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On radial zeros of Blaschke products

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1. Introduction

Let $\mathcal{B}$ be the class of all Blaschke products defined on the open unit disc C, i.e. all functions of the form

$$B(z)=e^{i\theta}z^m\prod_k \frac{\bar{a}_k(a_k-z)}{|a_k|(1-\bar{a}_k z)},$$

where θ is real, m a nonnegative integer and $\{a_k\}$ a set of nonzero complex numbers in C, such that the series $\sum(1-|a_k|)$ converges.

A point ζ on the boundary of C (henceforth denoted by ∂C) is a radial zero of a Blaschke product B if

$$B(\zeta)=\lim_{r\to 1-0} B(r\zeta)=0.$$

In his thesis, Frostman ([2], p. 109) gave an example of a Blaschke product B; namely,

$$B(z)=\prod_{k=1}^{\infty}\frac{1-k^{-2}-z}{1-(1-k^{-2})z},$$

which has zero radial limit at $\zeta=1$. More recently, Somadasa [11] and Tanaka [12] obtained sufficient conditions in terms of the sequence $\{a_k\}$ for the corresponding Blaschke product to have a zero angular limit at a point $\zeta\in\partial C$. In the following section we will give a different sufficient condition for a point $\zeta\in\partial C$ to be a radial zero of a Blaschke product. It turns out that the conditions given by Somadasa and Tanaka are stronger than ours. We will also establish a necessary condition for ζ to be a radial zero of a Blaschke product B. In fact, we will investigate the radial and angular growth of $-\log|B(z)|$ as z approaches a radial zero of B.

In Section 3 the local results of Section 2 are used to obtain global results, while in Section 5 we improve a uniqueness theorem given in [10], p. 199.

Section 4 contains two simple lemmas.

2. Radial behavior of Blaschke products

Before stating the main result of this section, let us introduce some notation.

Let $\mathcal{H}$ be the class of functions h, continuous and nondecreasing on the interval $[0,\infty)$, such that $h(t)>0$ if $t>0$ and $t^{-1}h(t)$ is nonincreasing on $(0,\infty)$.

The subclass of $\mathcal{H}$, consisting of functions h satisfying the additional condition $h(0)=0$ will be denoted by $\mathcal{H}_0$. In particular, we will be interested in the functions $h_\alpha \in \mathcal{H}_0$, $\alpha \leqslant 1$, defined by

$$h_\alpha(t) = \begin{cases} 0 & t=0, \\ t(-\log t)^{1-\alpha} \quad \text{if} & 0<t\leqslant t_\alpha, \\ t+h_\alpha(t_\alpha)-t_\alpha & t_\alpha < t, \end{cases}$$

where t_α is chosen in the interval $(0, e^{-1})$ so that

$$\log t_\alpha + (-\log t_\alpha)^\alpha + 1 - \alpha = 0.$$

If $\alpha < 1$, the number t_α is uniquely determined by this equation, while if $\alpha = 1$ the choice of t_1 is immaterial.

If $h \in \mathcal{H}$ and $B \in \mathcal{B}$, let $L(B,h)$ be the set

$$L(B,h) = \left\{\zeta \in \partial C;\ \liminf_{r\to 1-0} \frac{1-r}{h(1-r)} \log \frac{1}{|B(r\zeta)|} = +\infty\right\}.$$

In like manner, let $L_S(B,h)$ be the set of all points $\zeta \in \partial C$ such that

$$\liminf_{\substack{z\to\zeta \\ z\in S(\zeta,\alpha)}} \frac{|z-\zeta|}{h(|z-\zeta|)} \log \frac{1}{|B(z)|} = +\infty$$

for all Stolz domains $S(\zeta,\alpha)$ defined for $0<\alpha<1$ by

$$S(\zeta,\alpha) = \{z\in C;\ |z-\zeta| \leqslant \sqrt{1-\alpha^2},\ |\arg(1-\bar{\zeta} z)| \leqslant \arcsin \alpha\}.$$

In the particular cases of $h=h_1$ and $h=h_0$, we will use the notation $Z(B)=L(B,h_1)$, $L(B)=L(B,h_0)$ and $L_S(B)=L_S(B,h_0)$. Obviously $Z(B)$ is the set of all radial zeros of B. Moreover, it is a well-established fact that if a Blaschke product has a zero radial limit at $\zeta\in\partial C$, then it also has a zero angular limit at ζ ([7]), p. 5); and, therefore, $Z(B)=L(B,h_1)=L_S(B,h_1)$.

If $B\in\mathcal{B}$ and $\zeta\in\partial C$ let

$$\sigma(B,\zeta,t) = \sum_{|a_k-\zeta|\leqslant t} (1-|a_k|), \quad (t>0)$$

be the remainders of the convergent series $\sum(1-|a_k|)$. It is convenient to introduce the sets

$$\underline{\sum}(B,h) = \left\{\zeta\in\partial C;\ \liminf_{t\to+0} \frac{\sigma(B,\zeta,t)}{h(t)} = +\infty\right\}$$

and

$$\overline{\sum}(B,h) = \left\{\zeta\in\partial C;\ \limsup_{t\to+0} \frac{\sigma(B,\zeta,t)}{h(t)} = +\infty\right\}.$$

In this notation the main result of this section can be stated as follows.

Theorem 2.1. *Let $h \in \mathcal{H}$ and let $B \in \mathcal{B}$. Then*

$$\underline{\sum}(B,h) \subset L_S(B,h) \subset L(B,h) \subset \overline{\sum}(B,h).$$

Theorem 2.1 is an immediate consequence of the following lemma.

Lemma 2.2. *Let α be a fixed number such that $0<\alpha<1$. Then there exist positive constants A_1 and A_2 such that*

$$A_1 \liminf_{t\to+0} \frac{\sigma(B,\zeta,t)}{h(t)} \leqslant \liminf_{\substack{z\to\zeta\\ z\in S(\zeta,\alpha)}} \frac{|z-\zeta|}{h(|z-\zeta|)} \log \frac{1}{|B(z)|}$$

and

$$\liminf_{r\to1-0} \frac{1-r}{h(1-r)} \log \frac{1}{|B(r\zeta)|} \leqslant A_2 \limsup_{t\to+0} \frac{\sigma(B,\zeta,t)}{h(t)}$$

for all $\zeta \in \partial C$ all $B \in \mathcal{B}$ and all $h \in \mathcal{H}$.

