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Estimates of the age of a heat distribution

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ABSTRACT

The paper deals with the possibility to solve the heat equation backwards in time. More specifically, we treat the following problem. Given the temperature at a finite number of points of a homogeneous bar, how old can the heat distribution be? In the case that the temperature is given at equidistant points x_i, the problem is completely solved. In the case of nonequidistant x_i we find an upper bound for the age. Such a bound is also obtained when the information about the heat distribution is given by the value of a finite number of linear functionals.

I. Introduction

We consider the heat distribution (temperature distribution) in a homogeneous bar of infinite length (coordinate x) as a function of time (t). Our heat distributions will be considered as positive measures $u_t(x)$.

The fundamental solution of the heat equation ($\partial^2 u/\partial x^2 = \partial u/\partial t$) is

$$\psi_t(x) = (1/2\sqrt{\pi t}) \exp(-x^2/4t) \quad (t>0). \tag{1}$$

An "initial heat distribution" u_0 at $t=0$ gives the following distribution at the time t:

$$u_t = \psi_t * u_0. \tag{2}$$

We shall be concerned with problems connected with solving the heat equation backwards in time, viz. with the following problem: If v is a bounded positive measure, for which t does there exist a bounded positive measure u_0 satisfying

$$v = \psi_t * u_0? \tag{3}$$

When $t \to 0$, ψ_t approaches the Dirac measure at the origin, so for $t=0$, (3) has the solution $u_0 = v$. When $t \to \infty$, $\psi_t * u_0 \to 0$ for every x, so (3) has no solution for large t if $v \neq 0$. Furthermore, we have

$$\psi_{t_1} * \psi_{t_2} = \psi_{t_1+t_2}, \tag{4}$$

so if (3) has a solution u_0 for $t=\tau$, it has the solution $u_0 * \psi_{\tau-\varkappa}$ for a time $\varkappa \leqslant \tau$.

Thus, it is meaningful to ask for the largest interval $(0, t)$ in which (3) has a solution.

In our problems, however, the information about v will be incomplete and given by n real numbers only. In sections II–IV the information is the values $v(x_i)$ at n points x_i. In section V, we assume the values of n linear functionals of v to be known, which is a more realistic situation from a physical point of view.

II. Formulation of Problem I

Let the information about v be given by the values $a_i = v(x_i)$ at $x_1 < x_2 < x_3 < \dots < x_n$, so that we have the equations

$$a_i = \psi_t * u_0(x_i) \quad (1 \leq i \leq n). \tag{5}$$

If $u_0 \equiv 0$ we have $a_i = 0$ $(1 \leq i \leq n)$. If $u_0 \not\equiv 0$ and $t > 0$ we have $a_i > 0$, $(1 \leq i \leq n)$, since $\psi_t > 0$ for $t > 0$. Thus, if $a_i = 0$ for some i but not for all we must have $t = 0$. Further, for $n = 2$, (5) has a solution for any $t \geq 0$ consisting of a single Dirac measure of suitable size and position (both depending on t). We pose

Problem I. *For given $v(x_i) = a_i > 0$ $(1 \leq i \leq n;\ n \geq 3)$, find the supremum T of all t for which there exists a positive bounded u_0 satisfying* (5).

For t fixed, our problem is a finite moment problem. We take the following condition for existence of a solution to this problem from Rogosinski (1958), Theorem 1 and Corollary 1:

Theorem R. *There exists a positive u_0 satisfying* (5) *if and only if the point $a = (a_1, a_2, \dots, a_n) \in R^n$ is in the hull cone*[1] *P_t of the curve*

$$p_t(x) = (\psi_t(x_1 - x), \psi_t(x_2 - x), \dots, \psi_t(x_n - x)), \quad -\infty < x < +\infty. \tag{6}$$

By this theorem we have

$$T = \sup \{t : a \in P_t\}. \tag{7}$$

We shall investigate the properties of P_t.

Lemma 1. $P_{t_2} \subset P_{t_1}$ *if* $t_1 \leq t_2$. (8)

Proof. If $y \in P_{t_2}$ there exists a positive measure μ such that $y = \mu * p_{t_2}$. Then we have by (4) $y = \mu * p_{t_2} = \mu * \psi_{t_2 - t_1} * p_{t_1}$. Since $\mu * \psi_{t_2 - t_1}$ is a positive measure the lemma is proved.

