$, $e_3=e_1e_2$ and notice that e, e_1, e_2, e_3 satisfy exactly the multiplication rules of the four basis elements of Q. Let their linear span in A be Q_0. We prove that $Q_0=A$.

Assume that there exists a $z\notin Q_0$. As before we can take $z^2=-e$. From the quadratic law follows, for $i=1, 2, 3$:

$$e_iz+ze_i=(e_i+z)^2-e_i^2-z^2=\varkappa_i(e_i+z)+\lambda_ie,$$

$$e_iz+ze_i=-(e_i-z)^2+e_i^2+z^2=\mu_i(e_i+z)+\nu_ie.$$

The linear independence gives $\varkappa_i=\mu_i=0$ and $\nu_i=\lambda_i$. Take $u=\varepsilon(2z+\sum_1^3\lambda_i e_i)$. Then $e_i u+ue_i=0$ for $i=1, 2, 3$ and ε can be chosen so that $u^2=-e$. But the associative law is not satisfied; we compute $ue_1 e_2u$ in two ways:

$$u((e_1e_2)u)=u(e_3u)=u(-ue_3)=-u^2e_3=e_3,$$

$$(ue_1)\ (e_2u)=(-e_1u)\ (-ue_2)=e_1\, u^2e_2=-e_1e_2=-e_3.$$

Hence $Q_0=A$ and the theorem is proved.

Related results are given in [16] and [17].

4. *Gelfand representation*

The remarkable result of Gelfand that a complex commutative semi-simple Banach algebra is isomorphic to an algebra of continuous functions generalizes to real algebras in general. Some modifications of a technical nature are needed.

Throughout this section we let A be a real normed algebra. A left ideal I is called *modular* if there exists an element e_I such that $x-xe_I\in I$ for all $x\in A$. A direct consequence of Theorem 3.6 is that if A is commutative and M is a closed modular maximal ideal A/M is isomorphic to R or C. The (Jacobson) *radical* R_A is the intersection of all modular maximal left ideals. A is called *semi-simple* if $R_A=\{0\}$ and *radical* if $R_A=A$.

Given a real normed algebra A, let ϕ_A be the set of non-zero continuous real algebra homomorphisms of A into C. We will call ϕ_A the *Gelfand space* of A. The connection between ϕ_A and the set of closed modular maximal ideals, $\mathfrak{M}$, is slightly more complicated than in the complex case and will now be described.

Given $\varphi\in\phi_A$ we define $\tau\varphi$ by $\tau\varphi(x)=\overline{\varphi(x)}$ (complex conjugate). This τ, as a function from ϕ_A into ϕ_A, is called the conjugate mapping. Since any $\varphi\in\phi_A$ maps A onto either R or C we can distinguish a "real" and a "complex" part of ϕ_A

$$\phi_A^R=\{\varphi;\ \varphi(A)=R\}=\{\varphi;\ \tau\varphi=\varphi\},$$

$$\phi_A^C=\{\varphi;\ \varphi(A)=C\}=\{\varphi;\ \tau\varphi\neq\varphi\}.$$

The extreme cases when either ϕ_A^R or ϕ_A^C is empty will be discussed somewhat in Chapter II.

Two homomorphisms $\varphi,\psi\in\phi_A$ are called equivalent if $\varphi=\psi$ or $\varphi=\tau\psi$. If ϕ_A^τ is the set of equivalence classes we have

Proposition 4.1. *For a commutative real normed algebra A with Gelfand space ϕ_A and conjugate mapping τ, the set ϕ_A^τ is in one-to-one correspondence with $\mathfrak{M}$, the set of closed modular maximal ideals.*

Proof. We define the function $\varphi \to \operatorname{Ker}\varphi, \varphi \in \phi_A$. Since $\operatorname{Ker}\varphi = \operatorname{Ker}\tau\varphi$ and $\operatorname{Ker}\varphi$ is a closed modular maximal ideal this defines a function g from ϕ_A into $\mathfrak{M}$. From Theorem 3.6 follows that g is surjective. To prove that it is also injective assume $\operatorname{Ker}\varphi = \operatorname{Ker}\psi$. Then φ and ψ induce an automorphism α on R or C so that $\varphi = \alpha \circ \psi$. But for $R, \alpha =$ identity and for $C, \alpha =$ identity or complex conjugation. Hence φ and ψ are equivalent and g injective.

We now proceed to define the Gelfand representation and state the results in two theorems.

Definition 4.2. *For an element x of a normed algebra A, the Gelfand function $\hat{x}$ is a complex-valued function on ϕ_A defined by*

$$\hat{x}(\varphi) = \varphi(x), \ \varphi \in \phi_A.$$

Theorem 4.3. *If A is a real normed algebra and the Gelfand space ϕ_A is given the weakest topology in which all Gelfand functions $\hat{x}$, $x \in A$, are continuous, then*

(i) *ϕ_A is locally compact,*
(ii) *ϕ_A is compact if A has identity,*
(iii) *τ is a homeomorphism,*
(iv) *ϕ_A^R is a closed subset of ϕ_A.*

Proof. A^* denotes the real normed dual of A. We embed ϕ_A in $A^* \times A^*$ in a one-to-one manner $\varphi \to \varphi' = (\frac{1}{2}(\tau\varphi + \varphi), \frac{1}{2i}(\varphi - \tau\varphi))$. An element $(f,g) \in A^* \times A^*$ belongs to the image ϕ'_A if and only if $(f,g) \neq 0$ and

$$f(xy) = f(x)f(y) - g(x)g(y),$$

$$g(xy) = f(x)g(y) + f(y)g(x)$$

for all $x,y \in A$. We topologize A^* with the weak* topology and $A^* \times A^*$, ϕ'_A and ϕ_A accordingly. This topology on ϕ_A is the weakest in which the Gelfand functions are continuous. A natural norm on A induces a norm on A^* and it is easy to see that $\phi'_A \subset S_1 \times S_1$, where S_1 is the unit sphere in A. Since S_1 is weak* compact, the closure $\bar{\phi}'_A$ is compact. But a $(f,g) \in \bar{\phi}'_A$ must satisfy the multiplicative relation above and consequently either belong to ϕ'_A or be $=0$. Thus we have two cases:

1. $0 \notin \bar{\phi}'_A$; ϕ_A is compact. In particular if A has identity 0 cannot be an accumulation point.
2. $\bar{\phi}'_A = \phi'_A \cup \{0\}$; ϕ_A is equal to a compact Hausdorff space minus one point, hence locally compact.

Since τ corresponds to changing the sign of the second component in $A^* \times A^*$ (iii) is clear. Any $\varphi \in \bar{\phi}_A^R$ must satisfy $\tau\varphi = \varphi$, hence $\varphi \in \phi_A^R$ and ϕ_A^R is closed.

Remark. Since τ is a homeomorphism it is easy to verify that $\mathfrak{M}$ in a commutative A, given the topology of ϕ_A^τ, is compact or locally compact as ϕ_A [7, § 10, No. 6, 10].

Now we can give the Gelfand representation theorem for real commutative Banach algebras. We recall that in a Banach algebra all the modular maximal ideals are closed [24, p. 43], in particular $R_A = \bigcap_{M \in \mathfrak{M}} M$. If Ω is a locally compact Hausdorff space $C_0(\Omega)$ denotes the Banach algebra of all complex continuous functions that "tend to 0 at infinity" (i.e. tend to 0 after the filter generated by the complements of compact sets); the topology being defined by the maximum norm. If Ω is compact $C_0(\Omega) = C(\Omega)$, the algebra of *all* continuous functions on Ω.

Theorem 4.4. *Let A be a real commutative Banach algebra and ϕ_A its Gelfand space. The algebra homomorphism $h: x \to \hat{x}$ of A into $C_0(\phi_A)$ has the properties*

(i) *Ker $h = R_A$,*
(ii) $\sigma_A(x) = \hat{x}(\phi_A)$ *(the range of $\hat{x}$) for all x, with the possible exception of* 0 *if A does not have identity,*
(iii) $\max_{\varphi \in \phi_A} |\hat{x}(\varphi)| = \nu(x)$,
(iv) *h is continuous.*

Proof. From the way the topology of ϕ_A is defined follows that $\hat{x} \in C_0(\phi_A)$. From Proposition 4.1 follows that $\operatorname{Ker} h = \bigcap_{\varphi \in \phi_A} \operatorname{Ker} \varphi = \bigcap_{M \in \mathfrak{M}} M = R_A$ and (i) is proved. For given x and $\alpha + i\beta \neq 0$ put $x_0 = (\alpha^2 + \beta^2)^{-1} (x^2 - 2\alpha x)$. If $\alpha + i\beta \notin \sigma_A(x)$ then there exists a y so that $x_0 + y = x_0 y$, $\hat{x}_0 + \hat{y} = \hat{x}_0 \hat{y}$ and $\hat{x}_0(\varphi) \neq 1$, $\hat{x}(\varphi) \neq \alpha + i\beta$ for all φ. If, on the other hand, $\alpha + i\beta \in \sigma_A(x)$ the set $I = \{y - yx_0;\, y \in A\}$ is a proper modular ideal with $x_0 = e_I$. But I is contained in some maximal modular ideal M with $e_M = e_I = x_0$. According to Proposition 4.1 there exists $\varphi \in \phi_A$ so that $\hat{x}_0(\varphi) = \hat{x}_0(\tau\varphi) = 1$; hence $\hat{x}$ takes the value $\alpha + i\beta$ at φ or $\tau\varphi$. If A has identity every ideal is modular, and x is regular if and only if it does not belong to any maximal ideal; in other words $0 \notin \sigma_A(x)$ if and only if $\hat{x}(\varphi) \neq 0$ for all $\varphi \in \phi_A$. Now (ii) is proved. From (ii) and Proposition 3.5 follows (iii). Since $\nu(x) \leqslant \|x\|$ for any natural norm $\|\cdot\|$, h is continuous and the proof is finished.

Chapter II. Reality conditions

The main object of this chapter is to present a certain classification of real Banach algebras. In sec. 6 the class of real algebras that can also be complex is characterized and four "reality conditions", R_1–R_4, are introduced. It is shown (Theorems 6.5 and 6.8) that these conditions stand for increasing degrees of "reality" and also that, when an identity is adjoined to an algebra, the reality properties are preserved (Lemma 6.9). The significance of the conditions in various parts of the theory is discussed in sec. 7. It is finally shown that, under suitable finiteness assumptions, an algebra can be decomposed in a "complex" and a "real" part (Theorem 8.1). This result, like several notions and results of the chapter, is purely algebraic.