Proof. Let $\{a_k\}$ be the nonzero zeros of a Blaschke product B. If $|a_k-\zeta| \leqslant |z-\zeta|$, then $|1-\bar{a}_k z| \leqslant 2|z-\zeta|$. Moreover, there exists $K(\alpha)$ such that $1-|z| \geqslant K(\alpha)|z-\zeta|$ for all $z \in S(\zeta,\alpha)$. Hence, if $z \in S(\zeta,\alpha)$

$$\log \frac{1}{|B(z)|} \geqslant -\frac{1}{2} \sum_{|a_k-\zeta|\leqslant|z-\zeta|} \log\left(1-\frac{(1-|z|^2)(1-|a_k|^2)}{|1-\bar{a}_k z|^2}\right)$$

$$\geqslant \frac{1}{2}(1-|z|) \sum_{|a_k-\zeta|\leqslant|z-\zeta|} \frac{1-|a_k|}{|1-\bar{a}_k z|^2}$$

$$\geqslant \frac{K(\alpha)}{8} \frac{\sigma(B,\zeta,|z-\zeta|)}{|z-\zeta|}.$$

The first part of Lemma 2.2 follows from this inequality. To prove the second inequality in Lemma 2.2 we use the following lemma.

Lemma 2.3. *Let t be a fixed number such that $0<t<\frac{1}{3}$ and let I_t be the closed interval $[1-3t, 1-2t]$. Suppose that $h \in \mathcal{H}$, $B \in \mathcal{B}$, $B(0) \neq 0$ and $\sigma(B,1,x) \leqslant h(x)$ for $x>0$. Then there exists an absolute constant $A>0$, such that*

$$\inf_{r\in I_t} \log \frac{1}{|B(r)|} \leqslant At^{-1}h(t).$$

Proof. The proof of this lemma is based on two simple estimates of Green's potential of the mass t uniformly distributed over the interval I_t. Let

$$g(z,w) = \log\left|\frac{1-\bar{z}w}{w-z}\right|, \quad (z, w \in C)$$

be Green's function with logarithmic pole at z, and let

$$G(z) = \int_{I_t} g(z, r)\, dr$$

be the potential at the point $z \in C$.

We prove the existence of two absolute constants A_1 and A_2, such that, for $z \in C$,

$$G(z) \leqslant A_1(1-|z|) \quad \text{and} \quad G(z) \leqslant A_2 t^2(1-|z|)/|1-z|^2. \tag{2.4}$$

Simple calculations show, that if $r \in I_t$, then

$$g(z,r) = \frac{1}{2}\log\left(1+\frac{(1-r^2)(1-|z|^2)}{|r-z|^2}\right) \leqslant \frac{1}{2}\log\left(1+\frac{12t(1-|z|)}{|r-z|^2}\right), \tag{2.5}$$

If $\min_{r\in I_t}|r-z| \geqslant t$ and $r \in I_t$, then by (2.5), $g(z,r) \leqslant 6t^{-1}(1-|z|)$ and thus $G(z) \leqslant 6(1-|z|)$. If $\min_{r\in I_t}|r-z| < t$, then $t < 1-|z|$ and thus, by (2.5)

$$\begin{aligned} G(z) &\leqslant \frac{1}{2}\int_{I_t}\log\left(1+\frac{16(1-|z|)^2}{(r-|z|)^2}\right)dr \\ &\leqslant \int_{|z|}^{\infty}\log\left(1+\frac{16(1-|z|)^2}{(r-|z|)^2}\right)dr \\ &= 4(1-|z|)\int_0^\infty \log(1+x^2)\,\frac{dx}{x^2}. \end{aligned}$$

This completes the proof of the first part of (2.4).

To prove the second part of (2.4) let us first consider $z \in C$ such that $|1-z| > 4t$. If $|1-z| > 4t$ and $r \in I_t$, then $|z-r| > \frac{1}{4}|1-z|$. Hence $g(z,r) \leqslant 96t(1-|z|)/|1-z|^2$, by (2.5), and thus $G(z) \leqslant 96t^2(1-|z|)/|1-z|^2$. If $|1-z| \leqslant 4t$, then, by the first part of (2.4), $G(z) \leqslant 16A_1 t^2(1-|z|)/|1-z|^2$. The proof of (2.4) is complete.

Lemma 2.3 follows readily from (2.4). Let $\{a_k\}$ be the zeros of the Blaschhe product B. Then, by (2.4),

$$\begin{aligned} \inf_{r\in I_t}\log\frac{1}{|B(r)|} &\leqslant t^{-1}\sum_{|a_k-1|\leqslant t} G(a_k) + t^{-1}\sum_{|a_k-1|>t} G(a_k) \\ &\leqslant A_1 t^{-1}\sigma(B,1,t) + A_2 t \sum_{|a_k-1|>t}\frac{1-|a_k|}{|1-a_k|^2}. \end{aligned}$$

Since $\sigma(B,1,2^n t) \leqslant h(2^n t) \leqslant 2^n h(t)$, we have

$$\begin{aligned} \sum_{|a_k-1|>t}\frac{1-|a_k|}{|1-a_k|^2} &= \sum_{n=1}^{\infty}\sum_{2^{n-1}t<|a_k-1|\leqslant 2^n t}\frac{1-|a_k|}{|1-a_k|^2} \\ &\leqslant \sum_{n=1}^{\infty} 2^{-2n+2}t^{-2}\sigma(B,1,2^n t) \leqslant 4t^{-2}h(t). \end{aligned}$$

Hence

$$\inf_{r \in I_t} \log \frac{1}{|B(r)|} \leqslant (A_1 + 4A_2)\, t^{-1} h(t)$$

and Lemma 2.3 is proved.

Let us now prove the second part of Lemma 2.2. Without loss of generality we may assume that $\zeta = 1$ and

$$\limsup_{t \to +0} \frac{\sigma(B, 1, t)}{h(t)} = l, \quad (0 \leqslant l < +\infty).$$

Given $\varepsilon > 0$ there exists $t_0 > 0$ such that

$$\sigma(B, 1, t) \leqslant (l + \varepsilon)\, h(t) \quad \text{for} \quad 0 < t \leqslant t_0.$$

Let $\{a_k\}$ be the sequence of nonzero zeros of B. Put

$$B_1(z) = \prod_{|a_k - 1| \leqslant t_0} \frac{\bar{a}_k (a_k - z)}{|a_k|\,(1 - \bar{a}_k z)}$$

and

$$B_2(z) = B(z)/B_1(z).$$

Then $\sigma(B_1, 1, x) \leqslant (l + \varepsilon)\, h(x)$ for $x > 0$. Hence, by Lemma 2.3 applied to B_1 and the function $(l + \varepsilon)\, h$,

$$\inf_{r \in I_t} \frac{1 - r}{h(1 - r)} \log \frac{1}{|B_1(r)|} \leqslant \frac{3t}{h(3t)} \cdot \frac{A(l + \varepsilon)\, h(t)}{t} \leqslant 3A(l + \varepsilon)$$

whenever $0 < t < \frac{1}{3}$. It follows readily from this inequality that

$$\liminf_{r \to 1-0} \frac{1 - r}{h(1 - r)} \log \frac{1}{|B_1(r)|} \leqslant 3A(l + \varepsilon).$$

