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On the lower order of $f(z)\,e^{\varphi(z)}$

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CONTENTS

Introduction

This paper deals with generalizations of the following problem, which was given as one (No. 7) of 25 research problems in [1].

"If $f_1(z)$ and $f_2(z)$ are two entire functions of lower order less than one and if $f_1(z)$ and $f_2(z)$ have the same zeros, is $f_1(z)/f_2(z)$ a constant?"

The solution of the research problem is that the quotient $f_1(z)/f_2(z)$ need not be a constant. The proof is given in [8] by the present author. The same result can also be obtained as a direct consequence of Theorem 6.1 or 6.2 or 7.1 of this paper.

In Chapter I of this paper we give the definitions of some set functions suitable for function theoretic applications. Of special interest is the set function $\mu(A)$ of Definition 1.2. This set function was originally introduced (in [9]) by the present author.

In Chapter II there are given results analogous to the following: For every entire function $f(z)$ it holds true that the lower order of the product

$$f(z)\, e^{az}$$

is at least equal to $+1$, except for a nullset of a-values. The main part of this paper is an investigation of this nullset.

In Chapter III we try to give results converse to those in Chapter II. One special result (cf. Theorem 6.2) is that the quotient $f_1(z)/f_2(z)$ in the original research problem [1] can be any entire function without zeros.

In Chapter IV we deal with various problems, which are in some way connected with the earlier parts of this paper. Especially the result in Section 9 indicates that the (very small) nullsets $A_t^p(f)$ are by no means "almost countable".

Chapter I. Set functions

In this chapter the definitions of three set functions are given. Each one of the three definitions contains an auxiliary function which is continuous and monotonic. These set functions are used for classifying noncountable nullsets in the complex plane.

1. *The set function $m(A)$*

Let A be a set of complex numbers. The sequence $\{d_n\}_{n=1}^{\infty}$ of real numbers is called a *majorizing sequence* for the set A if there exists some sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers such that the inequality

$$|a - a_n| < d_n$$

holds for an infinity of values of n whenever $a \in A$.

Definition 1.1. *Let A be a set of complex numbers and $g(x)$ a monotonic decreasing real function, defined for $x > 0$, which satisfies*

$$\lim_{x\to\infty} \frac{g(x+\varepsilon)}{g(x)} = 0 \tag{1}$$

for every $\varepsilon > 0$.

The measure $m(A) = m(g(x), A)$ is then defined as the lower bound of real numbers $1/k > 0$ for which $\{g(kn)\}_{n=1}^{\infty}$ is a majorizing sequence for the set A.

The set function $m(A)$ is subadditive ([9], Satz 3). For the special case that

$$g(x) = \theta^{e^x}, \quad 0 < \theta < 1$$

the set function $m(g(x), A)$ is denoted $\mu(A)$. Notice that changes in θ are inessential, since they do not affect the value of $\mu(A)$. In particular, for $\theta = e^{-1}$ the definition of $\mu(A)$ can be written:

Definition 1.2. *For a set A of complex numbers, $\mu(A)$ is defined as the lower bound of positive numbers $1/k$ for which*

$$d_n = e^{-e^{kn}}, \quad n = 1, 2, \ldots$$

is a majorizing sequence for the set A.

2. *The Hausdorff measure $h^*(A)$*

In the following the real function $h(t)$ is any continuous and monotonic increasing function, defined for $t > 0$, and with $\lim_{t\to 0} h(t) = 0$.

Definition 2.1. *Let A be a set of complex numbers. The value $h^*(A) = h^*(h(t), A)$ of the Hausdorff measure of A is defined as*

$$h^*(A) = \lim_{\delta\to 0}\left(\inf_{(2)} \sum_{i=1}^{\infty} h(d_i)\right)$$

where d_i denotes the radius of the circle C_i, and the infimum is taken over all coverings

$$\bigcup_{i=1}^{\infty} C_i \supset A; \quad d_i < \delta. \tag{2}$$

This definition is due to Hausdorff [6]. We now deduce a relation between the set functions $m(A)$ and $h^*(A)$.

Proposition 2.2. *Let*

$$m(g(x), A) < \infty$$

and

$$\sum_{n=1}^{\infty} h(g(n)) < \infty,$$

then

$$h^*(h(t), A) = 0.$$

Proof. The meaning of

$$m(g(x), A) < \infty$$

is that there exists some k, $0 < k < \infty$, such that the sequence

$$\{g(kn)\}_{n=1}^{\infty}$$

is a majorizing sequence for the set A.

The functions $h(t)$ and $g(x)$ are monotonic, and therefore, it follows from

$$\sum_{n=1}^{\infty} h(g(n)) < \infty$$

that

$$\sum_{n=1}^{\infty} h(g(kn)) < \infty.$$

For an arbitrary $\varepsilon > 0$, let the number $N(\varepsilon)$ satisfy

$$\sum_{n=N(\varepsilon)}^{\infty} h(g(kn)) < \varepsilon.$$

From the fact that $\{g(kn)\}_{n=1}^{\infty}$ is a majorizing sequence for the set A, it follows that the corresponding sequence of circles C_n, for $n \geqslant N(\varepsilon)$, can give a covering of the set A. This implies

$$h^*(h(t), A) \leqslant \sum_{n=N(\varepsilon)}^{\infty} h(g(kn)) < \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the proof is complete.

3. *The generalized capacity $C(A)$*

For a survey on capacities and Hausdorff measures we refer to Taylor, [10].

In Definition 3.1 below the real function $\Phi(t)$ is continuous and strictly decreasing for $t > 0$, and

$$\lim_{t \to 0} \Phi(t) = +\infty.$$

Definition 3.1. (Frostman [5]). *Let A be a bounded Borel set and let ν denote a positive mass distribution with*

$$\nu(A) = \int_A d\nu = 1.$$

Then $C(A) = C(\Phi(t), A)$ is defined by

$$C(\Phi(t), A) = \Phi^{-1}\left(\inf_{\nu} \sup_{z} \int_A \Phi(|z - \zeta|)\, d\nu(\zeta)\right). \tag{3}$$

Here Φ^{-1} denotes the inverse function, and $\Phi^{-1}(+\infty) = 0$.

In (3) the supremum is taken over all complex numbers z, and the infimum is taken over all positive mass distributions ν with $\nu(A) = 1$. The value of the set function $C(A) = C(\Phi(t), A)$ is called the generalized capacity of the set A with respect to the kernel function $\Phi(t)$.