5. *The modified exponential function*

In a Banach algebra without identity we cannot define the usual exponential function. It is, however, always possible to define the function

$$x \to \operatorname{ixp} x = -\sum_{n=1}^{\infty} \frac{x^n}{n!}$$

which is defined and continuous for all x and moreover satisfies $\operatorname{ixp}(x+y)=\operatorname{ixp}x \circ \operatorname{ixp}y$ for commuting x and y. If the algebra A is commutative ixp is a semi-group homomorphism of $(A,+)$ into $(A,\circ)$. The function

$$x \to l(x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$$

is defined and continuous for all x with $\nu(x)<1$ and satisfies $\operatorname{ixp} l(x)=x$. If there is an identity e we clearly have

$$\exp x = e - \operatorname{ixp} x,$$

$$\log(e-x) = l(x).$$

For future reference we make two technical remarks concerning the ixp function.

Lemma 5.1. *If, for a topologically nilpotent element x in a Banach algebra A,* ixp αx *is a bounded function of the real variable $\alpha(-\infty<\alpha<\infty)$ then $x=0$.*

Proof. For a continuous linear functional f, the function

$$\varphi : \varphi(\alpha) = f(\operatorname{ixp}\alpha x) = -\sum_{n=1}^{\infty} \frac{\alpha^n}{n!} f(x^n)$$

can be extended to an entire function ψ on the whole complex plane. Since $f(x^n)^{1/n} \leqslant K^{1/n} \|x^n\|^{1/n}$ we have that

$$|\psi(z)| = O(\exp\delta|z|),\ |z|\to\infty,$$

for every $\delta>0$. Hence ψ is at most of order one, minimum type, and since it is bounded on the real axis a Phragmén-Lindelöf theorem [5, p. 84] tells that it must be bounded. Hence $f(x)=0$ for arbitrary f and $x=0$.

Lemma 5.2. *If for some x there exists a convergent sequence $\alpha_n, n=1,2,\ldots,$ of real numbers such that* ixp $\alpha_n x=0$ *for all n then $x=0$.*

Proof. Let f be a linear functional and φ, ψ as in the proof of Lemma 5.1. Now ψ is an analytic function with a non-isolated zero, hence $\psi=0, f(x)=0$ for all f and $x=0$.

6. *Reality conditions*

As we have already pointed out (sec. 1) the complex (normed) algebras can be regarded as a subclass of the real (normed) algebras. Next we look into this situation in some detail.

Definition 6.1. *A real (normed) algebra is said to be of complex type if it is possible to extend the scalar multiplication to complex scalars so that the algebra becomes a complex (normed) algebra.*

A somewhat more technical description can be given:

Proposition 6.2. *A real (normed) algebra A is of complex type if and only if there exists a (continuous) linear operator J on A satisfying*

$$J(ab) = Ja \cdot b = a \cdot Jb$$

and such that $-J^2$ is the identity map.

Proof. The "only if" part is trivial; $a \to ia$ is a map with the desired properties.

Assume J given with the properties above. A complex scalar multiplication can then be defined

$$(\alpha + i\beta)a = \alpha a + \beta Ja.$$

If A is normed, let $\|\cdot\|$ be an admissible (real) norm. Following Kaplansky [20] we can construct an equivalent, complex norm

$$|||x||| = \max_\varphi \|\cos\varphi \cdot x + \sin\varphi \cdot Jx\|.$$

Corollary 6.3. *A real (normed) algebra A with identity e is of complex type if and only if there exists an element j in the center of A, satisfying $j^2 = -e$.*

Proof. If such a j exists, take $J: x \to jx$. If A is of complex type, $j = Je$ with the J of Proposition 6.2 has the desired properties.

The question whether scalar multiplication can be extended to complex numbers or not is not trivial even if the compatibility with multiplication is disregarded, i.e. for real vector spaces. If the dimension is finite complex scalars can be introduced if and only if it is even. Any infinite-dimensional real space can be given a complex structure in the algebraic sense. There are, however, infinite-dimensional real Banach spaces on which no complex multiplication exists, making them complex Banach spaces; Dieudonné [10].

Our main concern in the sequel will be real algebras that are not of complex type. We introduce conditions, signifying different degrees of "reality" of an algebra A. The first set is such that it can be used for algebras with or without topology:

Definition 6.4. *A real (normed) algebra A is said to be*

R_1, *of real type, if A is not of complex type.*
R_2 *if A does not contain any subalgebra of complex type with identity.*
R_4, *of strictly real type, if $-x^2$ is quasi-regular for every x.*

It is clear that the two following conditions are equivalent to R_2:

R_2'': A does not contain any subalgebra isomorphic to the complex numbers.
R_2''': The equation $x^3 + x = 0$ has no solution in A except 0.

Theorem 6.5. *For a real algebra A we have*

(*a*) $R_4 \Rightarrow R_2$,
(*b*) $R_4 \Rightarrow R_2 \Rightarrow R_1$ *if A has identity.*

Proof. If A is not R_2 we have some $k \neq 0$ satisfying $k + k^3 = 0$. If A is also R_4 there exists a y such that

$$0 = y \circ (-k^2) = y - k^2 + yk^2.$$

Multiplying by k gives $k^3 = y(k + k^3) = 0$ and $k = 0$ against the assumption. This proves (*a*). In (*b*) it is trivial that $R_2 \Rightarrow R_1$ if A has identity.

For a Banach algebra we have alternative formulations and one additional condition.

Definition 6.6. *A real Banach algebra A is said to be*

R_2' *if* $\operatorname{ixp} x = 0$ *implies* $x = 0$,
R_3' *if* $\operatorname{ixp} \alpha x$ *is a bounded function of* α, $-\infty < \alpha < \infty$, *only if* $x = 0$,
R_4' *if* $\sigma_A(x)$ *is real for every* $x \in A$.

Proposition 6.7. *For a Banach algebra A*

$$(a)\ R_2 \Leftrightarrow R_2',$$

$$(b)\ R_4 \Leftrightarrow R_4'.$$

Proof. (*a*) Assume that A is not R_2', $\operatorname{ixp} x = 0$ with $x \neq 0$. Then, according to Lemma 5.2, $\operatorname{ixp} 2^{-n} x \neq 0$ but $\operatorname{ixp} 2^{-n+1} x = 0$ for some $n > 0$. Put $y = 2^{-n-1} x$ and take $e_0 = 1/2 \operatorname{ixp} 2y$ and $i_0 = e_0 - e_0 \operatorname{ixp} y$. Then $e_0^2 = e_0$ and $i_0^2 = -e_0$ and so the algebra generated by e_0 and i_0 is isomorphic to the complex numbers, and A is not R_2. If A is not R_2 we have an element $k \neq 0$ such that $k + k^3 = 0$. Then $\operatorname{ixp} 2\pi k = -k \sin 2\pi - k^2(1 - \cos 2\pi) = 0$ and A is not R_2'.

(*b*) It is well known that R_4 and R_4' are equivalent if A is commutative [24, p. 119]. For a given x let B be a maximal, commutative subalgebra containing x. Then quasi-inverses of elements in B are already in B; in particular spectra of elements do not change (except possibly for 0) when restricted to B. If A is R_4, B is also R_4, hence $\sigma_B(x)$ and $\sigma_A(x)$ are real. Thus $R_4 \Rightarrow R_4'$ and since $R_4' \Rightarrow R_4$ is obvious from Definition 3.1 the proof is complete.

From now on we leave out the primes in the conditions and use the different formulations interchangeably.

Theorem 6.8. *For a real Banach algebra A*

(*a*) $R_4 \Rightarrow R_3 \Rightarrow R_2$,
(*b*) $R_4 \Rightarrow R_3 \Rightarrow R_2 \Rightarrow R_1$ *if A has identity,*
(*c*) $R_4 \Rightarrow R_1$ *if A is not radical.*
(*d*) *Any radical A is R_4.*

Proof. Assume that A is R_4. The quotient algebra A/R_A (R_A is the radical) is semisimple and R_4, hence commutative (Theorem 7.1). If, for $z \in A/R_A$, $\operatorname{ixp} \alpha z$ is bounded this is true also for its Gelfand function $h(\operatorname{ixp} \alpha z) = \operatorname{ixp} \alpha \hat{z}$. But since $\hat{z}$ is real $\hat{z} = 0$ and $z = 0$. Hence, if $x \in A$ and $\operatorname{ixp} \alpha x$ is bounded, $x \in R_A$ and $\nu(x) = 0$. But then, according to Lemma 5.1, $x = 0$ and A is R_3. $R_3 \Rightarrow R_2$ is obvious since if $\operatorname{ixp} x = 0$, $\operatorname{ixp} \alpha x$ is periodic in α, hence bounded, and $x = 0$. Thus (*a*) is proved. Then (*b*) follows from Theorem 6.5. For (*c*) let A be strictly real and P a primitive ideal (see [18, Ch. I]). Since P is closed A/P is a primitive Banach algebra. An argument by Kaplansky [20, p. 405] shows that A/P is isomorphic to the real numbers. If A is also of complex type then, for every x, $x^2 = -(Jx)^2$. Then x must be mapped into 0 of A/P and $x \in P$ for every P. But this implies that every x is in the radical, contrary to the assumption. (*d*), finally, is trivial since all elements in a radical algebra are quasi-regular.

Theorems 6.5 and 6.8 apparently give us reason to regard R_1 through R_4 as successively stronger reality conditions. It is easy to see by examples that none of the four are equivalent (see [15, p. 30]).

We also notice that the reality conditions are preserved when an identity is adjoined. Given A, A_1 is the algebra defined in sec. 2.

Lemma 6.9. *For a real (normed) algebra A*

(a) A_1 is always R_1,
(b) A_1 is R_2 if and only if A is R_2.
If A is a Banach algebra
(c) A_1 is R_3 if and only if A is R_3,
(d) A_1 is R_4 if and only if A is R_4.

Proof. (a) and (b) are trivial. For (c) we notice that for $(\xi,x)\in A_1$

$$\operatorname{ixp}\alpha(\xi,x)=(1-\exp\alpha\xi,\ \exp\alpha\xi\cdot\operatorname{ixp}\alpha x).$$

For this to be bounded it is necessary that $\xi=0$; thus it is bounded if and only if $\xi=0$ and $\operatorname{ixp}\alpha x$ is bounded in A.