Since all nonzero zeros a_k of B_2 satisfy the inquality $|1 - a_k| > t_0$, the limit

$$B_2(1) = \lim_{r \to 1} B_2(r)$$

exists and $|B_2(1)| = 1$. Hence,

$$\liminf_{r \to 1-0} \frac{1 - r}{h(1 - r)} \log \frac{1}{|B(r)|} = \lim_{r \to 1-0} \frac{1 - r}{h(1 - r)} \log \frac{1}{|B_2(r)|}$$

$$+ \liminf_{r \to 1-0} \frac{1 - r}{h(1 - r)} \log \frac{1}{|B_1(r)|} \leqslant 3A(l + \varepsilon);$$

and since $\varepsilon > 0$ is arbitrarily chosen, we have

$$\liminf_{r\to 1-0} \frac{1-r}{h(1-r)} \log \frac{1}{|B(r)|} \leqslant 3\,Al.$$

This completes the proof of Lemma 2.2.

Let us single out two special cases of Theorem 2.1 corresponding to $h=h_1$ and $h=h_0$, respectively.

Corollary 2.6. *If* $B\in\mathcal{B}$ *then* $\underline{\sum}(B,h_1)\subset Z(B)\subset\overline{\sum}(B,h_1)$.

Corollary 2.7. *If* $B\in\mathcal{B}$ *then* $\underline{\underline{\sum}}(B,h_0)\subset L_S(B)\subset L(B)\subset\overline{\overline{\sum}}(B,h_0)$.

Theorem 2.1 gives a sufficient condition for $\zeta\in\partial C$ to be in $L_S(B,h)$. In the following theorem we establish a simpler, but stronger sufficient condition.

Theorem 2.8. *Let* $h\in\mathcal{H}$ *and* $B\in\mathcal{B}$. *If there exists a subsequence* $\{\alpha_k\}_1^\infty$ *of zeros of the Blaschke product* B, *such that* $\alpha_k\to\zeta\in\partial C$ *as* $k\to+\infty$ *in such a manner that*

$$\lim_{k\to+\infty} h(|\alpha_k-\alpha_{k+1}|)/(1-|\alpha_k|)=0, \tag{2.9}$$

then $\zeta\in L_S(B,h)$.

Theorem 2.8 with $h=h_1$ is due to Somadasa ([11], p. 296).

Proof. It follows from 2.9 that $(1-|\alpha_k|)/(1-|\alpha_{k+1}|)\to 1$ as $k\to+\infty$, so we may assume that

$$h(|\alpha_k-\alpha_{k+1}|)/(1-|\alpha_{k+1}|)|\to 0$$

as $k\to+\infty$. Given $\varepsilon>0$, there exists an integer n such that $h(|\alpha_k-\alpha_{k+1}|)\leqslant \varepsilon(1-|\alpha_{k+1}|)$ for all $k\geqslant n$. Given $0<t<|\alpha_n-\zeta|$, let m be the smallest integer such that $|\alpha_k-\zeta|\leqslant t$ for all $k\geqslant m$. Then for $k\geqslant m-1$ we have

$$|\alpha_k-\alpha_{k+1}|\leqslant|\alpha_k-\zeta|+|\alpha_{k+1}-\zeta|<2\,|\alpha_{m-1}-\zeta|.$$

Hence

$$\frac{h(|\alpha_k-\alpha_{k+1}|)}{|\alpha_k-\alpha_{k+1}|}\geqslant\frac{h(2|\alpha_{m-1}-\zeta|)}{2\,|\alpha_{m-1}-\zeta|}$$
$$\geqslant\frac{h(|\alpha_{m-1}-\zeta|)}{2\,|\alpha_{m-1}-\zeta|}$$

and therefore

$$\sigma(B,\zeta,t)\geqslant\sum_{k=m-1}^{\infty}(1-|\alpha_{k+1}|)$$
$$\geqslant\varepsilon^{-1}\sum_{k=m-1}^{\infty}h(|\alpha_k-\alpha_{k+1}|)$$
$$\geqslant(2\,\varepsilon)^{-1}\sum_{k=m-1}^{\infty}|\alpha_k-\alpha_{k+1}|\,\frac{h(|\alpha_{m-1}-\zeta|)}{|\alpha_{m-1}-\zeta|}$$
$$\geqslant(2\,\varepsilon)^{-1}h(|\alpha_{m-1}-\zeta|)\;\geqslant(2\,\varepsilon)^{-1}h(t).$$

Thus $\sigma(B,\zeta,t)\geqslant(2\varepsilon)^{-1}h(t)$ if $0<t<|\alpha_n-\zeta|$, i.e. $\zeta\in\underline{\sum}(B,h)$. By Theorem 2.1, $\zeta\in L_S(B,h)$ and Theorem 2.8 is proved.

To see that Theorem 2.8 is weaker than Theorem 2.1, let us consider Frostman's example, quoted earlier. Application of Theorem 2.1 shows that $1\in L_S(B,h)$ for all $h\in\mathcal{H}$, where $h=o(\sqrt{t})$ as $t\to+0$, while $1\notin L(B,\sqrt{t})$. However, if $h(t)=t^\alpha$, $\frac{1}{2}<\alpha\leqslant\frac{2}{3}$, then there exists no subsequence of zeros satisfying the hypothesis of Theorem 2.8.

To see that Somadasa's result (Theorem 2.8 with $h=h_1$) is weaker than Corollary 2.6, consider the Blaschke product B with zeros $1-e^{-k}$ of multiplicity k, $k=1,2,3,\dots$. It follows easily from Corollary 2.6 that $1\in Z(B)$. However, it is impossible to find a subsequence $\{\alpha_k\}$ of zeros of B such that (2.9) with $h=h_1$ holds.

Tanaka's result ([12], p. 472), is contained in Theorem 2.8 with $h=h_1$.

The relation between the radial growth of $-\log|B(r\zeta)|$ and the remainder $\sigma(B,\zeta,t)$ bears a close resemblance to the relation between the radial growth of a nonnegative harmonic function u and its Poisson-Stieltjes measure $d\mu$ (cf. Lemma 2.2 and Lemma 4.2). Our next theorem emphasizes this resemblance.