When the kernel function $\Phi(t)$ and the mass distribution ν are given, the potential $u(z)$ is defined as

$$u(z) = \int \Phi(|z - \zeta|)\, d\nu(\zeta).$$

For a given set A, the relation $C(A) > 0$ is equivalent to the existence of some positive mass distribution ν, with $\nu(A) > 0$ such that

$$u(z) = \int \Phi(|z - \zeta|)\, d\nu(\zeta) < k < \infty.$$

In order to establish a relation between $m(A)$ and $C(A)$, we prove the following proposition:

Proposition 3.2. *Let A be a given Borel set of complex numbers, and let $\Phi(t)$ be a given positive kernel function. If*

$$m(g(x), A) < \infty$$

and also

$$\sum_{n=1}^{\infty} \Phi(g(n))^{-1} < \infty,$$

then

$$C(\Phi(t), A) = 0.$$

Proof. The assumption $m(g(x), A) < \infty$ implies that there exists some positive number k, such that $\{d_n = g(kn)\}_{n=1}^{\infty}$ is a majorizing sequence for the set A (cf. Definition 1.1).

Hence there exists some sequence $\{a_n\}_{n=1}^{\infty}$ of complex numbers such that the circles

$$C_n = \left\{z \,\middle|\, |z - a_n| < g(kn) = d_n\right\}$$

cover the set A in the sence that

$$A \subset \bigcup_{n=N}^{\infty} C_n$$

for every N.

Let ν be an arbitrary non-negative mass distribution with

$$\int_A d\nu(\zeta) > 0.$$

Denote

$$\nu_n = \int_{C_n} d\nu(\zeta).$$

Then, for every $N > 0$ we have the inequality

$$\sum_{N}^{\infty} \nu_n \geqslant \int_{\bigcup_{n=N}^{\infty} C_n} d\nu(\zeta) \geqslant \int_A d\nu(\zeta) > 0$$

and therefore

$$\sum_{n=1}^{\infty} \nu_n = +\infty.$$

The kernel function $\Phi(t)$ is monotonic and positive and ν is non-negative, and therefore $u(a_n)$ is easy to estimate (cf. Frostman [5] p. 89). In fact

$$u(a_n) = \int_A \Phi(|a_n - \zeta|)\, d\nu(\zeta)$$

$$\geqslant \int_{C_n} \Phi(|a_n - \zeta|)\, d\nu(\zeta)$$

$$\geqslant \Phi(g(kn)) \cdot \int_{C_n} d\nu(\zeta) = \Phi(g(kn)) \cdot \nu_n.$$

This estimate of the potential $u(a_n)$ gives

$$\sum_{n=1}^{\infty} u(a_n)\, \Phi(g(kn))^{-1} \geqslant \sum_{n=1}^{\infty} \nu_n = +\infty .$$

The functions $\Phi(t)$ and $g(x)$ are monotonic, and therefore

$$\sum_{n=1}^{\infty} \Phi(g(n))^{-1} < \infty$$

implies

$$\sum_{n=1}^{\infty} \Phi(g(kn))^{-1} < \infty .$$

The upper bound for the potential $u(z)$ therefore turns out to be

$$\sup u(z) \geqslant \sup u(a_n) = +\infty$$

and this upper bound is independent of ν, if $\int_A d\nu(\zeta) > 0$. The capacity of the set A is therefore zero,

$$C(\Phi(t), A) = 0.$$

This completes the proof of Proposition 3.2.

Chapter II. Coverings of the set $E_t(f)$

The main aim of this chapter is to prove and interpret a general result regarding coverings of the set $E_t(f)$ (Theorem 5.3).

We first define the generalized lower order of an entire function. This generalization gives new classes of small sets $A_t^p(F(r), f(z))$ of complex numbers. It is an open question which set function (e.g. which function $g(x)$ in Definition 1.1) is best suited for characterizing these sets $A_t^p(F(r), f(z))$ for general $F(r)$.

4. *The generalized lower order*

The function $F(r)$ is continuous and strictly increasing from $-\infty$ to $+\infty$ for $r_0 < r < +\infty$ $(r_0 \geqslant -\infty)$. $F(r)$ also satisfies

$$\lim_{r\to\infty} (F(2r) - F(r)) = 0 \tag{4}$$

and

$$\lim_{r\to\infty} (F(r) - F(\log r)) = +\infty . \tag{5}$$

The inverse function is denoted by $F^{-1}(x)$. By $M(r)$ we denote $\max_{|z|=r} |f(z)|$.

Definition 4.1. *The generalized lower order of the function $f(z)$ with respect to $F(r)$ is denoted by*

$$\lambda = \lambda(f(z)) = \lambda(F(r), f(z)).$$

It is defined by $$\liminf_{r\to\infty} (F(\log M(r)) - F(r)) = \log \lambda.$$

If the limit equals $-\infty$, *then* $\lambda = 0$.

The same λ can be obtained as the lower bound of positive numbers t for which the inequality

$$\log M(r) \leqslant F^{-1}(F(r) + \log t)$$

is valid for an unbounded set of r-values.

Here $F^{-1}(F(r)+\log t)$ plays the role of r^t. We denote

$$g(r,t) = F^{-1}(F(r) + \log t)$$

and this function satisfies the following functional equation

$$g(g(r,t_1),t_2) = g(r, t_1 \cdot t_2) \tag{6}$$

(Proof of (6): Apply F to both sides of (6)).

The usual definition of lower order is obtained for $g(r,t) = r^t$, i.e. $F(r) = \log\log r + \text{const.}$

Definition 4.2. *When* $F(r)$, $f(z)$ *and* t *are given, the set* $E_t(f) = E_t(F(r), f(z))$ *is defined as*

$$E_t(f) = \{\varphi(z) \mid \varphi(0) = 0, \liminf_{r\to\infty} (F(\log M_\varphi(r)) - F(r)) < \log t\}$$

where $$M_\varphi(r) = \max_{|z|=r} |f(z)\, e^{\varphi(z)}|$$

i.e. the set of entire functions $\varphi(z)$ *for which the generalized lower order, with respect to* $F(r)$, *of the product* $f(z)\,e^{\varphi(z)}$ *is less than* t, *and* $\varphi(0) = 0$.

5. *A covering theorem*

In this section we shall obtain a covering of the set $E_t(f)$ of entire functions. We therefore need a distance function (say D) in the space of all entire functions.

Definition 5.1. *Let* $\varphi(z)$ *and* $\psi(z)$ *be two entire functions. For each* $r > 0$ *we define the following distance function* D.

$$D(r,\varphi,\psi) = \frac{1}{4\pi}\int_0^{2\pi} |\operatorname{Re}(\varphi(r\,e^{i\theta}) - \varphi(0) - \psi(r\,e^{i\theta}) + \psi(0))|\, d\theta.$$

Let $M_\varphi(r)$ denote the following maximum modulus

$$M_\varphi(r) = \max_{|z|=r} |f(z)\, e^{\varphi(z)}|$$

where $f(z)$ is the entire function for which the set $E_t(F(r), f(z))$ is to be examined.