Since $\psi_t(x) \geq 0$, P_t is a subset of the positive orthant in R^n. If $x \neq 0$, $\psi_t(x) \to 0$ when $t \to 0$, implying that the ray from the origin through the point $p_t(x_i)$ approaches the y_i-axis of R^n as $t \to 0$. Thus, P_t is monotonically increasing to the whole positive orthant when $t \to 0$. Since we have assumed $a_i > 0$ $(1 \leq i \leq n)$, there exists an $\varepsilon > 0$ such that $a \in P_\varepsilon$.

When $t \to \infty$, P_t decreases to a subcone, say P_∞ of the positive orthant. The cone P_∞ is the set of points a for which (5) has a solution for all t. P_∞ is described by theorem 1 in the case of equidistant x_i.

By Lemma 1 we have

[1] The hull cone of a set A is defined as the smallest convex cone with vertex 0 that contains A.

$$\sup \{t : a \in P_t\} = \inf \{t : a \notin P_t\} \tag{9}$$

if we interpret the right-hand side of (9) as ∞ when $a \in P_\infty$.

III. Equidistant data

Assume that the values $a_i = v(x_i)$ are obtained at equidistant points, i.e. assume $x_i = b + ih$ $(1 \leqslant i \leqslant n)$, where b and h are constants. Since the position of the origin on the x-axis is immaterial, we put $b = 0$.

Theorem 1. *Assume $a_i > 0$ $(1 \leqslant i \leqslant n)$ and define $\tau = \exp(h^2/4t)$ for $t > 0$ so that $t = h^2/4 \log \tau$. Consider the quadratic forms*

$$Q_1(\tau) = \Sigma_i \Sigma_m a_{i+m} \tau^{(i+m)^2} \xi_i \xi_m, \quad 1 \leqslant i,\, m \leqslant [n/2],$$

$$Q_2(\tau) = \Sigma_i \Sigma_m a_{i+m-1} \tau^{(i+m-1)^2} \xi_i \xi_m, \quad 1 \leqslant i,\, m \leqslant [(n+1)/2].$$

Let τ_0 be the smallest $\tau \geqslant 1$ such that the forms Q_1 and Q_2 are both positive semidefinite for $\tau_0 \leqslant \tau < \infty$. If $\tau_0 > 1$, we have $T = h^2/4 \log \tau_0$. If $\tau_0 = 1$, then $T = \infty$, that is $a \in P_\infty$ and the equations (5) are solvable for all $t \geqslant 0$.

Proof. A symmetric matrix is positive definite if all its diagonal subdeterminants are positive. For the matrices of the forms Q_1 and Q_2 these determinants are polynomials in τ and it is easily shown that their leading coefficients are positive. By the definition of τ_0 we then know that there exists a τ_1 such that $Q_1(\tau)$ and $Q_2(\tau)$ both are strictly positive for $\tau_0 < \tau < \tau_1$.

Now, we write out the equations (5) with $x_i = ih$

$$a_i = (1/2\sqrt{\pi t}) \int_{-\infty}^{+\infty} \exp(-(ih - x)^2/4t)\, u_0(dx). \tag{5'}$$

Rearranging (5′), we get

$$a_i = \exp(-i^2h^2/4t) \int_{-\infty}^{+\infty} \exp(ihx/2t)(1/2\sqrt{\pi t}) \exp(-x^2/4t)\, u_0(dx).$$

Now, the measure $w = (1/2\sqrt{\pi t}) \exp(-x^2/4t) u_0$ is positive if and only if u_0 is positive, so we have the question: for which t does there exist a positive w satisfying

$$a_i = \exp(-i^2h^2/4t) \int_{-\infty}^{+\infty} \exp(ihx/2t)\, w(dx).$$

We make a change of variable in the integral by putting $\exp(hx/2t) = \eta$. Since η is a monotonic function of x, the positive measure $w(dx)$ changes to a positive measure, say $\mu(d\eta)$, and we get

$$a_i = \exp(-i^2h^2/4t) \int_0^{+\infty} \eta^i \mu(d\eta),$$

or

$$a_i \tau^{i^2} = \int_0^{+\infty} \eta^i \mu(d\eta) \quad (1 \leqslant i \leqslant n). \tag{10}$$