In (d), it is clear that if A_1 is R_4 then A is R_4. For the "if" part a technique due to Civin and Yood [9] is used. We assume that A is R_4 and prove that $\sigma_{A_1}((\xi,x)^2)$ is non-negative real for any $(\xi,x)\in A_1$. Take $h=2\xi x+x^2$ and let B be a maximal commutative subalgebra of A containing h. B is a commutative Banach algebra and, thanks to the maximality, $\sigma_B(x)=\sigma_A(x)$ (except possibly for 0) for $x\in B$. If B is radical the conclusion is clear, hence assume that B has a non-empty Gelfand space ϕ_B and a Gelfand representation $x\to\hat{x}$. For every $\varphi\in\phi_B$ there is a $u\in B$ such that $u(\varphi)=1$. With $y=\xi u+xu$ we have $y^2=\xi^2u^2+hu^2\in B$ and

$$0\leqslant\hat{y}^2(\varphi)=\xi^2\hat{u}^2(\varphi)+\hat{h}(\varphi)\,\hat{u}^2(\varphi)=\xi^2+\hat{h}(\varphi).$$

Hence
$$\sigma_{A_1}((0,h))=\sigma_A(h)=\sigma_B(h)\geqslant-\xi^2$$

and finally
$$\sigma_{A_1}((\xi,x)^2)=\sigma_{A_1}((\xi^2,h))=\xi^2+\sigma_A(h)>0$$

and the proof is finished.

7. *Significance of the conditions*

Strict reality, R_4, was used already in Gelfand's original paper [11] in order to make sure that the representation of a real commutative Banach algebra only consists of real-valued functions. It is clear from section 4 that, for a commutative real Banach algebra, R_4 and $\phi_A^R=\phi_A$ are equivalent conditions, and that, for a R_4 algebra, the Gelfand space ϕ_A can be identified with the set of modular maximal ideals. R_4 is in fact a very strong condition, which is seen for instance from the remarkable result that a R_4 Banach algebra is commutative modulo its radical:

Theorem 7.1. (Kaplansky) *A strictly real semi-simple Banach algebra is commutative.*

The proof [20, p. 405] amounts to showing (by means of the density theorem) that any primitive R_4 Banach algebra is isomorphic to the real numbers. Since a subdirect sum of commutative rings is commutative the conclusion follows.

In Chapter IV some criteria for strict reality are given. In that context the closely related problem to characterize those real algebras whose Gelfand representation consists of all real functions in $C_0(\phi_A)$ is discussed.

The R_3 condition was introduced by the present author in connection with studies of the geometrical properties of the unit sphere [15]. A point on the boundary of a convex set is called a *vertex* if no straight line through the point is a tangent of the set. A real Banach algebra with identity e is said to have the *vertex property* if e is a vertex of every unit sphere that belongs to an admissible natural norm. We have [15, Theorem 2]:

Theorem 7.2. *A real Banach algebra with identity has the vertex property if and only if it is R_3.*

It would be interesting to know if there is an algebraic condition equivalent to R_3 (cf. Proposition 6.7).

Since any algebra without non-zero idempotents is R_2, and so quite a few complex algebras are in fact R_2, this condition seems rather weak. From Chapter III, however, it will be clear that R_2 is the "correct" reality condition for the quasi-regular group; in R_2 real algebras this group has distinctively different connectivity properties from that of complex (type) algebras.

8. *A decomposition theorem*

In this section, which is purely algebraic, we give a theorem to the effect that a commutative real algebra satisfying some suitable finiteness condition can be split in a direct sum of a complex type algebra and a R_2-algebra, thus enabling us to a certain extent to discriminate between the "complex" and the "real" properties of the algebra.

We consider the two conditions on an algebra A:

DCS = Any descending chain

$$A = A_0 \supset A_1 \supset \ldots \supset A_n \supset A_{n+1} \supset \ldots$$

of ideals of A, such that A_{n+1} is direct summand in A_n, has only a finite number of distinct members.

FI = The center of A contains only a finite number of idempotents.

Clearly, DCS is implied by the more familiar Artin descending chain condition for ideals, *a fortiori* by A being finite dimensional. The FI condition, on the other hand, is often satisfied by function algebras. Moreover, DCS implies FI.

Theorem 8.1. *A commutative real algebra, satisfying the FI (or the DCS) condition, is the direct sum of an algebra of complex type with identity and a R_2 algebra.*

Proof. If A is not R_2, we have elements e_1 and k such that $-k^2 = e_1$ and $e_1^2 = e_1$. With $A_1 = e_1 A$ and $A_2 = \{a - e_1 a;\ a \in A\}$ we have

$$A = A_1 \oplus A_2,$$

where A_1 is of complex type. If A is R_2 we take $A_1 = \{0\}$. Then we can split up A_2 in the same manner, $A_2 = A_3 \oplus A_4$, then A_4 and so on. The result can be described by the diagram

$$\begin{array}{ccccccc} A & \to & A_2 & \to & A_4 & \to & A_6 \to \dots \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ A_1 & & A_3 & & A_5 & & A_7 \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & & 0 & & 0 & & 0 \end{array}$$

where all the maps are algebra homomorphisms and

1. All horizontal maps are onto and split.
2. Sequences containing only one horizontal map are exact.
3. All members of the middle row are of complex type with identity.

Since each non-zero $A_1, A_3 \dots$ contains a different idempotent, the FI condition guarantees that only a finite number of them are $\neq \{0\}$ and the upper row becomes stationary from some element on. (The DCS tells directly that only a finite number of the A, $A_2 \dots$ can be different.) Hence, for some n, we have

$$A = A_1 \oplus A_3 \oplus \dots \oplus A_{2n-1} \oplus A_{2n},$$

where it is impossible to split up A_{2n} non-trivially, hence A_{2n} is R_2. Since a direct sum of complex type algebras with identity is again of complex type with identity, the conclusion follows.

Without any finiteness assumption of the type DCS or FI, the conclusion of Theorem 8.1 is no longer valid. To see this, let c_R be the algebra of all sequences of complex numbers converging to *real* limits (with component-wise multiplication). Assume that

$$c_R = A_c \oplus A_R$$

with A_c of complex type with identity. Its imaginary unit $k = \{\varkappa_n\}$ satisfies $k + k^3 = 0$, hence we must have $\varkappa_n = \pm i$ or 0, and consequently there is an N such that $\varkappa_n = 0$, $n \geqslant N$. Hence $\zeta_n = 0$, $n \geqslant N$ for all $\{\zeta_n\} \in A_c$. But A_R is never R_2: the elements with an arbitrary complex number in the Nth position and 0's elsewhere belong to A_R, but they clearly form a subalgebra isomorphic to the complex numbers.

Without the commutativity condition the theorem does not hold; for a finite-dimensional (i.e. DCS) counterexample take the algebra of all 2×2 matrices with real entries. This algebra is obviously not of complex type, but it contains a subalgebra isomorphic to the complex numbers, namely matrices of the form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix};$$

hence it is not R_2. But it is simple, thus indecomposable.

Remark 1. Since the decomposition in Theorem 8.1 is in fact effected by an idempotent it is clear that, if A is a topological algebra, both summands are closed. In particular if A is a Banach algebra the summands are Banach algebras.

Remark 2. There exists a purely ring-theoretic analogue to Theorem 8.1. A ring $(R, +, \cdot)$ is said to be of complex type if there exists a group endomorphism J on

$(R, +)$, satisfying $J(ab)=J(a)\cdot b=a\cdot J(b)$ and such that $-J^2$ is the identity. It is called R_2 if the equation $x^3+x=0$ has no solution $x \neq 0$ in R. If the conditions DCS and FI are read with "ring" instead of "algebra" we have:

A commutative ring, satisfying FI (or DCS), is the direct sum of a ring of complex type with identity and a R_2 ring.

Chapter III. The quasi-regular group

In this chapter the main object of study is the group of quasi-regular elements of a commutative Banach algebra. We first show that for a "sufficiently real" (i.e. R_2) algebra the principal component of the quasi-regular group is simply connected (Theorem 9.3). This depends on the extension of a result by Blum [4] to the general case of real algebras without identity.

A theorem by Lorch [21] says that in a complex commutative Banach algebra with identity the (quasi-) regular group either is connected or has an infinite number of components. This is obviously not true for real algebras in general but we obtain a satisfactory analogue for real commutative R_2 algebras: the components of the quasi-regular group are at least as many as the idempotents of the algebra, equal in number if the set of components is finite (Theorem 10.3). It is also shown that the Lorch result is true even without the assumption of identity. Finally (sec. 12) we make some remarks on how a recent result on the cohomology of the maximal ideal space is related to the questions discussed in the chapter.

9. *Structure of G_0^q*

The set of quasi-regular elements, which we have called G^q, is a group under the circle operation ($x \circ y = x+y-xy$) and a topological group in the topology of the algebra (sec. 2). Its principal component (the maximal connected subset of G^q containing 0) is called G_0^q. The following lemma, which is well known for the case with identity [24, p. 14], is technically important.

Lemma 9.1. *In a Banach algebra A, G_0^q is the subgroup generated by ixp (A). If A is commutative $x \to \operatorname{ixp} x$ is a homomorphism of $(A, +)$ onto $(G_0^q, \circ)$.*

Proof. Since $\operatorname{ixp} tx$, $0 \leqslant t \leqslant 1$, is a continuous path in G^q between 0 and $\operatorname{ixp} x$ we have $\operatorname{ixp}(A) \subset G_0^q$. If G' is the group generated by $\operatorname{ixp}(A)$ we show that G' is open and closed in G^q.

There exists a neighbourhood U of 0 consisting only of elements $\operatorname{ixp} a$, $a \in A$ (sec. 5). For a $x \in G'$ the set $x \circ U$ is a neighbourhood of x and belongs to G', hence x is an interior point and G' is open. If $x \in \bar{G}'$ then $(x \circ U) \cap G'$ is not empty, hence an a exists so that $x \circ \operatorname{ixp} a = g \in G'$ and $x \in G'$. (As by-products we get that G_0^q is pathwise connected and open in G^q.)

From here on we restrict our attention to commutative algebras. Let $\Pi_1(G_0^q)$ (or only Π_1) denote the fundamental group of G_0^q. The following theorem generalizes a result by Blum [4] for complex algebras with identity. We take $P=\{x; \operatorname{ixp} x=0\}$, which is a subgroup of $(A, +)$.

Theorem 9.2. *In a commutative Banach algebra $\Pi_1(G_0^q)$ is isomorphic to P.*

Proof. P is the kernel of the epimorphism $\mathrm{ixp}: A \to G_0^q$, hence A/P is isomorphic to G_0^q. We show that P is discrete, which is the case if 0 is an isolated point of P. Assume that $x \in P$ and $\|x\| < 1$. Since $\mathrm{ixp}\, x = 0$ the spectrum of x only consists of points $2\pi i N$, $N = 0, \pm 1, \pm 2 \ldots$ Since $\|x\| < 1$ we have that the spectrum of x is $\{0\}$ and so x is topologically nilpotent. But $\varphi(\alpha) = \mathrm{ixp}\, \alpha x$ is continuous and periodic in α, hence bounded, and Lemma 5.1. shows that $\varphi(\alpha) = 0$ and $x = 0$. Since A is locally connected, simply connected and locally simply connected, P discrete and ixp an open map, a theorem by Schreier [28] applies and shows that the fundamental group of (the torus) A/P is isomorphic to P.