We first prove a lemma.

Lemma 5.2. *Let* $f(z)$, $\varphi(z)$ *and* $\psi(z)$ *be given entire functions with*

$$f(0)=1, \quad \varphi(0)=0, \quad \psi(0)=0.$$

Then, for every $r>0$ *it holds true that*

$$\log M_\psi(r) \leqslant D(r,\varphi,\psi) \Rightarrow \log M_\varphi(r) \geqslant D(r,\varphi,\psi). \tag{7}$$

Proof of lemma 5.2. Let $z=r\cdot e^{i\theta}$. The following inequality is obtained from Jensen's formula and Definition 5.1.

$$\begin{aligned}\int_0^{2\pi} &\max(\log|f(z)\,e^{\varphi(z)}|, \log|f(z)\,e^{\psi(z)}|)\,d\theta \\ &= \tfrac{1}{2}\int_0^{2\pi} \log|f(z)\,e^{\varphi(z)}\cdot f(z)\,e^{\psi(z)}|\,d\theta + \tfrac{1}{2}\int_0^{2\pi} |\operatorname{Re}(\varphi(z)-\psi(z))|\,d\theta \\ &\geqslant \pi\log|f^2(0)\,e^{\varphi(0)+\psi(0)}| + 2\pi D(r,\varphi,\psi) = 2\pi D(r,\varphi,\psi). \end{aligned} \tag{8}$$

Now, assume that (7) is not true. Then we have

$$\log M_\varphi(r) < D(r,\varphi,\psi)$$

and

$$\log M_\psi(r) \leqslant D(r,\varphi,\psi).$$

These inequalities applied to (8) give

$$\log|f(z)\,e^{\varphi(z)}| < D(r,\varphi,\psi)$$

and

$$\log|f(z)\,e^{\psi(z)}| = D(r,\varphi,\psi)$$

for $|z|=r$.

Using $\varphi(0)=\psi(0)$ in the Jensen formula we obtain

$$2\pi D(r,\varphi,\psi) = \int_0^{2\pi}\log|f(z)\,e^{\psi(z)}|\,d\theta = \int_0^{2\pi}\log|f(z)\,e^{\varphi(z)}|\,d\theta < 2\pi D(r,\varphi,\psi).$$

This is a contradiction which proves the lemma.

We now proceed to the covering theorem.

Theorem 5.3. *Let* $f(z)$ *be an entire function which is not a constant. Let the numbers* $x_n>1$, $n=1,2,\ldots$, *be given such that*

$$y_m = \prod_{n=1}^{m-1} x_n \quad \textit{has} \quad \lim_{m\to\infty} y_m = +\infty.$$

Take a number r_1 *such that* $F(r_1) > -\infty$ *and then let*

$$r_n = g(r_1, y_n), \quad n = 2, 3, \ldots .$$

Let $t > 0$. *Then there exists some sequence of entire functions* $\varphi_n(z)$ *so that the function sets* S_n *defined by*

$$S_n = \{\varphi(z) \mid D(r_n, \varphi, \varphi_n) < g(r_n, x_n t)\} \tag{9}$$

cover the set $E_t(F(r), f(z))$ *in the sense that*

$$E_t(F(r), f(z)) \subset \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} S_n. \tag{10}$$

F, g *and* E_t *are defined in Section 4,* D *in Definition 5.1.*

To get a "strong" result in this theorem, we must let the choice of the numbers $\{x_n\}_{n=1}^{\infty}$ depend on the given $g(r, t)$.

Proof of Theorem 5.3. The function $F(r)$ satisfies

$$\lim_{r\to\infty} (F(2r) - F(r)) = 0. \tag{4}$$

The entire functions $f(z)$ and $k \cdot z^n \cdot f(z)$ have the same lower order with respect to $F(r)$ since it follows from (4) that

$$\lim_{r\to\infty} (F(\log |k| + n \log r + \log M(r)) - F \log M(r)) = 0$$

when $f(z)$ is not a constant. In the sequel we assume that

$$f(0) = 1, \quad \varphi(0) = 0, \quad \varphi_n(0) = 0.$$

When the sequence $\{r_n\}_{n=1}^{\infty}$ is given, we define the functions sets B_n by

$$B_n = \{\varphi(z) \mid \varphi(0) = 0, F(\log M_\varphi(r)) - F(r) \leqslant \log t \text{ for some } r \text{ in } r_n \leqslant r < r_{n+1}\}. \tag{11}$$

Then it follows from the Definition (4.2) of $E_t(f)$ that $\{B_n\}_{n=1}^{\infty}$ cover the set $E_t(f)$ in the sense that

$$E_t(f) \subset \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} B_n.$$

This is because $r_n \to +\infty$, which follows from

$$F(r_n) = F(r_1) + \log y_n \to +\infty.$$

To prove (10) it is now sufficient to prove that it is possible to choose the centre $\varphi_n(z)$ of S_n so that

$$B_n \subset S_n. \tag{12}$$

For the special case that the set B_n is empty, there is nothing to prove. If the function set B_n is not empty, let $\varphi(z)$ and $\psi(z)$ denote arbitrary elements in B_n. Let r and ϱ denote corresponding r-values (cf. (11)). i.e.

$$\log M_\varphi(r) \leqslant g(r, t), \quad \log M_\psi(\varrho) \leqslant g(\varrho, t).$$

The notations can be chosen so that $\varrho \leqslant r$, i.e.

$$r_n \leqslant \varrho \leqslant r < r_{n+1}. \tag{13}$$

If we can now prove the inequality

$$D(r_n, \varphi, \psi) < g(r_n, x_n t), \tag{14}$$

then we have proved Theorem 5.3, and we have also proved that any element $\psi(z)$ in B_n can be chosen as the centre of the sphere S_n in (9).

The proof of (14) is indirect. We assume

$$D(r_n, \varphi, \psi) \geqslant g(r_n, x_n t). \tag{15}$$

The Lemma 5.2 is applied to the inequality

$$\log M_\psi(r_n) \leqslant \log M_\psi(\varrho) \leqslant g(\varrho, t) < g(r_{n+1}, t) = g(r_1, y_{n+1} t) = g(r_1, y_n x_n t) =$$
$$= g(r_n, x_n t) \leqslant D(r_n, \varphi, \psi),$$

and we obtain

$$\log M_\varphi(r_n) \geqslant D(r_n, \varphi, \psi).$$

The final estimate becomes

$$g(r, t) \geqslant \log M_\varphi(r) \geqslant \log M_\varphi(r_n) \geqslant D(r_n, \varphi, \psi) \geqslant g(r_n, x_n t) = g(r_{n+1}, t)$$

and

$$r \geqslant r_{n+1}.$$

This contradicts (13), and the assumption (15) therefore was false. Hence (14) is proved, and the proof of Theorem 5.3 is complete.