This setting of the problem is known as Stieltjes' moment problem. A sufficient condition for the possibility of representing the quantities $a_i\tau^{i^2}$ by a positive measure as in (10) is the strict positivity of the forms Q_1 and Q_2. (See e.g. Krein, 1951.) Thus, if $\tau_0<\tau<\tau_1$ we have a representation (10), proving the solvability of (5) for τ in the open interval (τ_0, τ_1). By Lemma 1, however, (5) is then solvable for every $\tau>\tau_0$. Theorem R shows that the solution u_0 can be taken as a finite sum of Dirac measures. Conversely, if (5) has a solution $\bar{\mu}$ for $\bar{\tau}>1$ we have

$$Q_1(\bar{\tau}) = \int_0^\infty (\sum_m \xi_m \eta^m)^2 \bar{\mu}(d\eta) \geqslant 0,$$

and

$$Q_2(\bar{\tau}) = \int_0^\infty (\sum_m \xi_m \eta^m)^2 (1/\eta) \bar{\mu}(d\eta) \geqslant 0.$$

implying $\bar{\tau}\geqslant\tau_0$. In the case $\tau_0>1$ there is consequently no solution for τ in the open interval $(1, \tau_0)$, proving that $T=h^2/4 \log \tau_0$. In the case $\tau_0=1$ the eqs. (5) are solvable for every $t\geqslant 0$ implying $T=\infty$.

Remark. The positivity of Q_1 or Q_2 is a condition on an odd number of consecutive a_i. For n odd, $Q_1\geqslant 0$ is a condition on $a_2, a_3, \ldots, a_{n-1}$ and $Q_2\geqslant 0$ a condition on $a_1, a_2, \ldots, a_n$. For n even, $Q_1\geqslant 0$ is a condition on $a_2, a_3, \ldots, a_n$ and $Q_2\geqslant 0$ a condition on $a_1, a_2, \ldots, a_{n-1}$.

Example 1. Let $n=3$ so that a_1, a_2 and a_3 are given. The positivity of Q_1 and Q_2 then gives the inequalities $a_2\tau^4\geqslant 0$ and $a_1a_3\tau^{10}-a_2^2\tau^8\geqslant 0$. If $a_2^2>a_1a_3$, we get the estimate $\tau_0=a_2/\sqrt{a_1a_3}$ or $T=h^2/4 \log (a_2/\sqrt{a_1a_3})$. If $a_2^2\leqslant a_1a_3$, $T=\infty$.

Example 2. This example shows that there does not necessarily exist a solution of (5) for $t=T$. Let $n=4$ so that a_1, a_2, a_3 and a_4 are given. The positivity of Q_1 and Q_2 gives

$$a_1a_3\tau^{10}-a_2^2\tau^8\geqslant 0 \quad \text{and} \quad a_2a_4\tau^{20}-a_3^2\tau^{18}\geqslant 0.$$

Suppose $a_2^2/a_1a_3>\max(1, a_3^2/a_2a_4)$ so that $\tau_0^2=a_2^2/a_1a_3$. Krein (1951) shows that if there exists a representation (10), there exists one of the form

$$a_i\tau^{i^2} = \varrho_1\eta_1^i + \varrho_2\eta_2^i \ (1\leqslant i\leqslant 4),\ \varrho_1>0,\ \varrho_2>0,\ \eta_1\leqslant\eta_2. \tag{11}$$

The four eqs. (11) are actually just sufficient to determine the four quantities ϱ_1, ϱ_2, η_1 and η_2. Combining the eqs. (11) for $(1\leqslant i\leqslant 3)$ we get

$$\varrho_1\varrho_2\eta_1\eta_2(\eta_1-\eta_1)^2 = \tau^8(a_1a_3\tau^2-a_2^2).$$

For $\tau=\tau_0$ these expressions equal 0. Thus, either $\eta_1=0$ or $\eta_1=\eta_2$, so the representation has only one pointmass and has the form $\varrho_2\eta_2^i$. This representation does not satisfy the equation for $i=4$.

IV. Nonequidistant data

When the points x_i are not equidistant, we only give an estimate that takes into account three points $a_i=v(x_i)$. Of course, if more than three values are known, those three that give the smallest estimate T should be used.

Theorem 2. *Let $a_i = v(x_i) > 0$ $(1 \leqslant i \leqslant 3)$, be the values of v at three points $x_1 < x_2 < x_3$. Then,*

$$T = \frac{(1/4)(x_3 - x_1)(x_3 - x_2)(x_2 - x_1)}{(x_3 - x_1)\log a_2 - (x_3 - x_2)\log a_1 - (x_2 - x_1)\log a_3}$$

if this expression is positive. In other cases $T = \infty$.