Theorem 9.3. *For a commutative Banach algebra A the following statements are equivalent:*

(*a*) $\mathrm{ixp}: A \to G_0^q$ *is an isomorphism,*
(*b*) *A is* R_2,
(*c*) G_0^q *is simply connected,*
(*d*) G_0^q *is torsion free.*

Proof. From Lemma 9.1, Proposition 6.7 and Theorem 9.2, respectively, follows that (*a*), (*b*) and (*c*) are all equivalent to $P = \{0\}$. We prove that (*b*) and (*d*) are equivalent.

Assume that A is not R_2. Then $\mathrm{ixp}\, x = 0$ for some $x \neq 0$. From Lemma 5.2 follows that there exists an integer $n \geq 2$ such that $y_n = \mathrm{ixp}\, n^{-1} x \neq 0$. Then $y_n \in G_0^q$ and $y^{\circ n} = 0$ so G_0^q is not torsion free. Now assume that there is a $z \in G_0^q$ such that $z \neq 0$ and $z^{\circ N} = 0$. According to Lemma 9.1 $z = \mathrm{ixp}\, u$ for some u, and so $\mathrm{ixp}\, N u = z^{\circ N} = 0$, and A is not R_2, which completes the proof.

Thus we have that for "sufficiently real" Banach algebras G_0^q is simply connected. For every non- R_2 algebra, in particular every complex algebra with identity, however, Π_1 has a subgroup isomorphic to the additive group of integers. For complex algebras without identity it can very well happen that G_0^q is simply connected. For every radical Banach algebra, for instance, we have $A = G^q = G_0^q$.

We conclude this section with a technical remark on commutative complex type algebras.

Theorem 9.4. *A commutative real Banach algebra A with identity e is of complex type if and only if $-e$ belongs to the principal component of the regular group.*

Proof. If A is of complex type, $\exp(i\varphi e)$, $0 \leq \varphi \leq \pi$, is a path in G between e and $-e$; hence $-e$ belongs to the principal component G_e. If $-e \in G_e$ then (Lemma 9.1) there is a $u \in A$ such that $\exp u = -e$. But then $j = \exp \frac{1}{2} u$ is an imaginary unit and A is of complex type.

10. *Components of G^q*

In this section we study the quasi-regular group G^q of a commutative Banach algebra, in particular the number of components of G^q. For complex Banach algebras with identity we have a result by Lorch [21, Theorem 12]:

G^q (and G) either is connected or has an infinite number of components.

It is immediate that this does not hold for real algebras in general. For instance for the real numbers R, $G^q = R - \{1\}$ and has two components. A counterpart to Lorch's theorem will be given for real R_2-algebras. As a preparation we prove two lemmas.

Lemma 10.1. *Every element of finite odd order in G^q belongs to its principal component G_0^q.*

Proof. Let $r^{\circ n}=0$, n odd. The line segment

$$A(\tau)=(1-\tau)r,\ 0\leqslant\tau\leqslant 1,$$

connects 0 and r. But with

$$B(\tau)=[(1-\tau)^n-(-\tau)^n]^{-1}\sum_{k=1}^{n-1}(1-\tau)^k(-\tau)^{n-k-1}r^{\circ k}$$

we have $A(\tau)\circ B(\tau)=0$. Hence $A(\tau)\in G^q$ and $r\in G_0^q$. (This result obviously holds in any power associative real topological algebra, i.e. not necessarily commutative, associative or normed.)

Lemma 10.2. *Let Λ be an abelian topological group such that its principal component is torsion free. If x and y are two elements of finite order in Λ they belong to different components unless $x=y$.*

Proof. Assume x and y are in the same component, Λ'. $z\to x^{-1}z$ maps Λ' homeomorphically onto the principal component Λ_e. Then $x^{-1}y\in\Lambda_e$ and $x^{-1}y$ is of finite order, hence $x^{-1}y=e$ and $x=y$.

For the formulation of the main theorem we introduce the set of idempotents $I=\{y;\ y^2=y\}$. G^q/G_0^q, the set of components of G^q, is called K.

Theorem 10.3. *In a commutative R_2 Banach algebra holds*

(a) card $I\leqslant$ card K,
(b) card $I=$ card K if K is finite.

Proof. We first introduce the subset S of G^q consisting of all elements of order 2, $S=\{x;\ x^{\circ 2}=2x-x^2=0\}$. Evidently $x\to\frac{1}{2}x$ is a one-to-one map of S onto I, so card $I=$ card S.

To every element $x\in S$ we associate the component of G^q, Γ_x, in which it lies, defining a function

$$f: x\to\Gamma_x$$

of S into K. But G_0^q is torsion free (Theorem 9.3) and Lemma 10.2 shows that f is injective, which proves (a).

Now assume that K is finite, take an arbitrary element $\Gamma\in K$ and let $k\in\Gamma$. The powers $k^{\circ n}$, $n=1,2,\ldots$, cannot all lie in different components. If $k^{\circ m}$ and $k^{\circ(m+l)}$ are in the same component, $k^{\circ l}=k^{\circ(-m)}\circ k^{\circ(m+l)}\in G_0^q$ and there exists $u\in A$ such that $\operatorname{ixp}u=k^{\circ l}$ (Lemma 9.1). The element $v=k\circ\operatorname{ixp}(-u/l)$ satisfies $v^{\circ l}=0$ and $v\in\Gamma$. Lemma 10.1 together with the fact that G_0^q is torsion free shows that there are no elements in G^q of odd, finite order. Hence $l=2^n$ and $v^{\circ 2^n}=0$ for some $n\geqslant 0$. But in a R_2 algebra, $x^{\circ 2^n}=0$ for some $n>1$ implies that already $x^{\circ 2}=0$. To see this, assume that $x^{\circ 4}=0$ but $x^{\circ 2}\neq 0$. Then

$$k=\tfrac{1}{2}(2x-3x^2+x^3)=\tfrac{1}{2}(x^{\circ 3}-x)$$

is $\neq 0$, because otherwise $x^{\circ 2} = x \circ x^{\circ 3} = x^{\circ 4} = 0$ against the assumption. But k also satisfies $k + k^3 = 0$ (notice the formal identities $k = \frac{1}{2}(1-x)\,[1-(1-x)^2]$ and $(1-x)^4 = 1$) and the algebra would not be R_2. Hence we have $v^{\circ 2} = 0$, $v \in S$ and $\Gamma = \Gamma_v$ so f is onto and the proof of (b) is complete.

The inequality in (a) above can be strict. An example (slightly modified after S. Kakutani) is the following. A is the real R_2 algebra of all continuous complex functions on the unit circle $|z| = 1$ that take only real values at $z = 1$. The elements f_n, with $f_n(z) = z^n, n = 0, 1, 2\ldots$, lie in different components of G, but the number of idempotents is 2.

11. *The regular group of* A_1

In this section we investigate the connection between the properties of the quasiregular group of A and that of A_1 (A with adjoined identity, sec. 2). Here we reserve the notation G and G_1 for the regular group of A_1 and its principal component, while G^q and G_0^q stand for the quasiregular group and its principal component in A.

For a given component Γ of G^q we define the subsets

$$\Gamma^+ = \{\alpha(1, -g);\ \alpha > 0,\ g \in \Gamma\}, \quad \Gamma^- = -\Gamma^+$$

of A_1.

Theorem 11.1. *If A is a real Banach algebra then*

(a) to every component Γ of G^q in A correspond two components Γ^+ and Γ^- of G in A_1,

(b) card $G/G_1 = 2$ card G^q/G_0^q,

(c) the fundamental groups of G_0^q and G_1 are isomorphic,

$$\Pi_1(G_1) \simeq \Pi_1(G_0^q).$$

Proof. The map $x \to (1, -x)$ is a homeomorphism of G^q onto the subset $(1, -G^q) \subset G$. If Γ is a maximal connected subset of G^q then $(1, -\Gamma) \subset \Gamma^+$ and Γ^+ is clearly maximal connected. Γ^+ and Γ^- are homeomorphic and lie in disjoint homeomorphic half-spaces, hence Γ^- is also a component. Since for any component Λ of G either Λ or $-\Lambda$ must contain an element $(1, -g)$, $g \in G^q$, (a) and (b) are proved.

For (c), we rely on Theorem 9.2. Since $\operatorname{ixp}(\xi, x) = 0$ implies $\xi = 0$ we have

$$\Pi_1(G_1) \simeq \{(\xi, x);\ \operatorname{ixp}(\xi, x) = 0\} = \{(0, x);\ \operatorname{ixp} x = 0\} \simeq \Pi_1(G_0^q).$$

For an algebra of complex type we can, as an alternative to the construction used above, adjoin a complex identity; take

$$A_{1C} = C \oplus A,$$

direct sum as normed vector spaces, with the algebra operations defined as in sec. 2. We use the notations G, G_1, G^q, G_0^q as above, and for a component Γ of G^q we define the subset

$$\Gamma^\circ = \{\zeta(1, -g);\ \zeta \neq 0,\ g \in \Gamma\}$$

of A_{1C}.

Theorem 11.2. *If A is a Banach algebra of complex type and a complex identity is adjoined, then*

(*a*) *to every component Γ of G^q in A corresponds a component Γ° of G in A_{1C},*

(*b*) *card $G/G_1 =$ card G^q/G_0^q,*

(*c*) *the fundamental group of G_1 is isomorphic to the direct sum of the fundamental group of G_0^q with the integers Z,*

$$\Pi_1(G_1) = \Pi_1(G_0^q) \oplus Z.$$

Proof. (*a*) and (*b*) follow as in Theorem 11.1. For (*c*) we notice that $\operatorname{ixp}(\zeta, x) = (1-\exp\zeta,\ \exp\zeta \cdot \operatorname{ixp} x) = 0$ if and only if $\zeta = 2\pi i n$, $n \in Z$, and $\operatorname{ixp} x = 0$. Hence Theorem 9.2 gives

$$\Pi_1(G_1) \simeq \{(\zeta, x);\ \operatorname{ixp}(\zeta, x) = 0\} = \{(2\pi i n, x);\ n \in Z,\ \operatorname{ixp} x = 0\} \simeq Z \oplus \Pi_1(G_0^q).$$

A simple consequence of Theorem 11.2 (*b*) is that Lorch's theorem (see sec. 10) generalizes to the case without identity:

Theorem 11.3. *In a Banach algebra of complex type the quasi-regular group either is connected or has an infinite number of components.*

12. *Remarks on a cohomology result*

A recent result due to Arens [2] and Royden [25], [26] is that, for a complex commutative Banach algebra with identity, G/G_1 (as a group) is isomorphic to $H^1(\mathfrak{M}, Z)$, the first Čech cohomology group of the maximal ideal space $\mathfrak{M}$ with the integers as coefficient group. We can make two remarks on this, due to the fact that *for any compact space Ω, $H^1(\Omega, Z)$ is torsion free.*

This is immediately clear when Ω is a finite simplicial complex. Then $H^1(\Omega, Z)$ is isomorphic to the direct sum of the free (Betti) part of $H_1(\Omega, Z)$ and the torsion part of $H_0(\Omega, Z)$ [8, p. 127]. Since H_0 is always free $H^1(\Omega, Z)$ is torsion free.