Definition 5.4. *Let $f(z)$ be an entire function, t a real number and p a natural number. The set $A_t^p(f) = A_t^p(F(r), f(z))$ is defined as the set of complex numbers a for which*

$$\liminf_{r\to\infty} (F(\log M_a^p(r)) - F(r)) < \log t$$

where

$$M_a^p(r) = \max_{|z|=r} |f(z) \cdot e^{az^p}|.$$

Thus, $A_t^p(f)$ is the set of complex numbers a, for which the generalized lower order of $f(z)\cdot e^{az^p}$ with respect to $F(r)$ is less than t.

Interpretation of Theorem 5.3. Let p be a given natural number. We introduce the assumption that all functions denoted $\varphi(z)$, $\varphi_n(z)$ and $\psi(z)$ are of the form $z^p\cdot$constant. Under this assumption we could once again formulate and prove (the old proof works) Theorem 5.3.

However, for the present situation, we can actually compute the distance function D, as follows.

$$D(r, az^p, bz^p) = \frac{1}{4\pi}\int_0^{2\pi} \left|\mathrm{Re}\,(a\cdot r^p\cdot e^{ip\theta} - b\cdot r^p\cdot e^{ip\theta})\right| d\theta = \frac{1}{\pi} r^p \left|a-b\right|.$$

For a given b, $bz^p = \varphi_n(z)$, the sphere S_n in (9) corresponds to a circle for the a-values. The sphere of functions az^p for which

$$D(r, az^p, bz^p) < g(r_n, x_n t)$$

holds, corresponds to the following circle for the a-values:

$$\frac{1}{\pi} r_n^p \left|a-b\right| < g(r_n, x_n t)$$

or

$$\left|a-b\right| < \pi\cdot r_n^{-p}\cdot g(r_n, x_n t) = d_n.$$

We now define the numbers x_n by

$$r_n^{-p}\cdot g(r_n, x_n(t+\varepsilon)) = 1$$

for some arbitrary $\varepsilon > 0$.

For the special case $g(r,t) = r^t$ ($F(r) = \log\log r$) it then follows that all the x_n are equal:

$$x_n = \frac{p}{t+\varepsilon} > 1.$$

From Theorem 5.3 we recall that

$$r_n = g(r_1, y_n) = g(r_1, x_1^{n-1}) = r_1^{x_1^{n-1}}$$

which gives

$$r_{n+1} = r_n^{x_1}.$$

Together with

$$\frac{1}{\pi} d_n = r_n^{x_1 t - p}$$

this becomes
$$\frac{1}{\pi} d_{n+1} = \left(\frac{1}{\pi} d_n\right)^{x_1}. \tag{16}$$

We notice that $r_1 > 1$ $(F(r_1) > -\infty)$ implies

$$d_n = \pi \cdot r_1^{-\varepsilon \cdot x_1^n} \to 0$$

as $n \to \infty$.

As a consequence of (16) there exists a number $N > 0$ such that

$$d_{n+N} < e^{-x_1^n} = e^{-e^{n \log x_1}},\; n > 0.$$

The sequence $\{d_n\}_{n=1}^{\infty}$ is a majorizing sequence for the set $A_t^p(f)$ and it now follows from the definition of majorizing sequences, in Section 1, that

$$\left\{e^{-e^{n \cdot \log x_1}}\right\}_{n=1}^{\infty}$$

is also such a majorizing sequence for the set $A_t^p(f)$. From Definition 1.2 it then follows

$$\mu(A_t^p(f)) \leqslant (\log x_1)^{-1} = \left(\log \frac{p}{t+\varepsilon}\right)^{-1}.$$

Since ε is arbitrary, it follows that

$$\mu(A_t^p(f)) \leqslant \left(\log \frac{p}{t}\right)^{-1}.$$

This result can be formulated as a theorem.

Theorem 5.5. *Let $f(z)$ be an entire function, and let $p > 0$ be a given integer. The set $A_t^p(f)$ of a-values for which the entire function*

$$f(z)\, e^{az^p}$$

is of lower order less than t $(0 < t < p)$ satisfies

$$\mu(A_t^p(f)) \leqslant \left(\log \frac{p}{t}\right)^{-1}. \tag{17}$$

μ is the set function of Definition 1.2.

The phrase "lower order" is here used in its ordinary sense.

Chapter III. Subsets of the set $E_t(f)$

In Section 6 of this chapter we prove that any given countable set $\{\varphi_n(z)\}_{n=1}^{\infty}$ of entire functions can be contained in a set $E_t(F(r), f(z))$. This statement (Theorem 6.1) holds true for every admissible $F(r)$.

The result in Section 7 concerns the special case $F(r) = \log\log r$ (i.e the lower order has the usual meaning). The sets of a-values for which $az^p \in E_t(f)$ are investigated, and this gives a theorem (Theorem 7.1) converse to Theorem 5.5.

6. *Countable subsets of $E_t(f)$*

The function $F(r)$ satisfies the assumptions in Section 4.

$$\lim_{r\to\infty} (F(2r) - F(r)) = 0 \tag{4}$$

$$\lim_{r\to\infty} (F(r) - F(\log r)) = +\infty. \tag{5}$$

Theorem 6.1. *For an arbitrary countable set $\{\varphi_n(z)\}_{n=1}^{\infty}$ of entire functions there exists some corresponding entire function $f(z)$ with*

$$\lambda(F(r), f(z)\cdot e^{\varphi_n(z)}) = 0$$

for every n.

This means that

$$\liminf_{r\to\infty} (F(\log M_{\varphi_n}(r)) - F(r)) = -\infty, \tag{18}$$

where

$$M_{\varphi_n}(r) = \max_{|z|=r} |f(z)\cdot e^{\varphi_n(z)}|.$$

The proof of Theorem 6.1 makes use of the following theorem.