Proof. By (7) and (9) T can be characterized by the fact that if $t > T$, then there exists a plane in R^3 separating a from P_t. Let

$$\alpha^T y = \alpha_1 y_1 + \alpha_2 y_2 + \alpha_3 y_3 = 0$$

be a plane in R^3. We normalize it by putting $\alpha_2 = -1$. We shall try to find α_1 and α_3 so that

$$f(x) = \alpha^T p_t(x) \geqslant 0 \quad \text{for all } x$$

and
$$\alpha^T a \leqslant 0.$$

If this is possible, we obtain a plane that separates (not strictly) a and P_t. Further, we shall choose α_1 and α_3 so that the plane supports the cone P_t along the ray through $p_t(\Theta)$ for some Θ, giving the equation

$$f(\Theta) = \alpha^T p_t(\Theta) = 0. \tag{12}$$

Since we require $f(x) \geqslant 0$, we must also have

$$f'(\Theta) = \alpha^T p_t'(\Theta) = 0. \tag{13}$$

The equations (12) and (13) determine α_1 and α_3 as functions of t and Θ:

$$\alpha_1(t, \Theta) = [(x_3 - x_2)/(x_3 - x_1)] \exp((x_1 - x_2)(x_1 + x_2 - 2\Theta)/4t),$$

$$\alpha_3(t, \Theta) = [(x_2 - x_1)/(x_3 - x_1)] \exp((x_3 - x_2)(x_3 + x_2 - 2\Theta)/4t).$$

To see if there exist t and Θ so that

$$\alpha^T a = \alpha_1(t, \Theta) a_1 - a_2 + a_3(t, \Theta) a_3 \leqslant 0$$

we seek $\min_\Theta \alpha^T a$ by solving $(\partial/\partial\Theta)\alpha^T a = 0$. We find a unique solution

$$\Theta_0 = \tfrac{1}{2}x_1 + \tfrac{1}{2}x_3 + (2t \log a_3/a_1)/(x_3 - x_1) \tag{14}$$

and get

$$\min_\Theta \alpha^T a = [a_1^{(x_3 - x_2)} a_3^{(x_2 - x_1)}]^{1/(x_3 - x_1)} \exp\left(\frac{(x_2 - x_1)(x_3 - x_2)}{4t}\right) - a_2.$$

For some a_1, a_2 and a_3, this expression is not negative even when $t \to \infty$, meaning that there does not exist any plane separating a and P_t even for large t. From such a_i, our theorem gives no estimate. For other a_i, the expression is negative for large t. Then T is the value of t for which $\min_\Theta \alpha^T a = 0$.

Remark. For three equidistant points, Theorem 2 gives the same estimate as Theorem 1. Cf. example 1.

Remark. For equidistant x_i, the method of proof for theorem 2 can be generalized to handle 5, 7, etc. points by introducing 2, 3, etc. points of contact like Θ. Such a generalization gives exactly the estimate of theorem 1. Also for nonequidistant x_i, the generalization of the above method is usable, but explicit expressions corresponding to (14) are not obtained.

V. Practical measurements

Since it is hard to construct a thermometer that gives the temperature $v(x_i)$ at a single point, we shall consider the situation that the information about v is given by the values of n ($\geqslant 3$) linear functionals

$$d_i = \int v(x_i - x)\nu(dx) \quad (1 \leqslant i \leqslant n), \tag{15}$$

where ν is a positive measure with total mass equal to 1. The measure ν is thought to describe the measuring instrument and the values d_i are obtained when the instrument is "centered" at the points x_i. Physicist often assume

$$\nu(dx) = (1/\sqrt{2\pi}\,\sigma)\exp(-x^2/2\sigma^2)\,dx = \psi_{\sigma^2/2}(x)\,dx \tag{16}$$

for some $\sigma > 0$. The limit case when $\sigma \to 0$ corresponds to $\nu \to \delta$ and $d_i \to v(x_i) = a_i$, that is the case in secs II–IV. We pose

Problem II. *For given $d_i > 0$ ($1 \leqslant i \leqslant n$, $n \geqslant 3$), find an upper bound for the supremum $\tilde{T}$ of all t for which there exists a positive bounded u_0 satisfying* (15).

Remark. This problem has not always a solution since there may not exist any t for which the d_i ($1 \leqslant i \leqslant n$) is a conceivable set of data. Cf. the corollary of Lemma 3.