For Ω an arbitrary, compact space $H^1(\Omega, Z)$ is the direct limit of $H^1(\Omega_\sigma, Z)$, where Ω_σ are finite simplicial complexes: the finite open coverings (or "nerves" of such coverings) of Ω (for terminology, see [14, p. 132]). Since a direct limit of torsion free groups is torsion free, the conclusion follows.

Hence, if $H^1(\Omega, Z)$ is not trivial, it is infinite. Since we also know of real commutative Banach algebras in which G/G_1 is finite but non-trivial, we can make the two remarks:

1. For real, commutative Banach algebras in general it is not true that $G/G_1 \simeq H^1(\Omega, Z)$ for any compact space Ω.
2. For complex, commutative Banach algebras with identity the result that $G/G_1 \simeq H^1(\mathfrak{M}, Z)$ (Arens, Royden) implies that G/G_1 has either one or an infinite number of elements (Lorch).

Chapter IV. Strict reality and full function algebras

The first part (sec. 13) of this chapter contains criteria for a real Banach algebra to be strictly real (R_4). One of the conditions (Theorem 13.1) is analytic in character, the other (Theorem 13.3) deals with geometric properties of the unit sphere and complements previous results by the author [15]. These conditions can also be formulated in terms of an ordering of the algebra (sec. 14). In section 15 we give criteria for a Banach algebra A to be homeomorphically (or isometrically) isomorphic to a $C_0^R(\Omega)$, the algebra of all continuous real functions on a locally compact space that tend to 0 at infinity. The key to these results is the observation (Theorem 15.2) that if, in a strictly real algebra, the spectral radius $\nu(x)$ is in a certain sense compatible with the topology, the algebra is isomorphic to $C_0(\phi_A)$. The results generalize a theorem by Segal [29] that has been used in his work on the foundations of quantum mechanics.

13. *Conditions for strict reality*

We recall the definition $\operatorname{ixp} x = -\sum_{n=1}^{\infty}(n!)^{-1}x^n$ and start with an analytic criterion of strict reality.

Theorem 13.1. *If in a real Banach algebra* $\operatorname{ixp}(-\alpha x^2), \alpha \geqslant 0$, *is a bounded function of* α *for every* x, *then the algebra is strictly real.*

Proof. Take an arbitrary element x of the algebra A and let B be a maximal, commutative subalgebra containing x. B is closed and the spectrum of x in B coincides with the spectrum of x in A (except possibly for 0). If $\hat{x}$ is the image of x in the Gelfand representation of B, $\operatorname{ixp}(-\alpha\hat{x}^2)$ is a bounded (complex function-valued) function of α. Then $\operatorname{Re}\,\hat{x}^2 \geqslant 0$ and the spectrum of x is contained in the "double wedge" $|\operatorname{Im}\zeta| \leqslant |\operatorname{Re}\zeta|$ of the complex plane. Since this holds for every element it follows from the spectral mapping theorem (for powers) that the spectrum of every element must be real.

The condition that $\operatorname{ixp}(-\alpha x^2)$ is bounded for all x, which is sufficient for strict reality, is not necessary, which will be shown by examples. We first make an observation about "topologically very nilpotent" elements.

Lemma 13.2. *If* x *is an element of a Banach algebra such that* $\|x^n\|^{1/n} = o(n^{-1})$, $n \to \infty$, *and* $\operatorname{ixp} \alpha x$ *is bounded for* $\alpha \geqslant 0$ *then* $x = 0$.

Proof. For any continuous linear functional f, $f(\operatorname{ixp} \alpha x) = \varphi(\alpha)$ is an analytic function on the real line that can be extended to an entire function on the whole complex plane. As such, it is at most of order $\frac{1}{2}$, minimum type, and since it is also bounded on the positive real line a Phragmén-Lindelöf theorem tells that it must be constant (cf. Lemma 5.1). Hence $f(x) = 0$ for all f and $x = 0$.

Remark. Results of this type are obtained by Lumer and Phillips [22]. They also disprove the conjecture by Bohnenblust and Karlin [6] that, in a Banach algebra with identity, no ray $e + \alpha x$, $\alpha \geqslant 0$, $x \in R_A$, can be a tangent of a natural unit sphere. It is true, however, that no full straight line $e + \alpha x$, $-\infty < \alpha < \infty$, x topologically nilpotent, can be a tangent of a natural unit sphere ([15], cf. also Lemma 5.1 and Theorem 7.2).

In view of Lemma 13.2 it is sufficient to exhibit an element $x \neq 0$ in a Banach algebra satisfying $\|x^{2n}\|^{1/n} = o(n^{-1})$. The closed subalgebra generated by x will then be a radical Banach algebra and automatically strictly real (Theorem 6.8, (d)) but $\operatorname{ixp}(-\alpha x^2)$ must be unbounded. A trivial example is then a nilpotent element x, where $\|x^{2n}\| = 0$

from some n on. A little less trivial is the following. The algebra is the vector space $C\,[0,1]$ with the maximum norm $\|\cdot\|$ and the multiplication

$$x * y(t) = \int_0^t x(t-s)\,y(s)\,ds.$$

Let c be the function with $c(t)=1$. It is easily verified that

$$c^n(t) = \frac{t^{n-1}}{(n-1)!} \quad \text{and} \quad \|c^{2n}\|^{\frac{1}{n}} = \left(\frac{1}{(2n-1)!}\right)^{\frac{1}{n}} = n^{-2}O(1).$$

Next we give a condition for strict reality in terms of the geometry of the unit sphere. For an algebra with identity e, we define the *enveloping cone* at e of a certain natural unit sphere to be the collection of all rays $e+\alpha x$, $\alpha \geqslant 0$, that either are tangents to the unit sphere or cut through it. In terms of the Gateau differential $\phi(x) = \lim_{\alpha\to+0}\alpha^{-1}(\|e+\alpha x\|-1)$ the rays in the enveloping cone are precisely those belonging to x such that $\phi(x)\leqslant 0$. The author has proved that if a Banach algebra is strictly real then the enveloping cone at e does not contain a full line through e for any natural unit sphere (vertex property [15, Theorem 4], cf. Theorem 7.2). The following theorem, which also deals with the local properties of a unit sphere at e, is in a way complementary to this result.

Theorem 13.3. *If a Banach algebra with identiy e has a natural norm such that every ray $e-\alpha x^2$, $\alpha\geqslant 0$, is in the enveloping cone at e, the algebra is strictly real.*

Proof. The assumption is equivalent to $\phi(-x^2)\leqslant 0$ for every x for some natural norm $\|\cdot\|$. But we also have $\phi(x)=\lim_{\alpha\to+0}\alpha^{-1}\log\|\exp\alpha x\|$, [15, p. 25]. With $h(\alpha)=\log\|\exp(-\alpha x^2)\|$, $\alpha\geqslant 0$, we have $h(\alpha+\beta)\leqslant h(\alpha)+h(\beta)$. From this follows

$$\frac{h\left(\frac{\alpha}{n}\right)}{\frac{\alpha}{n}} \geqslant \frac{h(\alpha)}{\alpha}$$

for any natural number n. If $h(\alpha)>0$ for some α, this inequality shows that we cannot have

$$\phi(-x^2) = \lim_{\alpha\to+0}\frac{h(\alpha)}{\alpha} \leqslant 0$$

and so we conclude that $h(\alpha)\leqslant 0$ for all $\alpha\geqslant 0$. But then $\|\exp(-\alpha x^2)\|\leqslant 1$ and Theorem 13.1 shows that the algebra is strictly real.

14. *Order-theoretic formulation*

We call a set K in a real vector space a *cone* (with vertex at 0) if $x,y\in K$ implies $x+y\in K$ and $\alpha x\in K$ for every $\alpha\geqslant 0$. K is called *proper* if $-x\in K$ and $x\in K$ together imply $x=0$. Given a cone K we can define a partial ordering (the K-ordering) of the space by defining $x \geq y$ as $x-y \in K$. Then $x \geq y$ and $y \geq x$ imply $x=y$ if and only if K is proper; K consists of all positive elements, $x \geq 0$.

The image of an enveloping cone at e under the mapping $x \to e-x$ is a cone (with vertex at 0) which we call N. N is proper if and only if the unit sphere has a vertex at e (cf. Theorem 7.2). Theorem 13.3 can now be reformulated in terms of N-ordering.

Theorem 14.1. *If a Banach algebra with identity has a natural norm such that in the corresponding N-ordering every square is positive then it is strictly real.*

It follows that under the assumptions of the theorem N is proper [15, Theorem 4].

For a commutative algebra it is obvious that the set of all x, such that $\operatorname{ixp}(-\alpha x)$ is bounded for $\alpha \geq 0$, form a cone. If we call this cone E we get directly from Theorem 13.1.

Theorem 14.2. *If in a commutative Banach algebra every square is positive in the E-ordering then the algebra is strictly real.*

E is proper if and only if A is R_3. From Theorem 6.8 it follows that E is proper if the algebra satisfies the assumptions of Theorem 14.2.

A discussion of ordered real algebras is found in Kadison [19]. (In our discussion the ordering has no *a priori* connection with the multiplicative structure, i.e., it is not necessarily an algebra ordering in the sense of [19].)

15. *Abstract characterization of $C_0^R(\Omega)$*

If Ω is a locally compact Hausdorff space, $C_0^R(\Omega)$ denotes the real Banach algebra of all real continuous functions that tend to 0 at infinity; the topology is defined by the sup norm. In the abundance of properties for $C_0^R(\Omega)$, we can ask for a set of topological or metric conditions on a Banach algebra A to guarantee that A is isomorphic and homeomorphic to a $C_0^R(\Omega)$. Largely two different sets of conditions of this kind can be found in the literature:

I. A is commutative and has a natural norm that satisfies $\|x\|^2 \leq \alpha\|x^2+y^2\|$ for all x and y and some fixed α.

This follows as a special case of a theorem by Arens and Kaplansky on real commutative *-algebras [1] (also [24, p. 191]) and depends on the fact that the complexification of A is (essentially) complex B^* (cf. also sec. 18).