Theorem 6.2. *Let $\{\varphi_n(z)\}_{n=1}^{\infty}$ be a given sequence of entire functions, and let $h(r)$ be a monotonic function with*

$$\lim_{r\to\infty} h(r) = +\infty.$$

Then there exists an entire function $f(z)$ such that for every $n > 0$,

$$\liminf_{r\to\infty} \frac{\log \max_{|z|=r} |f(z)\cdot e^{\varphi_n(z)}|}{h(r)\cdot \log r} = 0. \tag{19}$$

Proof of Theorem 6.2. Without loss of generality we may assume that every element of the sequence $\{\varphi_n(z)\}_{n=1}^{\infty}$ occurs an infinity of times in this sequence. Denote

$$k(r) = \sqrt{h(r)}\cdot \log r.$$

To prove Theorem 6.2 it is now sufficient to define an entire function $f(z)$ which satisfies

$$\log \max_{|z|=r_n} |f(z)\, e^{\varphi_n(z)}| < k(r_n) \tag{20}$$

for some sequence $\{r_n\}_{n=1}^{\infty}$ with

$$\lim_{n\to\infty} r_n = +\infty.$$

We now give the notations needed for the definition of the function $f(z)$. Let

$$\varphi_n(z) = \sum_{\nu=0}^{\infty} a_\nu z^\nu.$$

Then we denote

$$\varphi_n(N, z) = \sum_{\nu=0}^{N} a_\nu z^\nu.$$

$P_n(z)$ is the following polynomial

$$P_n(z) = \left(1 + \frac{\varphi_n(M_n, z)}{M_n}\right)^{M_n} \left(1 - \frac{\varphi_n(L_n, z)}{L_n}\right)^{L_n}$$

where M_n and L_n are defined as follows.

To begin with, let $\quad r_0 = 1, \quad L_0 = 1, \quad M_1 = 1.$

To obtain a recursive definition, assume that the $3n$ numbers

$$r_0,\ L_0,\ M_1,\ r_1,\ L_1,\ \ldots,\ r_{n-1},\ L_{n-1},\ M_n$$

are already known.

We then choose a number r_n which satisfies both

$$\log \prod_{\nu=1}^{n-1} |P_\nu(z)| + M_n \cdot \log \left| 1 + \frac{\varphi_n(M_n, z)}{M_n} \right| \leqslant \tfrac{1}{2} k(r_n) \tag{21}$$

for $|z| = r_n$, and $r_n > 2r_{n-1}$.

The natural number L_n is then chosen, so that for $|z| = r_m \leqslant r_n$ $(1 \leqslant m \leqslant n)$, $s = \pm 1$ there holds, both for $\nu = n$ and $\nu = n+1$

$$\left| \log \left(1 - \frac{s \cdot \varphi_\nu(L_n, z)}{L_n}\right)^{L_n} \cdot e^{s \cdot \varphi_\nu(z)} \right| < 2^{m-n-3} k(r_m). \tag{22}$$

Finally M_{n+1} is defined as $\quad M_{n+1} = L_n.$

We now consider a function $f(z)$ defined by

$$f(z) = \prod_{n=1}^{\infty} P_n(z). \tag{23}$$

The inequality (22) shows that the infinite product is convergent. For $|z| = r_m$ we then estimate $\log |f(z)\, e^{\varphi_m(z)}|$. Formula (21) gives:

$$\log \prod_{\nu=1}^{m-1} |P_\nu(z)| + M_m \cdot \log \left| 1 + \frac{\varphi_m(M_m, z)}{M_m} \right| \leqslant \tfrac{1}{2} k(r_m). \tag{24}$$

Now write $P_n(z) = e^{-\varphi_n(z)} \cdot P_n(z) \cdot e^{\varphi_n(z)}$ for $n > m$, in (23). First, let $s = 1$ and $\nu = n \geqslant m$ in (22),

$$\log \left| \left(1 - \frac{\varphi_n(L_n, z)}{L_n}\right)^{L_n} \cdot e^{\varphi_n(z)} \right| < 2^{m-n-3} k(r_m) \tag{25}$$

for $|z| = r_m$.

Then, let $s = -1$ and $\nu = n + 1 > m$, i.e. $M_\nu = L_n$ in (22). For $|z| = r_m$ we then get

$$\log \left| \left(1 + \frac{\varphi_\nu(M_\nu, z)}{M_\nu}\right)^{M_\nu} \cdot e^{-\varphi_\nu(z)} \right| < 2^{m-\nu-2} k(r_m). \tag{26}$$

The sum of (24), and (25) for all $n \geqslant m$, and (26) for all $\nu > m$, gives (20). Because $r_n \geqslant 2^n$ the condition

$$\lim_{n \to \infty} r_n = +\infty$$

of (20) is satisfied, and the proof of Theorem 6.2 is complete.

Proof of Theorem 6.1. The function $F(r)$ is given and we intend to define a function $h(r)$ ($h(r) + \infty$, and $h(r)$ depends on $F(r)$) so that (19) of Theorem 6.2 implies (18) of Theorem 6.1. When this is done, Theorem 6.1 follows from Theorem 6.2.

The problem is now reduced to that of defining $h(r)$ from $F(r)$ so that

$$\liminf_{r \to \infty} (F(\log M_{\varphi_n}(r)) - F(r)) = -\infty \tag{27}$$

whenever

$$\liminf_{r \to \infty} \frac{\log M_{\varphi_n}(r)}{h(r) \cdot \log r} = 0$$

or even when

$$\liminf_{r \to \infty} \frac{\log M_{\varphi_n}(r)}{\sqrt{h(r)} \cdot \log r} < +\infty. \tag{28}$$

Let the function $h(r)$ be defined by

$$F(\sqrt{h(r)} \cdot \log r) = 1 + F(\log r). \tag{29}$$

By repeated use of (4) we obtain, for each fixed n

$$\lim_{x \to \infty} (F(2^n \cdot x) - F(x)) = 0. \tag{4'}$$

It follows from (4′) and (29) that $\liminf_{r \to \infty} h(r) > 2^n$, i.e.

$$\lim_{r \to \infty} h(r) = +\infty.$$

From (28) and (4′) it follows that

$$\liminf_{r\to\infty}\,(F(\log M_{\varphi_n}(r)) - F(\sqrt{h(r)}\cdot\log r)) \leqslant 0 \tag{30}$$

but we also have:

$$\lim_{r\to\infty}\,(F(\sqrt{h(r)}\cdot\log r) - F(\log r)) = 1 \tag{29'}$$

and

$$\lim_{r\to\infty}\,(F(\log r) - F(r)) = -\infty. \tag{5'}$$

The sum of (30), (29′) and (5′) gives (27) and the proof of Theorem 6.1 is complete.

7. *Subsets of the set* $A_t^p(f)$

Our aim in this section is to show that the constant $(\log p/t)^{-1}$ in Theorem 5.5 cannot be replaced by any smaller real number. The question that will remain unsolved is whether we can replace the inequality by strict inequality in (17) or vice versa for (31).

As before, μ denotes the set function of Definition 1.2.

Theorem 7.1. *Let t be a given real number, and p a given natural number, $0<t<p$. A denotes a set of complex numbers. In order that there exist an entire function $f(z)$ with*

$$A \subset A_t^p(f(z)),$$

a necessary condition is (Theorem 5.5)

$$\mu(A_t^p(f)) \leqslant \left(\log\frac{p}{t}\right)^{-1}, \tag{17}$$

and a sufficient condition is

$$\mu(A) < \left(\log\frac{p}{t}\right)^{-1}. \tag{31}$$

The sufficiency condition remains to be proved. We first give the proof for the case $p=1$.