For $B\in\mathcal{B}$ let $R(B)$ be the set

$$R(B)=\{\zeta\in\partial C;\ \lim_{r\to1-0}|B(r\zeta)|=1\},$$

and let $\sum(B)$ be the set

$$\sum(B)=\left\{\zeta\in\partial C;\ \lim_{t\to+0}\frac{\sigma(B,\zeta,t)}{t}=0\right\}.$$

We have;

Theorem 2.10. *If $B\in\mathcal{B}$ then $\sum(B)=R(B)$.*

In the proof of this theorem we will use a lemma similar to Lemma 2.2. If ζ is a fixed boundary point and α is a fixed number such that $0<\alpha<1$ let $\mathcal{B}_{\zeta,\alpha}$ designate the class of all Blaschke products B such that $B(z)\neq0$ for $z\in S(\zeta,\alpha)$.

Lemma 2.11. *There exists a positive constant A_α, depending only on α, such that*

$$\limsup_{r\to1-0}\frac{1-r}{h(1-r)}\log\frac{1}{|B(r\zeta)|}\leqslant A_\alpha\limsup_{t\to+0}\frac{\sigma(B,\zeta,t)}{h(t)}$$

for all $B\in\mathcal{B}_{\zeta,\alpha}$ and all $h\in\mathcal{H}$.

Lemma 2.11 is proved exactly as Lemma 2.2 once we have established the existence of a positive constant A_α, depending only on α, with the following property; *for all $h\in\mathcal{H}$ and all $B\in\mathcal{B}_{\zeta,\alpha}$, such that $B(0)\neq0$, and $\sigma(B,\zeta,t)\leqslant h(t)$ for $t>0$, the inequality*

$$\log\frac{1}{|B(r\zeta)|}\leqslant A_\alpha\frac{h(1-r)}{1-r}$$

holds whenever $r>r_\alpha=1-(1-\alpha)^{\frac{1}{2}}(1+\alpha)^{-\frac{1}{2}}$.

We will omit the proof of Lemma 2.11, but prove the existence of a constant A_α with the above property.

If $B \in \mathcal{B}_{\zeta,\alpha}$ and $B(0) \neq 0$, then, for $r > r_\alpha$

$$\log \frac{1}{|B(r\zeta)|} = \frac{1}{2} \sum_k \log \left(1 + \frac{(1-|a_k|^2)(1-r^2)}{|a_k - r\zeta|^2}\right)$$

$$\leqslant 2(1-r) \sum_{|a_k - r\zeta| \geqslant \alpha(1-r)} \frac{1-|a_k|}{|a_k - r\zeta|^2}.$$

Let $\{u_n\}_1^\infty$ be an increasing sequence for which $\sum u_n^{-2}(u_{n+1}+1)$ converges, $u_1 = \alpha$, and $u_n \to +\infty$ as $n \to +\infty$. As in the proof of Lemma 2.3 one shows that, if $\sigma(B, \zeta, t) \leqslant h(t)$ for $t > 0$, then

$$\sum_{|a_k - r\zeta| \geqslant \alpha(1-r)} \frac{1-|a_k|}{|a_k - r\zeta|^2} \leqslant \frac{h(1-r)}{(1-r)^2} \sum_1^\infty u_n^{-2}(u_{n+1}+1).$$

Thus $A_\alpha = 2\sum_1^\infty u_n^{-2}(u_{n+1}+1)$ is a positive constant with the required property.

Let us now turn to the proof of Theorem 2.10. The inclusion $R(B) \subset \sum(B)$ follows from the inequality

$$8 \log \frac{1}{|B(r\zeta)|} \geqslant \frac{\sigma(B, \zeta, 1-r)}{1-r}$$

established in the proof of Lemma 2.2. It remains to prove that $\sum(B) \subset R(B)$.

Suppose that $\zeta \in \sum(B)$. Let α be a number such that $0 < \alpha < 1$. Then there exists $K(\alpha) > 0$ such that $1 - |z| \geqslant K(\alpha)|\zeta - z|$ for all $z \in S(\zeta, \alpha)$. Let $0 < \varepsilon < K(\alpha)$ and let $t_0 > 0$ be such that $\sigma(B, \zeta, t) \leqslant \varepsilon t$ for $0 < t \leqslant t_0$. Put

$$B_1(z) = \prod_{|a_k - \zeta| \leqslant t_0} \frac{\bar{a}_k(a_k - z)}{|a_k|(1 - \bar{a}_k z)},$$

and

$$B_2(z) = B(z)/B_1(z).$$

Then $B_1 \in \mathcal{B}_{\zeta,\alpha}$ and therefore, by Lemma 2.11 applied to $B = B_1$ and $h = h_1$, we have

$$\lim_{r \to 1} |B_1(r\zeta)| = 1.$$

Since the zeros a_k of B_2 satisfy the inequality $|\zeta - a_k| > t_0$, we have

$$|\lim_{r \to 1} B_2(r\zeta)| = 1,$$

and thus $\zeta \in R(B)$. This completes the proof of Theorem 2.10.

It should be noted that the condition $\sigma(B, \zeta, t) \to 0$ as $t \to +0$, does not imply the existence of the radial limit of B at $\zeta \in \partial C$. Frostman ([3], p. 176) proved that there exist Blaschke products B, which, for each $\zeta \in \partial C$, can be written as the product of two Blaschke products B_ζ and $\bar{B}_\zeta$ in such a manner that B_ζ does not have a radial limit of modulus 1 at $\zeta \in \partial C$. Let B be such a Blaschke product and let $\zeta \in \partial C$ be such that the radial limit $B(\zeta)$ exists and $|B(\zeta)|=1$. Then by Theorem 2.10 $\sigma(B,\zeta,t)/t \to 0$ as $t \to +0$. Obviously $\sigma(B_\zeta,\zeta,t)/t \to 0$ as $t \to +0$, while $\lim_{r\to 1-0} B_\zeta(r\zeta)$ does not exist.

3. Radial zeros and Hausdorff measures

Throughout this section we will consider Hausdorff measures induced by functions in $\mathcal{H}_0$. We will denote the Hausdorff measures induced by the functions $h, g_\alpha, h_\alpha, \ldots$, etc., by $H, G_\alpha, H_\alpha \ldots$, respectively.

We prove the following lemma.

Lemma 3.1. *Let $h \in \mathcal{H}_0$ and $B \in \mathcal{B}$. Let $\overline{\sum}_0(B,h)$ be the set*

$$\overline{\sum}_0(B,h)=\left\{\zeta \in \partial C;\ \limsup_{t\to+0} \frac{\sigma(B,\zeta,t)}{h(t)} > 0\right\}.$$

Then $H(\overline{\sum}_0(B,h))=0$.