II. A is commutative, has identity and a natural norm that satisfies

$$\|x^2\|=\|x\|^2 \quad \text{and} \quad \|x^2-y^2\| \leq \max \|x^2\|,\ \|y^2\|.$$

This is due to Segal [29, Theorem 1]. Conditions of type I have been used by Kadison [19]; different proofs of I and II have been given by Aurora [3].

We are going to give conditions that in a sense generalize II. Theorems 15.2 and 15.3 show that the conditions in II can be relaxed considerably. In the following we denote, as usual, $\nu(x)=\lim_{n\to\infty}\|x^n\|^{1/n}$ for some admissible norm (and hence for all). S_ν is the set of x for which $\nu(x) \leq 1$.

Lemma 15.1. *The conditions*

(i) *S_ν is bounded,*

(ii) *for some admissible norm (not necessarily natural) it holds that $\|x^2\| \geq \alpha\|x\|^2$ for all x and some fixed $\alpha>0$,*

(iii) *for every neighbourhood V of 0 there is a neighbourhood U of 0 such that $x \notin V$ implies $x^2 \notin U$,*

(iv) *for some admissible norm there exist positive constants m, M such that $m\|x\| \leqslant \nu(x) \leqslant M\|x\|$ for all x,*

on a normed algebra are equivalent.

Proof. (i) means that for some admissible norm $\|\cdot\|$, which we can choose as natural, $\nu(x) \leqslant 1$ implies $\|x\| \leqslant C$ for some fixed C. Hence for any x we have

$$\left\| \frac{x}{\nu(x)} \right\| \leqslant C \quad \text{and} \quad \nu(x) \geqslant C^{-1}\|x\|,$$

and since $\|\cdot\|$ is natural

$$C^{-1}\|x\| \leqslant \nu(x) \leqslant \|x\|.$$

Thus (i) and (iv) are equivalent.

(ii) and (iii) are easily seen to be equivalent for a normed algebra. If (ii) holds we have, for some admissible natural norm $|\cdot|$ and fixed $\beta > 0$,

$$|x^2| \geqslant \beta |x|^2.$$

From this follows by direct computation $\nu(x) \geqslant \beta |x|$ and

$$\beta |x| \leqslant \nu(x) \leqslant |x|.$$

But from (iv) follows

$$\|x^2\| \geqslant M^{-1}\nu(x^2) = M^{-1}\nu(x)^2 \geqslant M^{-1}m^2\|x\|^2$$

and so (ii) and (iv) are equivalent.

The following result will be used for the metric-topological characterization of $C_0^R(\Omega)$ but it has also independent interest. In a slightly weaker form (and with a different proof) it is found by Kadison [19, Theorem 6.6].

Theorem 15.2. *A strictly real Banach algebra A satisfying one of the conditions* (i)–(iv) *of Lemma 15.1 is isomorphic and homeomorphic to $C_0^R(\phi_A)$.*

Proof. Since $\nu(x) = 0$ implies $x = 0$, A is semi-simple. Then Theorem 7.1 (Kaplansky) applies and shows that A is commutative.

For any x and the Gelfand mapping (sec. 4), $h: x \to \hat{x}$, $\hat{x}$ is real and so h is a map into $C_0^R(\phi_A)$. A and $h(A)$ are isomorphic and, since $\nu(x) = \sup_{\varphi \in \phi_A} |\hat{x}(\varphi)|$, ν an admissible norm for A and h an isometry under ν and the sup norm. But from Stone-Weierstrass' theorem follows that $h(A)$ is dense in $C_0^R(\phi_A)$. Being complete it must be closed, we have $h(A) = C_0^R(\phi_A)$ and the theorem is proved.

By combining the criteria of strict reality from section 13, the result of Theorem 15.2 and the different conditions from Lemma 15.1 it is now possible to formulate various metric-topological conditions for an algebra to be a $C_0^R(\Omega)$. We choose a metric formulation that might be applicable.

Theorem 15.3. *A real Banach algebra A with identity e that has an admissible natural norm satisfying*

(i) $\|e - \alpha x^2\| \leqslant 1 + o(\alpha), \quad \alpha \to +0,$

(ii) $\|x^2\| \geqslant k\|x\|^2, \quad k > 0,$

for all $x \in A$, *is isomorphic and homeomorphic to* $C^R(\phi_A)$ *with* ϕ_A *compact Hausdorff; in particular* A *is commutative. If* $k=1$ *it is isometric with respect to* $\|\cdot\|$ *and the sup norm.*

Proof. Condition (i) and Theorem 13.3 tell that A is strictly real. But (ii) is condition (ii) in Lemma 15.1 and so Theorem 15.2 gives the conclusion.

Corollary 15.4. (*Segal.*) *A real (commutative) Banach algebra with identity and an admissible natural norm satisfying*

$$\|x^2 - y^2\| \leqslant \max(\|x^2\|, \|y^2\|),$$

$$\|x^2\| = \|x\|^2$$

is isometrically isomorphic to $C^R(\phi_A)$.

Chapter V. Real algebras with involution

One of the main results in the theory of complex Banach algebras is that a B^*-algebra (a *-algebra with norm satisfying $\|x\|^2 = \|x^*x\|$) is isomorphic and isometric to a C^*-algebra (a self-adjoint algebra of bounded operators on a Hilbert space). This chapter deals with the corresponding problem for real algebras.

After some introductory discussion of real *-algebras and some conditions relating the involution to the topology (sec. 16) we obtain a negative answer to the question whether a real B^*-algebra is necessarily C^*. It is shown, however, that a symmetric ($-x^*x$ is quasi-regular for all x) real Banach *-algebra with a norm satisfying $\|x\|^2 \leqslant \beta \|x^*x\|$ is isomorphic and homeomorphic to a C^*-algebra (Thorem 17.6). If the norm condition is strengthened, symmetry will follow and so C^*-algebras can be characterized by norm conditions only (Theorem 18.6):

A real Banach *-algebra with identity and a norm satisfying $\|x\|^2 = \|x^*x\|$ for all x and $\|x\|^2 \leqslant \alpha \|x^*x + y^*y\|$ for all normal, commuting x, y and a constant α, is isomorphic and isometric to a C^*-algebra.

The same result is obtained (under slightly stronger hypotheses) even for non-complete normed algebras and algebras without identity (Theorem 18.7).

In some parts of this chapter the technique can be borrowed, with no or little change, from the complex case. Brief mention is always made, however, in order to make the presentation self-contained.

16. *Definitions and preliminaries*

All algebras treated here have the real numbers as their scalar field. With a *-algebra is meant an algebra on which is defined an involution, $x \to x^*$, which is an involutive ($x^{**} = x$) linear operator and moreover satisfies $(xy)^* = y^*x^*$. For a complex algebra, regarded also as a real algebra, real involution clearly is a more general concept than complex involution. A *-algebra is called *symmetric* if $-x^*x$ is quasi-regular for all x.

An element x of a *-algebra is called *hermitian* if $x^* = x$, *antihermitian* if $x^* = -x$ and *normal* if $x^*x = xx^*$. The sets of hermitian, antihermitian and normal elements will be denoted H, K and N respectively.

We make some remarks about decomposition of elements and linear functionals. Since $x=\frac{1}{2}(x+x^*)+\frac{1}{2}(x-x^*)$ and $0^*=0$, it is clear that every element has a unique decomposition in a hermitian and an antihermitian part. A linear functional F is called *hermitian* if $F(x^*)=F(x)$ and *antihermitian* if $F(x^*)=-F(x)$. A hermitian functional satisfying $F(x^*x)\geqslant 0$ for all x is called *positive*. If we take $F=H+K$, $H(x)=\frac{1}{2}F(x+x^*)$ and $K(x)=\frac{1}{2}F(x-x^*)$, we see that every linear functional can be expressed uniquely as a sum of a hermitian and an antihermitian functional. If $x=h+k$, $h\in H$ and $k\in K$, we also have $F(x)=H(h)+K(k)$ with H and K as above.

In a normed $*$-algebra it may or may not happen that involution is continuous. If it is, it is also isometric with respect to a norm:

Proposition 16.1. *If a normed algebra has a continuous involution there exists an admissible natural norm such that* $\|x^*\|=\|x\|$ *for all* x.

Proof. Let $|\cdot|$ be a natural, admissible norm. Then $\|x\|=\max(|x|,|x^*|)$ is natural, equivalent to $|\cdot|$ and has the desired property.

We define, for later use, some conditions linking norms with the involution.

Definition 16.2. *A natural norm on a $*$-algebra is called*

b^* if $\|x\|^2\leqslant\beta\|x^*x\|$ *for all x and some constant β,*

B^* if $\|x\|^2=\|x^*x\|$ *for all x,*

Dc^* if $\|x\|^2=\|x^*x\|$ *for all x and* $\|x\|^2\leqslant\alpha\|x^*x+y^*y\|$ *for all normal, commuting x, y and some constant α,*

D^* if $\|x\|^2=\|x^*x\|$ *and* $\|x\|^2\leqslant\alpha\|x^*x+y^*y\|$ *for all x, y and some constant α.*

A $*$-algebra with an admissible $b^*(B^*, Dc^*, D^*)$ norm will be called a b^*-(B^*-, Dc^*-, D^*-) algebra.

It is clear that these conditions are successive strengthenings of each other, i.e. every D^*-algebra is Dc^*, every Dc^* is B^* and every B^* is b^*. In all four cases involution is continuous: for b^* we get $\|x^*\|\leqslant\beta\|x\|$ and hence (Proposition 16.1) we have a b^*-norm with $\|x\|=\|x^*\|$, for the other three this identity is automatic from the conditions.

If A is a real $*$-algebra, A_1 (sec. 2) can be made a real $*$-algebra by defining $(\xi,x)^*=(\xi,x^*)$. For later use we state two technical lemmas about A_1.

Lemma 16.3. *If A is a b^*-(B^*-, D^*-) algebra then A_1 is also $b^*(B^*, D^*)$.*

Proof. For $(\xi,x)\in A_1$ and $\|\cdot\|$ a natural norm on A, we define

$$|||(\xi,x)|||=\sup_{\|z\|=1}\|\xi z+xz\|.$$

Since $|||\cdot|||$ is an operator norm, it is natural on A_1. Assume that $\|x\|^2\leqslant\alpha\|x^*x+y^*y\|$ and $\|x^*\|=\|x\|$ for all $x,y\in A$. Then, for $x_1=(\xi,x), y_1=(\eta,y)\in A_1$

$$\|\xi z+xz\|^2\leqslant\alpha\|(\xi z+xz)^*\,(\xi z+xz)+(\eta z+yz)^*\,(\eta z+yz)\|$$

$$\leqslant\alpha\|z^*\|\cdot|||x_1{}^*x_1+y_1{}^*y_1|||\cdot\|z\|$$

and $$|||x_1|||^2\leqslant\alpha|||x_1{}^*x_1+y_1{}^*y_1|||.$$

In the same way, if $\|x\|^2 \leqslant \beta \|x^* x\|$ and $\|x^*\| = \|x\|$

$$|||x_1|||^2 \leqslant \beta |||x_1{}^* x_1|||.$$

This proves the b^* part. If $\beta = 1$ it follows $|||x_1{}^*||| = |||x_1|||$ and $|||x_1|||^2 = |||x_1{}^* x_1|||$. The D^* case, finally, follows from B^* and the calculation above.