Because A and t are given, the strict inequality in (31) implies the existence of x such that

$$\mu(A) < \left(\log\frac{1}{x}\right)^{-1} = \frac{1}{k} < \left(\log\frac{1}{t}\right)^{-1}. \tag{32}$$

We shall now try to find an entire function $f(z)$ such that, whenever $a\in A$, the lower order of

$$f(z)\,e^{az}$$

is less than t. For this purpose we define four sequences, with elements

$$d_n,\; r_n,\; a_n \text{ and } c_n.$$

Let the first two sequences be

$$d_n = e^{-x^{-n}} = e^{-e^{kn}}, \quad n = 1, 2, \ldots$$

$$r_n = d_n^{-(x/1-x)}.$$

We notice that $\qquad r_{n+1} = r_n^{1/x}, \; d_{n+1} = d_n^{1/x} \text{ and } d_n = r_{n+1}^{x-1}.$

As a consequence of $\mu(A) < (1/k)$ in (32) and Definition 1.2 there exist sequences of complex numbers

$$\{a_n\}_{n=1}^{\infty}$$

such that every $a \in A$ satisfies

$$|a - a_n| < e^{-e^{kn}} = d_n$$

for an infinity of values of n. We take one such sequence to be our third sequence. An other sequence of this kind is

$$\{c_n\}_{n=1}^{\infty}$$

where $$c_n = a_n \text{ if } |a_n| < b_n = \frac{n}{10}$$

and $$c_n = 0 \quad \text{if } |a_n| \geqslant b_n = \frac{n}{10}.$$

This is because those n for which

$$|a - a_n| < d_n$$

but not $$|a - c_n| < d_n$$

holds, constitute a finite set. Thus for every $a \in A$ the inequality

$$|a - c_n| < d_n$$

holds for an infinity of values of n. We take $\{c_n\}_{n=1}^{\infty}$ as our fourth sequence.

We now introduce the following notation for MacLaurin polynomials of $e^{c_n z}$

$$e(N, c_n z) = \sum_{m=0}^{N} \frac{(c_n z)^m}{\underline{|m}} = e^{c_n z} - \sum_{m=N+1}^{\infty} \frac{(c_n z)^m}{\underline{|m}}. \tag{33}$$

If N is not an integer, the first sum is to be taken over $0 \leqslant m < N$. The function $f(z)$ is then defined by

$$f(z) = \prod_{n=1}^{\infty} P_n(z)$$

where $$P_n(z) = e(nr_n, c_n z) \cdot e((n+1)\, r_{n+1}, -c_n z).$$

The convergence of the infinite product follows from the estimate of $|1 - P_m(z)|$ on page 451.

It now remains to give an estimate of the lower order of $f(z)\, e^{az}$ for an arbitrary $a \in A$. This $a \in A$ is kept constant from now on. Let $n > 2$ be one of those infinitely many natural numbers for which

$$|a - c_n| < d_n.$$

For $|z| = r_{n+1}$ we estimate $\log |f(z)\, e^{az}|$. The expression for $f(z)\, e^{az}$ is divided into five factors.

$$Q_1(z) = P_1(z) \cdot P_2(z) \cdot \ldots \cdot P_{n-1}(z)$$

$$Q_2(z) = e(nr_n, c_n z)$$

$$Q_3(z) = e^{az} \cdot e^{-c_n z}$$

$$Q_4(z) = e^{c_n z} \cdot e((n+1)\, r_{n+1}, -c_n z)$$

$$Q_5(z) = P_{n+1}(z) \cdot P_{n+2}(z) \cdot \ldots$$

For the estimate of $Q_1(z)$, let

$$1 \leqslant m \leqslant n-1 \quad \text{and} \quad |z| = r_{n+1}.$$

$$\begin{aligned} \log |P_m(z)| &= \log |e(mr_m, c_m z) \cdot e((m+1)\, r_{m+1}, -c_m z)| \\ &< 2 \cdot \log e(nr_n, nr_{n+1}) \\ &< 2 \cdot \log ((nr_{n+1})^{nr_n}) = 2nr_n \log (nr_{n+1}). \end{aligned}$$

For $Q_1(z)$ this becomes

$$\log |Q_1(z)| < 2n(n-1)\, r_n \log (nr_{n+1}).$$

The same estimate for $Q_2(z)$ gives

$$\log |Q_2(z)| < nr_n \log (nr_{n+1}).$$

For $Q_3(z)$ we use $$|a - c_n| < d_n = r_{n+1}^{x-1}$$

which gives $$\log |Q_3(z)| = |a - c_n| \cdot r_{n+1} < r_{n+1}^{x}.$$

With $N = (n+1)\, r_{n+1} + 1$, $Q_4(z)$ becomes

$$Q_4(z) = 1 - e^{c_n z} \sum_{m=N}^{\infty} \frac{(-c_n z)^m}{\underline{|m}}.$$

The inequalities $\underline{|m} > (m/3)^m$, $m \geqslant N > 10 b_{n+1} |z| > 10 |c_n z|$ give:

$$|1 - Q_4(z)| < e^{b_n r_n + 1} \cdot 2 \frac{(b_n r_{n+1})^N}{\underline{|N}} < 2 \left(e^{\frac{1}{10}} \cdot \frac{3}{10}\right)^N < \frac{1}{2}$$

and it follows $$\log |Q_4(z)| < 1.$$

For $Q_5(z)$, let $m > n$. The second part of (33) gives

$$P_m(z) = \left(1 - e^{-c_m z} \sum_{k=mr_m+1}^{\infty} \frac{(c_m z)^k}{\lfloor k}\right)\left(1 - e^{c_m z} \sum_{k=(m+1)r_m+1+1}^{\infty} \frac{(-c_m z)^k}{\lfloor k}\right)$$

$$|1 - P_m(z)| < 3 \cdot e^{b_m r_{n+1}} \sum_{k=mr_m}^{\infty} \frac{(b_m r_{n+1})^k}{\lfloor k} < 6 \cdot e^{b_m r_m} \cdot \left(\frac{mr_{n+1}}{10}\right)^{mr_m} \cdot \left(\frac{3}{mr_m}\right)^{mr_m} < 6 \cdot 2^{-m}.$$

For $m > n > 2$ it follows, for $|z| = r_{n+1}$

$$|\log |P_m(z)|| < 12 \cdot 2^{-m}$$

and
$$\log |Q_5(z)| < 2.$$

The final estimate of $f(z)\, e^{az}$ for $|z| = r_{n+1}$, $n > 2$, and $|a - c_n| < d_n$ becomes

$$\log |f(z)\, e^{az}| < (2n(n-1) + n)\, r_n \log (nr_{n+1}) + r_{n+1}^{x} + 1 + 2 = r_{n+1}^{x+o(1)}.$$

Therefore it follows from
$$\lim_{n\to\infty} r_n = +\infty$$

that the lower order of $f(z)\, e^{az}$ is at most equal to x.