Proof. Let $\{a_k\}_1^\infty$ be the sequence of zeros of B. Put for $a>0$

$$\overline{\sum}_a(B,h)=\left\{\zeta \in \partial C;\ \limsup_{t\to+0} \frac{\sigma(B,\zeta,t)}{h(t)} > a\right\}.$$

Obviously it suffices to prove that $H(\overline{\sum}_a(B,h))=0$ for all $a>0$. Given $\varepsilon>0$, there exists an integer $K>1$ such that $\sum_{k=K}^\infty (1-|a_k|)<\varepsilon a(22)^{-1}$. Let ϱ be a positive number such that $\varrho<\inf_{1\leqslant k\leqslant K-1}(1-|a_k|)$. For each $\zeta \in \overline{\sum}_a(B,h)$ consider all closed discs $C(\zeta,t)$ with center ζ and radii t, such that $h(t)<a^{-1}\sigma(B,\zeta,t)$ and $0<t\leqslant\varrho$. The family of all such discs is a covering of $\overline{\sum}_a(B,h)$ in the Vitali narrow sense (cf. [1], p. 104, and [10], p. 198); and, therefore, by Besicovitch's theorem ([1], pp. 104–106, and [10], p. 198), we can extract a subcovering of $\overline{\sum}_a(B,h)$ consisting of 22 countable subfamilies of disjoint discs. Then, if

$$\Gamma_i=\{C(\zeta_{i,n},t_{i,n})\}_n \quad (i=1,2,\ldots,22)$$

denotes the subfamilies of such a subcovering, we have

$$\sum_{i=1}^{22}\sum_n h(t_{i,n}) \leqslant a^{-1}\sum_{i=1}^{22}\sum_n \sigma(B,\zeta_{i,n},t_{i,n}) \leqslant a^{-1}\sum_{i=1}^{22}\sum_{k=K}^{\infty}(1-|a_k|)<\varepsilon.$$

Consequently, $H(\overline{\sum}_a(B,h))\leqslant\varepsilon$. Thus, since ε is arbitrarily chosen, $H(\overline{\sum}_a(B,h))=0$. This completes the proof of Lemma 3.1.

Combining Theorem 2.1 and Lemma 3.1, it is easily proved that $H(L(B,h))=0$ for all $h\in\mathcal{H}_0$ and all $B\in\mathcal{B}$. In fact, if $L_0(B,h)$ is the set

$$L_0(B,h)=\left\{\zeta\in\partial C;\ \liminf_{r\to 1-0}\frac{1-r}{h(1-r)}\log\frac{1}{|B(r\zeta)|}>0\right\},$$

Lemma 2.2 and Lemma 3.1 yield the following theorem.

Theorem 3.2. *If $B\in\mathcal{B}$ then the set $L_0(B,1)$ is empty. If $h\in\mathcal{H}_0$, then*

$$H(L_0(B,h))=0.$$

Incidentally, the first part of Theorem 3.2 is a simple consequence of results due to Heins ([4], pp. 193–196).

Before stating our next result let us introduce the following notation. Let $g_\alpha(t)=t^\alpha$, $0<\alpha\leqslant 1$ and let h_α be the functions introduced in Section 2. If E is a subset of the complex plane, for which $H_1(E)=0$, the Hausdorff dimension of E is defined by

$$\dim\ E=\inf\ \{\alpha;\ \alpha\leqslant 1,\ G_\alpha(E)=0\}.$$

Analogously, let $\dim_{H_0}E$ be defined by

$$\dim_{H_0}E=\inf\ \{\alpha;\ \alpha\leqslant 1,\ H_\alpha(E)=0\}.$$

In this notation our next result can be stated as follows.

Theorem 3.3. *If $B\in\mathcal{B}$ then $H_1(Z(B))=0$ and $H_0(L(B))=0$. Conversely, there exist Blaschke products B and B_0 such that* $\dim Z(B)=1$ *and* $\dim_{H_0}L_S(B_0)=0$.

The first part of this theorem follows from Theorem 3.2 with $h=h_1$ and $h=h_0$, respectively.

To prove the second part we use the following lemma.

Lemma 3.4. *Let $h, g\in\mathcal{H}_0$. Suppose that*

(i) *$t^{-1}h(t)$ is decreasing on $(0,\infty)$,*

(ii) *$\int_0^1 dt/h(t)$ converges,*

(iii) *$g(t^2)=O(tg(t))$ as $t\to+0$, and*

(iv) *$\frac{h(t)}{g(t)}\int_0^t d\tau/h(\tau)\to+\infty$ as $t\to+0$.*

Then there exists a Blaschke product B such that

$$H(L_S(B,g))>0.$$

Proof. Under the hypothesis of Lemma 3.4 there exists a sequence $\{\varrho_n\}_1^\infty$ such that

(α) $0<2\varrho_{n+1}<\varrho_n<\frac{1}{2}\quad(n=1,2,\ldots)$,

(β) $\sum_1^\infty 2^n\varrho_n$ converges,

(γ) $\liminf_{n\to+\infty} 2^n h(\varrho_n) > 0,$

(δ) $2^n g(\varrho_n) = o(\sum_{n+1}^{\infty} 2^k \varrho_k)$, as $n \to +\infty$.

To see this, define

$$2^n \varrho_n = K\, 2^{-n}/h(2^{-n}) \quad (n = 1, 2, \ldots),$$

where K is chosen so that $0 < K < 2\,h(2^{-1})$. Property (α) then follows from (i), and (β) follows from (ii) and from the inequality

$$2^n \varrho_n \leqslant 2\,K \int_{2^{-(n+1)}}^{2^{-n}} \frac{dt}{h(t)}.$$

Property (γ) is a consequence of the inequality

$$2^n h(\varrho_n) = K \frac{h(\varrho_n)}{\varrho_n} \cdot \frac{2^{-n}}{h(2^{-n})} > K,$$

which follows from (α) and (i). To prove (δ) assume that

$$g(t^2) \leqslant Atg(t) \quad \text{for} \quad 0 \leqslant t \leqslant 1.$$

Then, for n sufficiently large, we have

$$2^n g(\varrho_n) = 2^n g\left(\frac{K}{h(2^{-n})}\, 2^{-2n}\right)$$

$$\leqslant \frac{2^n K}{h(2^{-n})}\, g(2^{-2n})$$

$$\leqslant \frac{AKg(2^{-n})}{h(2^{-n})}.$$

Thus

$$\sum_{n+1}^{\infty} 2^k \varrho_k / 2^n g(\varrho_n) \geqslant \frac{K}{2^n g(\varrho_n)} \int_0^{2^{-n}} \frac{dt}{h(t)}$$

$$\geqslant A^{-1} \frac{h(2^{-n})}{g(2^{-n})} \int_0^{2^{-n}} \frac{dt}{h(t)}$$

and (δ) follows from hypothesis (iv).