The equations (15) can also be written

$$d_i = v * \nu(x_i) \quad (1 \leqslant i \leqslant n) \tag{15'}$$

or by (3)

$$d_i = u_0 * \psi_t * \nu(x_i) \quad (1 \leqslant i \leqslant n). \tag{15''}$$

Let

$$\varkappa_t = \psi_t * \nu \tag{17}$$

so that

$$d_i = \varkappa_t * u_0(x_i) \quad (1 \leqslant i \leqslant n). \tag{18}$$

By Rogosinski (1958), theorem 1 and corollary 1 we get

Theorem R′. *There exists a positive bounded u_0 satisfying* (18) *if and only if the point $d = (d_1, d_2, \ldots, d_n) \in R^n$ is in the hull cone K_t of the curve*

$$k_t(x) = (\varkappa_t(x_1 - x), \varkappa_t(x_2 - x), \ldots, \varkappa_t(x_n - x)), \quad -\infty < x < +\infty.$$

By this theorem we have

$$\tilde{T} = \sup \{t : d \in K_t\}. \tag{19}$$

Most of the results of section II have their counterparts here.

Lemma 2. $$K_{t_2} \subset K_{t_1} \quad \text{if} \quad t_1 \leqslant t_2. \tag{20}$$

Proof. If $y \in K_{t_2}$ there exists a positive measure μ such that $y = \mu * k_{t_2} = \mu * p_{t_2} * \nu = \mu * \psi_{t_2-t_1} * p_{t_1} * \nu = \mu * \psi_{t_2-t_1} * k_{t_1}$. Since $\mu * \psi_{t_2-t_1}$ is a positive measure the lemma is proved.

By this lemma we have, as in section II, that

$$\tilde{T} = \sup \{t : d \in K_t\} = \inf \{t : d \notin K_t\} \tag{21}$$

if we interpret the infimum as ∞ when $d \in K_t$ for all $t \geqslant 0$.

Define

$$\vartheta = \sup \{t : \nu = \psi_t * \lambda,\ \lambda \text{ positive measure}\}.$$

The total mass of λ equals that of ν so it is equal to 1. We can think of ψ_t, ν and λ as probability distributions corresponding to random variables X_ψ, X_ν and X_λ. Taking variances, we get

$$\operatorname{Var}(X_\nu) = 2\vartheta + \operatorname{Var}(X_\lambda).$$

Since $\psi_t \to \delta$ when $t \to 0$ and $\operatorname{Var}(X_\lambda) \geqslant 0$ we have

$$0 \leqslant \vartheta \leqslant \tfrac{1}{2} \operatorname{Var}(X_\nu).$$

Of course, $\vartheta > 0$ only when ν is infinitely differentiable. For the particular ν in (16) we get $\vartheta = \frac{1}{2} \operatorname{Var}(X_\nu) = \frac{1}{2}\sigma^2$.

Lemma 3. $$K_t \subset P_{t+\vartheta}.$$

Proof. If $y \in K_t$ there exists a positive measure μ such that $y = k_t * \mu = p_t * \nu * \mu = p_t * \psi_\vartheta * \lambda * \mu = p_{t+\vartheta} * \lambda * \mu$. Since $\lambda * \mu$ is a positive measure, the lemma is proved.

Corollary. *In general, $K_0 = \lim_{t \to 0} K_t$ is not the whole positive orthant. Since Problem II has a solution if and only if $d \in K_0$, the condition $d_i > 0$ $(1 \leqslant i \leqslant n)$ is not in general sufficient for Problem II to have a solution.*

Theorem 3. *Assume $d \in K_0$ and apply Theorem 1 or 2 with $a = d$ so that an estimate T is obtained. Then,*

$$\tilde{T} \leqslant T - \vartheta.$$

Proof. For every $t < \tilde{T}$ we have $d \in K_t \subset P_{t+\vartheta}$ by Lemma 3. Then, $t + \vartheta \leqslant T = \sup \{t : d \in P_t\}$ proving the theorem.

The physical interpretation of the last formula is that the dispersion of u_0 caused by heat conduction during the time $\tilde{T}$ and the dispersion caused by the measuring instrument, which is larger than if it had been aged ϑ, together form a dispersion corresponding to heat conduction during a time T.

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REFERENCES

KREIN, M. G., The ideas of P. L. Chebyshev and A. A. Markov in the theory of limiting values of integrals and their further development, *Amer. Math. Soc. Translations*, ser. 2, no 12, 3–120 (1951).

ROGOSINSKI, W. W., Moments of non-negative mass, *Proc. Roy. Soc.* (London), A *245*, 1–27 (1958)

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