Since $a \in A$ was arbitrarily chosen, it follows that

$$A \subset A_t^1(f)$$

and this proves Theorem 7.1 for the case $p = 1$.

For $p > 1$, put $t = \tau p$, $0 < \tau < 1$. Let A be a given set with

$$\mu(A) < \left(\log \frac{1}{\tau}\right)^{-1} = \left(\log \frac{p}{t}\right)^{-1}.$$

We have just proved that there exists an entire function $f(z)$ with

$$A \subset A_\tau^1(f(z)).$$

The function $f(z^p)$ gives the final solution, and this is because

$$A \subset A_\tau^1(f(z)) = A_t^p(f(z^p))$$

which follows from the definition of lower order. The proof of Theorem 7.1 is now complete.

Chapter IV. Miscellaneous results

8. *Strong subadditivity*

The strong subadditivity is a property of certain set functions. A set function (denoted m) is said to be strongly subadditive, if the inequality

$$m(A \cup B) + m(A \cap B) \leqslant m(A) + m(B)$$

holds for arbitrary sets A und B.

We first mention an example of a strongly subadditive set function. Let $C(A)$ denote the capacity (Definition 3.1). The G-capacity is then defined by

$$G\text{-cap}\,(A) = \Phi(C(A))^{-1}$$

(cf. [4] Def. 12 p. 43). Then the G-capacity is strongly subadditive ([4] Theorem 8 p. 47), for certain kernel functions $\Phi(t)$.

The result of this section concerns the subadditive ([9] Satz 3) set function $m(g(x), A)$ of Definition 1.1, which turns out not to be strongly subadditive.

Proposition 8.1. *Let m be the set function of Definition 1.1. Then there exist sets A_1 and B_1 of complex numbers with*

$$m(A_1 \cap B_1) = m(A_1) = m(B_1) = \tfrac{1}{2}$$

and

$$m(A_1 \cup B_1) = 1.$$

Proof of Proposition 8.1. We now use the same construction as in the proof of Hilfssatz 2, [9]. The constant k in Hilfssats 2 is here $k = 1$. The set A is defined by means of a sequence of circles C_n, and we here list all important properties of these sets.

1. $\{d_n = g(n)\}_{n=1}^{\infty}$ is a majorizing sequence for the set A. (This implies $m(g(x), A) \leqslant 1$).
2. $m(g(x), A) = m(A) = 1$.
3. The circles C_n uniquely define the set A by means of the following covering:

$$A = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} C_n.$$

   The radius of the circle C_n is $g(n)$.

   The center a_n of C_n is a real number, and the set A is thus restricted to the real axis.

4. For the circles C_n there also holds (for certain numbers $H(1)$, $H(2)$, ...)

$$\bigcup_{n=H(p)}^{n=H(p+1)-1} C_n \supset \bigcup_{n=H(p+1)}^{n=H(p+2)-1} C_n \supset \ldots \supset A$$

   if p is big enough ($p \geqslant m$).

5. For a given natural number N with

$$H(p) \leqslant N < H(p+1), \quad (p \geqslant m)$$

those n for which both

$$H(p+1) \leqslant n < H(p+2)$$

and
$$C_n \subset C_N$$

hold, are those given by the relation

$$2^{2^N} \leqslant n < (2^{2^N})^2.$$

6. For all n with
$$2^{2^\eta} \leqslant n < (2^{2^\eta})^2 - 1$$

we have
$$a_{n+1} - a_n = 2g(\eta) \cdot 2^{-2^{\eta+1}} \cdot (1 \pm \varepsilon_n)$$

where
$$\lim_{n \to \infty} \varepsilon_n = 0.$$

The sets A_1, B_1, A_2:

The definitions are:
$$A_1 = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} C_{2n+1}$$

$$B_1 = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} C_{2n}$$

$$A_2 = \bigcap_{p=m}^{\infty} \bigcup_{\substack{n=H(p) \\ n+p=\text{even number.}}}^{n=H(p+1)-1} C_n$$

For every point $x \in A$ there exists a sequence $n_1, n_2, \ldots$ such that

$$x \in C_{n_i} \quad i = 1, 2, \ldots$$

and
$$H(m) \leqslant n_1 < H(m+1) \leqslant n_2 < \ldots.$$

If $x \in A_2$ this sequence is of the form

$$\ldots \text{odd, even, odd, even, } \ldots \tag{34}$$

For each $x \in A_1$ the sequence $n_1, n_2, \ldots$ contains an infinity of odd numbers (an infinity of even numbers if $x \in B_1$). It follows

$$A_2 \subset A_1 \subset A, \quad A_2 \subset B_1 \subset A$$

and
$$A_2 \subset A_1 \cap B_1, \quad A = A_1 \cup B_1$$

and
$$m(A_1 \cup B_1) = m(A) = 1.$$

We deduce from the preceeding that the sequence $\{g(2n)\}_{n=1}^{\infty}$ is a majorizing sequence for the set A_1, and also for the set B_1. This gives (cf. Definition 1.1)

$$m(A_1) \leqslant \tfrac{1}{2}, \quad m(B_1) \leqslant \tfrac{1}{2}.$$

The subadditivity of m ([9], Satz 3) gives

$$m(A_1) + m(B_1) \geqslant m(A_1 \cup B_1) = 1,$$

and it follows $$m(A_1) = m(B_1) = \tfrac{1}{2}.$$

There remains to prove $$m(A_1 \cap B_1) = \tfrac{1}{2},$$

and for this purpose it is sufficient to prove

$$m(A_2) \geqslant \tfrac{1}{2}. \tag{35}$$

This is because $$A_2 \subset A_1 \cap B_1$$

and $$m(A_1 \cap B_1) \leqslant \tfrac{1}{2}.$$

The proof of (35) is indirect, and we assume that $g((2+\varepsilon)\,n)$ is a majorizing sequence for the set A_2. From this assumption it follows that there exists a sequence C_n' of circles of radii $g((2+\varepsilon)\,n)$ with

$$A_2 \subset \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} C_n'$$

i.e. $$A_2 \subset \bigcup_{n=N}^{\infty} C_n' \tag{36}$$

for every N.