The next lemma is proved under somewhat more general assumptions than will be needed in sec. 17, but seems to have independent interest. The technique is partly due to Civin and Yood [9].

Lemma 16.4. *If A is a real symmetric Banach *-algebra with continuous involution, A_1 is also symmetric.*

Proof. We show that $\sigma_{A_1}((\xi, x)^*(\xi, x))$ is non-negative. Let $h = \xi(x^* + x) + x^* x$ and B a subalgebra of A which is maximal with respect to the properties of being commutative, contain h and be contained in H. Then B is a strictly real Banach algebra and $\sigma_B(b) = \sigma_A(b)$ (except possibly for 0) for $b \in B$. We proceed as in the proof of Lemma 6.9 (d) to show that

$$\sigma_B(h) = \sigma_A(h) \geqslant -\xi^2 \quad \text{and} \quad \sigma_{A_1}((\xi, x)^* \, (\xi, x)) \geqslant 0.$$

By a *C^*-algebra* will be meant a *-subalgebra of $B(X_H)$, the algebra of all bounded linear operators on a Hilbert space X_H, with involution defined as taking the adjoint of an operator. A C^*-algebra is, unless otherwise stated, supposed to carry with it the operator norm of $B(X_H)$, $|A|^2 = \sup_{(x,x)=1} (Ax, Ax)$. This norm satisfies $|A|^2 \leqslant |A^*A + B^*B|$, hence a C^*-algebra is necessarily D^*.

17. *Representation of symmetric b^*-algebras*

The purpose of this section is to prove that every real, symmetric b^*-algebra is homeomorphically isomorphic to a C^*-algebra and, if it is also B^*, isometric with respect to the B^*-norm. For complex algebras it is a classical result [12] that a symmetric B^*-algebra is isometric to a C^*-algebra. It was conjectured and later, through the contributions of several authors, proved that the symmetry condition is redundant, see [27]. (Another conjecture of [12], that B^* could be weakened to $\|x^* x\| = \|x^*\| \cdot \|x\|$, was recently proved by Glimm and Kadison [13].)

Our first remark will be that in the real case the assumption of symmetry cannot be waived entirely. We notice:

Proposition 17.1. *Every complete C^*-algebra with identity is symmetric.*

Proof. Let the C^*-algebra be A and its identity p. This is a self-adjoint projection that commutes with every element of A. Then we can regard A as a closed *-subalgebra of $B(pX_H)$ rather than of $B(X_H)$. Since A now contains the identity of $B(pX_H)$ a standard argument, see [23, I, p. 299], gives the result.

The complex numbers C, as a real algebra, is commutative, hence involution can be defined as the identity map. The usual absolute value is a B^*-norm but C is obviously not symmetric. Then (Proposition 17.1) it cannot be C^*; the answer to the question [24, p. 181] if a real B^*-algebra is necessarily C^*, is negative.

The decisive step in the proof of the announced result is the extension Theorem 17.5. In preparation for this we prove three lemmas. The line of argument will be in principle that of the classical [12] but modifications, sometimes quite significant, have to be done to take care of the real scalar case.

The sets of "positive" and "semi-positive" elements are defined

$$P^+ = \{x;\quad x^* = x \quad \text{and} \quad \sigma_A(x) > 0\},$$

$$P = \{x;\quad x^* = x \quad \text{and} \quad \sigma_A(x) \geqslant 0\}.$$

Lemma 17.2. *In a symmetric real *-algebra with identity*

(*a*) *$\sigma_A(x)$ is real for every $x \in H$,*
(*b*) *$x^*x \in P$ for every x.*

Proof. We recall that in a real algebra A with identity e the complex number $\alpha + i\beta$ belongs to $\sigma_A(x)$ if and only if $x' = (x - \alpha e)^2 + \beta^2 e$ is singular. For $\beta \neq 0$ we have that

$$\frac{x'}{\beta^2} = e + \left(\frac{x}{\beta} - \frac{\alpha e}{\beta}\right)^2 = e + \left(\frac{x}{\beta} - \frac{\alpha e}{\beta}\right)^* \left(\frac{x}{\beta} - \frac{\alpha e}{\beta}\right)$$

which is regular according to the symmetry assumption. This proves (*a*). Now we know that x^*x, being hermitian, has real spectrum. If $-\varkappa^2 \in \sigma_A(x^*x)$, $\varkappa$ real, then with $x_1 = \varkappa^{-1}x$ we have $0 \in \sigma_A(e + x_1{}^* x_1)$ contrary to the symmetry assumption. Therefore $\sigma_A(x^*x)$ is non-negative.

Lemma 17.3. *In a Banach *-algebra with identity and continuous involution there exists to every element $x \in P^+$ an element $y \in P^+$ such that $y^2 = x$.*

Proof. We can assume that $\sigma_A(x) \subset (0,1)$ (open interval) and then, for $z = e - x$, $\sigma_A(z) \subset (0,1)$. We define the sequence of polynomials

$$P_n(\xi) = \sum_{i=0}^{n} (-1)^i \binom{\frac{1}{2}}{i} \xi^i$$

for which holds that

(*a*) $\lim\limits_{n\to\infty} P_n(\xi) = \sqrt{1-\xi}$ for $\xi \in (-1, 1)$,

the convergence being uniform on every inner subinterval.

(*b*) $P_0(\xi) = 1$ and $P_n(\xi)$ is a decreasing sequence for each positive ξ, since the coefficients of P_n (except the first) are negative.

If we define $v_n = P_n(z)$, v_n is hermitian and since $\nu(z) < 1$ there exists a y such that

$$\lim_{n\to\infty} v_n = y, \quad y^* = y \quad \text{and} \quad y^2 = e - z = x.$$

According to (*a*) and (*b*) above, P_n maps the interval $(0,1)$ into itself and the spectral mapping theorem gives

$$\sigma_A(v_n) = \sigma_A(P_n(z)) = P_n(\sigma_A(z)) \subset P_n((0,1)) \subset (0,1).$$

Since y and v_n commute we can apply a continuity theorem for the spectrum [24, p. 36] and obtain $\sigma_A(y) \geqslant 0$. But $0 \in \sigma_A(y)$ means that y is singular which is impossible when $y^2 = x$ is regular. Thus it is proved that $y \in P^+$.

Lemma 17.4. *In a symmetric Banach *-algebra with identity and continuous involution $x_1, x_2 \in P$ implies $x_1 + x_2 \in P$.*

Proof. (Well known [23, IX, p. 302].) We show that $z = x_1 + x_2 + 2\alpha e$ is regular for $\alpha > 0$. From Lemma 17.3 $z = (x_1 + \alpha e) + (x_2 + \alpha e) = y_1^2 + y_2^2 = y_1^2 (e + y_1^{-2} y_2^2)$ with y_1, $y_2 \in P^+$. But $\sigma_A(y_1^{-2} y_2^2) = \sigma_A(y_1^{-1} u^* u y_1) = \sigma_A(u^* u) \geq 0$, $(u = y_2 y_1^{-1})$ because of the symmetry. Thus z is the product of regular elements, hence regular.

We are ready to state the important extension theorem for positive functionals.

Theorem 17.5. *Let A be a symmetric Banach *-algebra with identity and continuous involution such that the only topologically nilpotent hermitian element is 0. To every element $a \in A$ belongs a positive functional F_a with norm 1, satisfying $F_a(a^*a) = \nu(a^*a)$.*

Remark. Since the norm of any positive functional F is $F(e)$ [23, p. 190] it is independent of the choice of (natural, admissible) norm for A.

Proof. If $x \in P$ and $-x \in P$ then $\sigma_A(x) = \{0\}$, x is topologically nilpotent and $x = 0$ according to the assumption. Together with Lemma 17.4 this shows that P is a proper convex cone in the real space H. Moreover, e is an interior element of P. Now let S_a be the subspace of H spanned by the elements e and a^*a. On S_a we define F_a as

$$F_a(\lambda e + \alpha a^* a) = \lambda + \alpha \nu(a^* a).$$

It is clear that F_a is non-negative on $S_a \cap P$ and since $S_a \cap P$ also contains the interior point e of P, we can apply a standard extension theorem [23, p. 63] and get F_a defined on the whole of H and taking non-negative values on P. Finally we extend F_a to all of A by defining

$$F_a(x) = F_a\left(\frac{x^* + x}{2}\right)$$

and this F_a clearly has the stated properties.

We can now give the main representation theorem.

Theorem 17.6. *A complete, symmetric real b^*-algebra is homeomorphically *-isomorphic to a C^*-algebra.*

Proof. Lemmas 16.3 and 16.4 show that we can adjoin identity, if necessary, without affecting the assumptions. Hence we assume identity from here on.

For a b^*-norm and $h \in H$, $\nu(h) \geq \beta^{-1} \|h\|$, hence $\nu(h) = 0$ implies $h = 0$ and the assumptions of Theorem 17.5 are satisfied. For a certain $a \in A$, take F_a according to that theorem and form the inner product $(x, y)_a = F_a(y^* x)$. From here we proceed in a well-known manner. Let $N_a = \{x;\ (x, x)_a = 0\}$, then N_a is a left ideal and we can form $X_a = A - N_a$ as left modules. The desired Hilbert space X_H is the completion of the l^2-normed direct sum of the X_a's. The left regular representation of A induces (via reduction to X_a, direct sum representation and extension by continuity) a *-representation $b \to T_b$ of A on X_H. We further get

$$\beta^{-\frac{1}{2}} \|b\| \leq |T_b| \leq \|b\|,$$

where $\|\cdot\|$ is any given b^*-norm and β its b^*-constant. This shows that the representation is a homeomorphism. Then A is homeomorphically *-isomorphic to its image by the representation and the theorem is proved.

Corollary 17.7. *A complete symmetric real B^*-algebra is isometrically (with respect to the B^*-norm) $*$-isomorphic to a C^*-algebra.*

Proof. If it is necessary to adjoin an identity, Lemma 16.3 tells that we still have a B^*-norm after doing this. The last inequality in the proof of Theorem 17.6 with $\beta=1$ gives $|T_b|=\|b\|$.