Given a sequence $\{\varrho^n\}_1^{\infty}$, with properties (α) through (δ), construct on ∂C the perfect symmetric set

(3.5) $$E = \{e^{ix};\ x = \sum_{n=1}^{\infty} \varepsilon_n r_n,\ \varepsilon_n = 0 \text{ or } 1\},$$

where $r_1 = 2\pi - \varrho_1$, $r_n = \varrho_{n-1} - \varrho_n$ $(n = 2, 3, \ldots)$. Let L be the Lebesgue–Cantor function constructed on E and let ω_L be its modulus of continuity. Then using property (γ) it is easily proved that

$$\omega_L(t) = O(h(t)) \quad \text{as} \quad t \to +0.$$

Consequently, since the Hausdorff measure of E induced by ω_L is positive (cf. [5], p. 30), we have $H(E) > 0$.

To prove Lemma 5.4 it therefore suffices to construct a Blaschke product B, such that $E \subset L_S(B, g)$. Let B be the Blaschke product having the zeros

$$a_{n, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n} = (1 - \varrho_n) \exp\{i \sum_{k=1}^{n} \varepsilon_k r_k\},$$

where $n = 1, 2, \ldots$, and $\varepsilon_k = 0$ or 1. If $2\varrho_{n-1} > t \geqslant 2\varrho_n$ and $\zeta \in E$, then the disc $|z - \zeta| \leqslant t$ contains at least 2^k zeros of modulus $1 - \varrho_{n+k}$ $(k = 0, 1, \ldots)$, and therefore

$$\sigma(B, \zeta, t)/g(t) \geqslant \sum_{0}^{\infty} 2^k \varrho_{n+k}/g(2\varrho_{n-1})$$

$$\geqslant \sum_{n}^{\infty} 2^k \varrho_k / 2^{n+1} g(\varrho_{n-1}).$$

Consequently, by property (δ)

$$\liminf_{t \to +0} \frac{\sigma(B, \zeta, t)}{g(t)} = +\infty$$

and thus, by Theorem 2.1, $\zeta \in L_S(B, g)$. Hence $E \subset L_s(B, g)$ and the proof of Lemma 3.4 is complete.

To prove the last part of Theorem 3.3, put $g = h_0$ and let h be any function satisfying the hypothesis of Lemma 3.4, such that for $\alpha < 0$

$$h(t) = o(h_\alpha(t)) \quad \text{as} \quad t \to +0. \tag{3.6}$$

Then, by Lemma 3.4, there exists a Blaschke product B such that $H(L_S(B)) > 0$. Hence, by (3.6), $H_\alpha(L_S(B)) = +\infty$ for all $\alpha < 0$, i.e. $\dim_{H_0} L_S(B) \geqslant 0$. Consequently, by the first part of Theorem 3.3, $\dim_{H_0} L_S(B) = 0$.

The existence of a Blaschke product B, such that $\dim Z(B) = 1$, is proved in like manner, by applying Lemma 3.4 to $g = h_1$ and $h = h_{-1}$. Although this result may be known, we state it here for completeness.

4. Two lemmas

In this section we prove two lemmas similar to Lemma 2.2 and Lemma 3.1, respectively.

Let u be a nonnegative function, harmonic on C. Then u has a Poisson–Stieltjes representation

$$u(re^{ix}) = \int_0^{2\pi} P_r(x-t)\,d\mu(t), \tag{4.1}$$

where
$$P_r(t) = \frac{1-r^2}{1-2r\cos t + r^2}$$

is the Poisson kernel and μ is a nondecreasing function defined on the interval $[0, 2\pi]$.

Lemma 4.2. *Let u be given by* (4.1). *Then there exist positive constants A_1 and A_2, such that*

$$A_1 \liminf_{t\to+0} \frac{\mu(x+t)-\mu(x-t)}{h(t)} \leqslant \liminf_{r\to1-0} \frac{1-r}{h(1-r)}\, u(re^{ix})$$

and

$$\limsup_{r\to1-0} \frac{1-r}{h(1-r)}\, u(re^{ix}) \leqslant A_2 \limsup_{t\to+0} \frac{\mu(x+t)-\mu(x-t)}{h(t)}$$

for $0 < x < 2\pi$ *and all* $h \in \mathcal{H}$.

In case of $h = h_0$ a similar result was proved in [9], p. 290.

Proof. The proof of this lemma is classical and therefore we restrict ourselves to the proof of the second part.

Assume that

$$\limsup_{t\to+0} \frac{\mu(x+t)-\mu(x-t)}{h(t)} = l \quad (0 \leqslant l < \infty).$$

Then, given $\varepsilon > 0$ there exists $\delta > 0$ $(\delta < \pi)$, such that

$$\mu(x+t) - \mu(x-t) \leqslant (l+\varepsilon)\, h(t) \quad \text{for} \quad 0 \leqslant t \leqslant \delta.$$

Integration by parts yields

$$\begin{aligned}
&\int_{x-\delta}^{x+\delta} P_r(x-t)\,d\mu(t) \\
&\quad = P_r(\delta)\,\{\mu(x+\delta+0) - \mu(x-\delta-0)\} + \int_0^{\delta} \{\mu(x+t)-\mu(x-t)\}\, P_r'(-t)\,dt \\
&\quad \leqslant P_r(\delta)\,\{\mu(x+\delta+0) - \mu(x-\delta-0)\} + (l+\varepsilon)\int_0^{\delta} h(t)\, P_r'(-t)\,dt,
\end{aligned}$$

where P_r' denotes the derivative of P_r with respect to t. However, if $1-r<\delta$, we have

$$\frac{1-r}{h(1-r)}\int_0^{\delta} h(t)\,P_r'(-t)\,dt$$

$$=(1-r)\int_0^{1-r}\frac{h(t)}{h(1-r)}\,P_r'(-t)\,dt+\int_{1-r}^{\delta}\frac{h(t)}{t}\cdot\frac{1-r}{h(1-r)}\,tP_r'(-t)\,dt$$

$$\leqslant(1-r)\int_0^{1-r}P_r'(-t)\,dt+\int_{1-r}^{\delta}tP_r'(-t)\,dt$$

$$\leqslant 2+2\pi.$$

Hence

$$\limsup_{r\to 1-0}\frac{1-r}{h(1-r)}\,u(re^{ix})=\limsup_{r\to 1-0}\frac{1-r}{h(1-r)}\int_{x-\delta}^{x+\delta}P_r(x-t)\,d\mu(t)$$

$$\leqslant(2+2\pi)(l+\varepsilon)$$

and, since $\varepsilon>0$ is arbitrarily chosen, the second part of Lemma 4.2 follows.