We now consider the following two families of circles, subsets of $\{C_n\}_{n=1}^{\infty}$ and $\{C_n'\}_{n=1}^{\infty}$:

$$\alpha_i = \left\{ C_n' \,\middle|\, (2^{2^{n_{i-1}}})^2 \cdot \frac{1}{2+\varepsilon} + 2 \leqslant n < (2^{2^{n_i}})^2 \frac{1}{2+\varepsilon} + 2 \right\}, \quad i = 1, 2, \ldots$$

where $$2^{2^{n_{i-1}}} \leqslant n_i < (2^{2^{n_{i-1}}})^2,$$

and

$$\beta_i = \{C_n \mid H(p_i) \leqslant 2^{2^{n_i}} \leqslant n < (2^{2^{n_i}})^2 \leqslant H(p_i+1) = H(p_{i+1}), \text{ and } n + p_i \text{ is even}\},$$
$$i = 1, 2, \ldots.$$

The same estimate as in [9] (the proof of Hilfssatz 2) gives that there exists one $C_n \in \beta_i$ (say $C_{n_{i+1}}$) which does not intersect any $C_n' \in \alpha_i$. This holds if n_i is big enough. Therefore $C_{n_i} \supset C_{n_{i+1}} \supset C_{n_{i+2}} \supset \ldots$ defines a point $x \in A_2$ with

$$x \notin \bigcup_{n=N}^{\infty} C_n'$$

(if N is big enough). This contradicts (36). Thus the sequence

$$\{g((2+\varepsilon)\,n)\}_{n=1}^{\infty}$$

is not a majorizing sequence for the set A_2 and

$$m(A_2) \geqslant \tfrac{1}{2}. \tag{35}$$

The proof of Proposition 8.1 is now complete.

9. *The vector sum of nullsets*

The sum, $A+B$, of two sets of complex numbers is defined as

$$A+B=\{z \mid z=a+b,\ a\in A,\ b\in B\}.$$

If the set A has $m(g(x), A)\leqslant(1/k)$ and if the set B is countable, then

$$m(g(x), A+B)\leqslant\frac{1}{k}$$

([9] Satz 5).

If the sets A and B are only assumed to have

$$m(g(x), A)\leqslant\frac{1}{k} \quad\text{and}\quad m(g(x), B)\leqslant\frac{1}{k},$$

then the vector sum $A+B$ can contain all complex numbers. This statement is independent of the choice of the function $g(x)$. An example where the vector sum $A+B$ contains all complex numbers is the following:

Let the set $\{a_n\}_{n=1}^{\infty}$ be dense and denote

$$C_n=\{z \,\big|\, |z-a_n|<g(kn)\}$$

and let
$$A=B=\bigcap_{m=1}^{\infty}\bigcup_{n=m}^{\infty} C_n.$$

For a given arbitrary complex number z we can define a sequence $n_1 n_2 \dots$ such that

$$C_{n_{i+1}}\subset\{\zeta \,\big|\, |z-\zeta-a_{n_i}|<g(kn_i)\}$$

This is possible since $\{a_n\}_{n=1}^{\infty}$ is dense and $\lim_{n\to\infty} g(kn)=0$.

It follows that
$$C_{n_{i+2}}\subset C_{n_i}$$

and the set
$$\bigcap_{m=1}^{\infty}\bigcup_{i=m}^{\infty} C_{n_i}$$

therefore defines two complex numbers, say $a\in A$ and $b\in B$, with $a+b=z$. Thus, we have shown that each complex number z belongs to $A+B$. This example originates from [3].

We now investigate the special case when the sets A and B are assumed to have capacity zero with respect to some arbitrary given positive kernel function $\Phi(t)$. How does this assumption on A and B restrict the vector sum $A+B$?

It is here sufficient to apply Proposition 3.2. The function $g(x)$ is at our disposal, and the result is that the sets A and B can be chosen so that each complex number z belongs to $A+B$.

10. *A functional equation*

In this section we study the functional equation from Section 4 of this paper:

$$g(g(r, t_1), t_2) = g(r, t_1 \cdot t_2). \tag{6}$$

This functional equation is important in the theory of iterated functions, and $\log t$ here indicates the "number of iterations" of the function $r \to g(r, e)$.

The most immediate solutions of (6) are $g(r, t) = rt$ and $g(r, t) = r^t$. In this paper the function $g(r, t)$ generalizes the function r^t in function theoretic applications (Section 4, 5, 6). We now give a solution of the functional equation (6), and then study the convexity of $g(r, t)$ for fixed values of $t, t > 1$. The function $g(r, t)$ is assumed to be twice differentiable with respect to both variables.

The solution of (6): The following assumptions are made

$$g(r, 1) = r$$

$$\frac{\partial g(r, t)}{\partial t} > 0.$$

For $t = 1$ we denote
$$\frac{\partial g(r, t)}{\partial t} = f(r). \tag{37}$$

We also assume that
$$f(r)^{-1} = \frac{dF(r)}{dr} \tag{38}$$

and
$$F(r_0) = -\infty, \quad F(R_0) = +\infty, \quad -\infty \leqslant r_0 < R_0 \leqslant +\infty.$$

The solution is then given for

$$r_0 < r < R_0,$$
$$0 < t < +\infty.$$

The functional equation (6) is applied with $t_1 = t$ and $t_2 = 1 + (dt/t)$.

$$g\left(g(r, t), 1 + \frac{dt}{t}\right) = g(r, t + dt).$$

The left and right hand sides are, up to first order terms

$$x + \frac{dt}{t} f(x) \quad \text{and} \quad x + dt \frac{\partial x}{\partial t},$$

where $x = g(r, t)$. From this we obtain

$$\frac{f(x)}{t} = \frac{\partial x}{\partial t}.$$

For $r = \text{const.}$ the relation to be integrated becomes

$$\frac{dt}{t} = \frac{dx}{f(x)},$$

which gives
$$\log t - \log 1 = F(g(r, t)) - F(r).$$

Then the solution of the functional equation is

$$g(r, t) = F^{-1}(F(r) + \log t).$$

A check shows that this solution fulfils all assumptions made.

Convex solutions:

A problem concerning the convexity of the solutions $g(r, t)$ of (6) is indicated in [2] (No. 5).

Let $g(r, t_1)$ be a convex function of r for some given $t_1 > 1$. Is $g(r, t)$ then convex for all $t > 1$?

The answer is in the negative

Proof. Differentiation of

$$F(g(r, t_1)) = F(r) + \log t_1$$

shows that the convexity condition $\dfrac{\partial^2 g(r, t_1)}{\partial r^2} > 0$ is equivalent to

$$\varphi(x) > \varphi(x + \log t_1) \tag{39}$$

where the function $\varphi(x)$ is defined by

$$F''(r)\,(F'(r))^{-2} = \varphi(F(r)).$$

The same method, or direct use of (37) and (38) shows that $g(r, t)$ is convex for every $t > 1$ if and only if the function $\varphi(x)$ is monotonic decreasing, or, which is the same, $F'(r)^{-1}$ is convex. Evidently (39) does not force the function $\varphi(x)$ to be monotonic.

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