18. *Characterization of real C^*-algebras*

In this section will be shown that if the b^*-condition is strengthened to Dc^*, symmetry follows and so real C^*-algebras can be characterized by a norm condition only.

We will repeatedly use the following theorem by Arens and Kaplansky [1, Theorem 9.1], also [24, p. 191]:

A real commutative Banach $*$-algebra A with identity that has an admissible norm satisfying $\|x\|^2 \leqslant \alpha\|x^*x+y^*y\|$ for all $x, y \in A$ and some constant α, is homeomorphically $*$-isomorphic to an algebra $C(\Omega,\gamma)$.

Here Ω is a compact Hausdorff space, γ a homeomorphism of Ω into itself such that $\gamma \circ \gamma =$ identity. $C(\Omega,\gamma)$ consists of all complex-valued continuous functions such that $\overline{f(\omega)}=f(\gamma\omega)$; involution on $C(\Omega,\gamma)$ is complex conjugation.

We give two lemmas based on this result.

Lemma 18.1. *Let A be a Banach $*$-algebra with identity, continuous involution and an admissible norm satisfying $\|x\|^2 \leqslant \alpha\|x^2+y^2\|$ for commuting $x,y \in H$ and a constant α. Then*

(*a*) *$x \in P$ if and only if $\nu(e-x/C) \leqslant 1$ for all sufficiently large positive numbers C;*
(*b*) *to every $x \in H$ belongs $u, v \in P$ such that $x=u-v$, $uv=0$.*

Proof. Take $x \in H$ and let B be a maximal commutative hermitian subalgebra containing x. Then $\sigma_B(x)=\sigma_A(x)$, and moreover B is closed, hence Banach, and isomorphic to the algebra of *all* real functions on a compact space Ω, according to the Arens-Kaplansky theorem. Let f_x be the function corresponding to x. For (*a*) we notice that if $x \in P$ then $f_x(\omega) \geqslant 0$ and $|1-C^{-1}f_x(\omega)| \leqslant 1$ for all $C > \frac{1}{2}\nu(x)$. If, for some $C>0, \nu(e-x/C) \leqslant 1$, $1-C^{-1}f_x(\omega) \leqslant 1$ and $f_x(\omega) \geqslant 0$.

For (*b*) take
$$f_1(\omega)=\max\,[f_x(\omega), 0]$$
$$f_2(\omega)=-f_x(\omega)+f_1(\omega).$$

Then $f_1(\omega) \geqslant 0$, $f_2(\omega) \geqslant 0$ and $f_1(\omega)\cdot f_2(\omega)=0$. But since every continuous function belongs to an element of B we have u, v so that $f_1=f_u$, $f_2=f_v$, $x=u-v$, $uv=0$ and $u, v \in P$.

Lemma 18.2. *Let A be a real Banach $*$-algebra with identity, continuous involution and an admissible norm satisfying $\|x\|^2 \leqslant \alpha\|x^*x+y^*y\|$ for all commuting, normal $x, y \in A$ and a constant α. If $k^*=-k$ then $-k^2 \in P$.*

Proof. Let B be a maximal commutative $*$-subalgebra containing k. Then B contains inverses, in particular $\sigma_B(k)=\sigma_A(k)$. But B is also closed, hence Banach, and isomorphic to a function algebra $C(\Omega,\gamma)$. The function corresponding to k has only imaginary values, hence $\sigma_B(k)=\sigma_A(k)$ is imaginary and $\sigma_A(-k^2)$ non-negative real.

In preparation for the announced result we prove two lemmas on Dc^*-algebras.

Lemma 18.3. *In a Banach Dc^*-algebra with identity $x,y \in P$ implies $x+y \in P$.*

Proof. For the Dc^*-norm and $x \in H$ we have $\|x^2\| = \|x\|^2$ and $\nu(x) = \|x\|$, hence ν is a norm on the real vector space H. Take $C \geqslant \nu(x)+\nu(y)$. Then

$$\nu\left(e-\frac{x+y}{2C}\right) \leqslant \tfrac{1}{2}\nu\left(e-\frac{x}{C}\right)+\tfrac{1}{2}\nu\left(e-\frac{y}{C}\right) \leqslant 1$$

according to Lemma 18.1 (a), and then also $x+y \in P$.

Lemma 18.4. *Let x be an element of a Banach Dc^*-algebra with identity. If $-x^*x \in P$ then $x^*x=0$.*

Proof. It is easy to see that x^*x and xx^* have the same spectrum except possibly for 0. If we write $x=h+k$ with $h \in H$, $k \in K$ (anti-hermitian) we have

$$x^*x=2h^2+2(-k^2)+(-xx^*)$$

and, according to Lemmas 18.2 and 18.3, $x^*x \in P$. Since both x^*x and $-x^*x$ belong to P we have $\sigma(x^*x)=\{0\}$. Then $\nu(x^*x)=0$ and $x^*x=0$.

Theorem 18.5. *A real Banach Dc^*-algebra with identity is symmetric.*

Proof. We will prove that, for every $x, \sigma(x^*x) \geqslant 0$. If this is true $(e+x^*x)^{-1}$ exists and the algebra is symmetric. The proof depends in a well-known manner [27] on the preceding lemmas. Since x^*x is hermitian we can write $x^*x=u-v$ according to Lemma 18.1. Then

$$-(vx)^*(vx)=-v\,x^*x\,v=-v(u-v)v=v^3 \in P$$

and Lemma 18.4 shows that $(vx)^*(vx)=0$. Then $v^3=0$, $v=0$ and $x^*x=u \in P$, that is $\sigma(x^*x) \geqslant 0$.

An immediate consequence of Theorem 18.5 and Corollary 17.7 is now

Theorem 18.6. *A complete real Dc^*-algebra with identity is isometrically (with respect to the Dc^*-norm) $*$-isomorphic to a C^*-algebra.*

We do not know whether one or both assumptions of identity and completeness in Theorem 18.6 can be removed. However, if Dc^* is replaced by the stronger D^*, this is possible. Given a D^*-algebra A, A_1 is a D^*-algebra (Lemma 16.3). The completion $\bar{A}_1$ is also D^* since the D^*-relations hold on a dense subset. Hence A is isometrically embedded in a complete D^*-algebra with identity, and Theorem 18.6 gives

Theorem 18.7. *A real D^*-algebra is isometrically (with respect to the D^*-norm) $*$-isomorphic to a C^*-algebra.*

Corollary 18.8. *If a real $*$-algebra has a natural norm satisfying $\|x\|^2 \leqslant \|x^*x+y^*y\|$ for all x,y then it is isometrically $*$-isomorphic to a C^*-algebra.*

REFERENCES

1. ARENS, R., and KAPLANSKY, I., Topological representation of algebras. Trans. Amer. Math. Soc. *63*, 457–481 (1948).
2. ARENS, R., The problem of locally-A functions in a commutative Banach algebra A. Trans. Amer. Math. Soc. *104*, 24–36 (1962).
3. AURORA, S., On connected normed algebras whose norm satisfy a reality condition. Illinois J. Math. *6*, 488–499 (1962).
4. BLUM, E. K., The fundamental group of the principal component of a commutative Banach algebra. Proc. Amer. Math. Soc. *4*, 397–400 (1953).
5. BOAS, R. P., JR., Entire Functions. Academic Press, New York, 1954.
6. BOHNENBLUST, H. F., and KARLIN, S., Geometrical properties of the unit sphere of Banach algebras. Ann. of Math. *62*, 217–229 (1955).
7. BOURBAKI, N., Topologie générale, Chap. I, Act. Sci. Ind. 1142, Paris, 1951.
8. CAIRNS, S. S., Introductory Topology. The Ronald Press Co., New York, 1961.
9. CIVIN, P., and YOOD, B., Involutions in Banach algebras. Pacific. J. Math. *9*, 415–436 (1959).
10. DIEUDONNÉ, J., Complex structure on real Banach spaces. Proc. Amer. Math. Soc. *3*, 162–164 (1952).
11. GELFAND, I. M., Normierte Ringe. Mat. Sbornik *9*, 3–24 (1941).
12. GELFAND, I. M., and NAIMARK, M. A., On the embedding of normed rings into the ring of operators in Hilbert space. Mat. Sbornik *12*, 197–213 (1943).
13. GLIMM, J. G., and KADISON, R. V., Unitary operators in C^*-algebras. Pacific. J. Math. *10*, 547–556 (1960).
14. HUREWITZ, W., and WALLMAN, H., Dimension Theory. Princeton University Press, Princeton, 1948.
15. INGELSTAM, L., A vertex property for Banach algebras with identity. Math. Scand. *11*, 22–32 (1962).
16. —— Hilbert algebras with identity. Bull. Amer. Math. Soc. *69*, 794–796 (1963).
17. —— Non-associative normed algebras and Hurwitz' problem, Ark. Mat. *5*, 231–238 (1964).
18. JACOBSON, N., Structure of rings. Amer. Math. Soc. Coll. Publ. XXXVII, Providence, 1956.
19. KADISON, R. V., A representation theory for commutative topological algebra. Memoirs of Amer. Math. Soc. No. 7, New York, 1951.
20. KAPLANSKY, I., Normed algebras, Duke. Math. J. *16*, 399–418 (1949).
21. LORCH, E. R., The theory of analytic functions in normed abelian vector rings. Amer. Math. Soc. *54*, 414–425 (1943).
22. LUMER, G., and PHILLIPS, R. S., Dissipative operators in a Banach space. Pacific. J. Math. *11*, 679–698 (1961).
23. NAIMARK, M. A., Normed Rings. Nordhoff, Groningen, 1959.
24. RICKART, C. E., General Theory of Banach Algebras. Van Nostrand, Princeton, 1960.
25. ROYDEN, H. L., On the multiplicative groups of function algebras. Technical Report No. 18, Stanford University, 1960.
26. —— Function algebras. Bull. Amer. Math. Soc. *69*, 281–298 (1963).
27. SCHATZ, J. A., Review of paper by Fukamiya. Math. Reviews *14*, 884 (1952).
28. SCHREIER, O., Die Verwandschaft stetiger Gruppen im Grossen. Hamburg. Abh. Math. Seminar *5*, 233–244 (1926).
29. SEGAL, I. E., Postulates for general quantum mechanics. Ann. of Math. *48*, 930–948 (1947).
30. YOOD, B., Seminar on Banach algebras. University of California Notes, Berkeley, 1956–57.

The Royal Institute of Technology, Stockholm

Tryckt den 11 mars 1964

Uppsala 1964. Almqvist & Wiksells Boktryckeri AB