Our next lemma deals with the sets

$$M_a(\mu,h)=\left\{e^{ix}\in\partial C;\ 0<x<2\pi,\ \limsup_{t\to+0}\frac{\mu(x+t)-\mu(x-t)}{h(t)}\geqslant a\right\},$$

where $0\leqslant a\leqslant+\infty$, μ is a nondecreasing function defined on $[0,2\pi]$ and h is a function in class $\mathcal{H}$.

Lemma 4.3. *Let $h\in\mathcal{H}_0$. Then $H(M_a(\mu,h))=0$ if $a=+\infty$, while $H(M_a(\mu,h))$ is finite if $a>0$.*

For $a=+\infty$ and $h=h_0$, Lemma 4.3 was proved in [10], p. 198. The same proof holds for any $h\in\mathcal{H}_0$. The second part of the lemma is proved in like manner with obvious modifications. We omit the proof.

The second part of Lemma 4.3 cannot be improved. For instance, if $h\in\mathcal{H}_0$ and $t^{-1}h(t)$ is strictly decreasing, if E is the perfect symmetric Cantor set given by (3.5) with $r_1=2\pi-\varrho_1$ and $r_n=\varrho_{n-1}-\varrho_n$ $(n=2,3,\ldots)$, where $h(\varrho_n)=2^{-n}$, then, by a theorem of Hausdorff, $0<H(E)<+\infty$. ([4], p. 30). On the other hand, if μ is the Lebesgue–Cantor function (constructed on E) multiplied by $2a$, then, using the technique developed in [8], pp. 226–227, it is readily shown that $M_a(\mu,h)=E\setminus\{1\}$. Hence, $0<H(M_a(\mu,h))<+\infty$.

5. Sets of uniqueness

Let $\mathcal{J}$ be the class of all functions, bounded and analytic in the open disc C. If $f\in\mathcal{J}$ and $f\neq 0$, then

(5.1) $$f=\|f\|\cdot B\cdot E,$$

where $\|f\|$ is the supremum norm of f, B is the normalized Blaschke product of f and E is a function in $\mathcal{J}$ with no zeros in C.

For $0 \leqslant a < +\infty$, $h \in \mathcal{H}$ and $f \in \mathcal{J}$, let $L_a(f, h)$ and $L(f, h)$ be the sets

$$L_a(f,h) = \left\{\zeta \in \partial C;\ \liminf_{r \to 1-0} \frac{1-r}{h(1-r)} \log \frac{1}{|f(r\zeta)|} > a\right\}$$

and

$$L(f,h) = \left\{\zeta \in \partial C;\ \liminf_{r \to 1-0} \frac{1-r}{h(1-r)} \log \frac{1}{|f(r\zeta)|} = +\infty\right\}.$$

In this notation our first result can be stated as follows.

Theorem 5.2. *If $f \neq 0$ is a function in $\mathcal{J}$, then the set $L(f, 1)$ is empty, while the set $L_a(f, 1)$ is finite for all $a > 0$.*

The first part of Theorem 5.2 is in [4], pp. 195–196, and the second part is a slightly stronger version of a result of Kegejan [(6], p. 245). More precisely Kegejan proves that, if the set

$$\{\zeta \in \partial C;\ |f(r\zeta)| \leqslant \exp\{(r-1)^{-1}\} \quad \text{for} \quad 0 \leqslant r < 1\}$$

is closed, then it is finite, unless $f = 0$.

Theorem 5.2 has the following corollary.

Corollary 5.3. *If $0 \neq f \in \mathcal{J}$, then $L_0(f, 1)$ is countable.*

Proof of Theorem 5.2. Let f be given by (5.1) and suppose that $h \in \mathcal{H}$. Then

$$L(f,h) \subset L_0(B,h) \cup \bar{L}(E,h), \tag{5.4}$$

where

$$\bar{L}(E,h) = \left\{\zeta \in \partial C;\ \limsup_{r \to 1-0} \frac{1-r}{h(1-r)} \log \frac{1}{|E(r\zeta)|} = +\infty\right\}.$$

By Theorem 3.2 the set $L_0(B, 1)$ is empty. Lemma 4.2, applied to $u = -\log|E|$ and $h(t) = 1$, shows that $\bar{L}(E, 1)$ contains no other points than possibly $\zeta = 1$. However, if $1 \in \bar{L}(E, 1)$, then $\zeta_0 \in \bar{L}(E_1, 1)$, where $E_1(z) = E(z\bar{\zeta}_0)$ and $1 \neq \zeta_0 \in \partial C$. This contradicts the fact that the only possible point in $\bar{L}(E_1, 1)$ is $\zeta = 1$. Hence, $\bar{L}(E, 1)$ is empty and therefore, by (5.4) the set $L(f, 1)$ is empty.

If $h \in \mathcal{H}$ and $t = o(h(t))$ as $t \to +0$ then

$$L_a(f,h) \subset L_0(B,h) \cup \bar{L}_a(E,h), \tag{5.5}$$

where

$$\bar{L}_a(E,h) = \left\{\zeta \in \partial C;\ \limsup_{r \to 1-0} \frac{1-r}{h(1-r)} \log \frac{1}{|E(r\zeta)|} \geqslant a\right\}.$$

Let $u = -\log|E|$ and let μ be the corresponding nondecreasing function in the Poisson-Stieltjes representation of u. Then, by Lemma 4.2

$$\bar{L}_a(E, h) \subset M_{a'}(\mu, h) \cup \{1\}, \tag{5.6}$$

where $a' = aA_2^{-1}$ and A_2 is the positive constant in Lemma 4.2. Obviously $M_{a'}(\mu, 1)$ is finite if $a' > 0$; and, therefore, by (5.6) the set $\bar{L}_a(E, 1)$ is finite. Hence, by Theorem 3.2 and (5.5), the set $L_a(f, 1)$ is finite. This completes the proof of Theorem 5.2.

Theorem 5.7. *Let $h \in \mathcal{H}_0$. If $f \neq 0$ is a function in $\mathfrak{F}$, then $H(L(f, h)) = 0$ while $H(L_a(f, h))$ is finite for all $a > 0$.*

Proof. Let f be given by (5.1), let $u = -\log |E|$ and let μ be the corresponding nondecreasing function in the Poisson–Stieltjes representation. Then, by Lemma 4.2

$$\bar{L}(E, h) \subset M_\infty(\mu, h) \cup \{1\}.$$

Hence, by Lemma 4.3 $H(\bar{L}(E, h)) = 0$; and, consequently, by (5.4) and Theorem 3.2, $H(L(f, h)) = 0$.

To prove the second part of Theorem 5.7, we may assume that $t = o(h(t))$ as $t \to +0$. If this is not the case, then $h(t) \sim \alpha t$ as $t \to +0$ for some $\alpha > 0$, and there is nothing to prove. However, by (5.6) and Lemma 4.3, $H(\bar{L}_a(E, h))$ is finite; and, consequently, by (5.5) and Theorem (3.2), $H(L_a(f, h))$ is finite.

Theorem 5.7 will be used to prove two uniqueness theorems. Before we state these theorems, let us introduce the following notation.

For functions f_1 and f_2 in $\mathfrak{F}$, let $D(f_1, f_2)$ be the set of all boundary points ζ such that

$$\lim_{r \to 1-0} f_1^{(k)}(r\zeta) = \lim_{r \to 1-0} f_2^{(k)}(r\zeta) \quad (k = 0, 1, 2, \ldots).$$

In like manner, let $D_S(f_1, f_2)$ be the set of all points $\zeta \in \partial C$ such that

$$\lim_{\substack{z \to \zeta \\ z \in S(\zeta, \alpha)}} f_1^{(k)}(z) = \lim_{\substack{z \to \zeta \\ z \in S(\zeta, \alpha)}} f_2^{(k)}(z) \quad (k = 0, 1, 2, \ldots),$$

for all Stolz domains $S(\zeta, \alpha)$. If $f_2 = 0$, we will simply write $D(f_1) = D(f_1, 0)$ and $D_S(f_1) = D_S(f_1, 0)$, respectively.

The following lemma was proved in [10], p. 195; we abbreviate $L_S(f, h_0)$ and $L(f, h_0)$, with $L_S(f)$ and $L(f)$, respectively.

Lemma 5.8. *If $f \in \mathfrak{F}$, then*

$$L_S(f) = D_S(f) \subset D(f) \subset L(f).$$

The following theorem is an immediate consequence of Lemma 5.8 and Theorem 5.7.

Theorem 5.9. *If $f_1, f_2 \in \mathfrak{F}$ and $H_0(D(f_1, f_2)) > 0$, then $f_1 = f_2$.*

Proof. Put $f = f_1 - f_2$ and assume that $f \neq 0$. Then by Lemma 5.8 $D(f_1, f_2) \subset D(f) \subset L(f) = L(f, h_0)$. Consequently, $H_0(L(f)) > 0$, contradicting Theorem 5.7. Hence, $f_1 = f_2$; and the theorem is proved.

Theorem 5.9 is best possible in the following sense: there exists a function $f \in \mathfrak{F}$, such that $\dim_{H_0} D(f)=0$. This follows immediately from Theorem 3.3 and Lemma 5.8.

For functions f_1 and f_2 in $\mathfrak{F}$, let $U_\alpha(f_1, f_2)$ be the set of points $\zeta \in \partial C$, such that

(i)

$$\lim_{r\to 1-0} f_1(r\zeta) = \lim_{r\to 1-0} f_2(r\zeta)$$

and

(ii)

$$|f_1'(r\zeta) - f_2'(r\zeta)| = O((1-r)^\alpha) \quad \text{as} \quad r\to 1-0.$$

The sets $U_\alpha(f_1, f_2)$ are sets of uniqueness in the following sense.

Theorem 5.10. *Let $f_1, f_2 \in \mathfrak{F}$. Then $f_1 = f_2$ if and only if there exists $\alpha > -1$, such that $H_0(U_\alpha(f_1, f_2)) = +\infty$.*

Proof. First let us assume that $f_1 = f_2$. Let E be the exceptional boundary set, where f_1 has no radial limits. Then $U_\alpha(f_1, f_2) = \partial C \setminus E$ for all α. By Fatou's theorem $H_1(\partial C\setminus E) > 0$. Hence, since $h_1(t) = o(h_0(t))$ as $t \to +0$, we have

$$H_0(U_\alpha(f_1, f_2)) = +\infty$$

for all α.

Next, assume that $f_1 \neq f_2$. Put $f = f_1 - f_2$. Then, if $\zeta \in U_\alpha(f_1, f_2)$, and $\alpha > -1$,

$$|f(r\zeta)| \leqslant \int_r^1 |f'(\varrho\zeta)|\, d\varrho$$

$$= O((1-r)^{1+\alpha}) \quad \text{as} \quad r\to 1-0.$$

Hence

$$\liminf_{r\to 1-0} \frac{1-r}{h_0(1-r)} \log \frac{1}{|f(r\zeta)|} \geqslant 1+\alpha,$$

i.e., $U_\alpha(f_1, f_2) \subset L_a(f, h_0)$, where $\alpha + 1 > a > 0$. Thus by Theorem 5.7 $H_0(U_\alpha(f_1, f_2)) < +\infty$ for all $\alpha > -1$. This completes the proof of Theorem 5.10.

Incidentally, if $\alpha \leqslant -1$, there exists $f \neq 0$ such that $H_0(U_\alpha(f, 0)) = +\infty$. To see this, let E be any closed set on the boundary ∂C, such that $H_0(E) = +\infty$ and $H_1(E) = 0$. Construct $f \in \mathfrak{F}$ such that $f \neq 0$ and $\lim_{r\to 1-0} f(r\zeta) = 0$ for all $\zeta \in E$ (cf. [7], p. 34). Since $|f'(r\zeta)| = O((1-r)^\alpha)$ as $r\to 1-0$ for any function $f\in\mathfrak{F}$, whenever $\alpha \leqslant -1$, we conclude that $E \subset U_\alpha(f, 0)$. And thus $H_0(U_\alpha(f, 0)) = +\infty$.

Theorem 5.10 has the following corollary. If $f_1, f_2 \in \mathfrak{F}$ let $D_1(f_1, f_2)$ be the set of boundary points ζ, such that

$$\lim_{r\to 1-0} f_1^{(k)}(r\zeta) = \lim_{r\to 1-0} f_2^{(k)}(r\zeta) \quad \text{for} \quad k=0, 1.$$

Obviously $D_1(f_1, f_2) \subset U_0(f_1, f_2)$. Thus, we have;

Corollary 5.11. *If $f_1, f_2 \in \mathfrak{F}$ and $H_0(D_1(f_1, f_2)) = +\infty$ then $f_1 = f_2$.*

Corollary 5.11 is equivalent to the statement (we abbreviate $D_1(f,0)$ with $D_1(f)$); *if* $0 \neq f \in \mathcal{J}$, *then* $H_0(D_1(f))$ *is finite.* For the class $\mathcal{B}$ a stronger result holds (cf. [10], p. 200); *if* $B \in \mathcal{B}$ *then* $H_0(D_1(B)) = 0$.

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