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Linear partial differential operators and generalized distributions

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CONTENTS

PREFACE

The book "Linear partial differential operators" (denoted by [H] in the list of references) by Hörmander is written in the language of Schwartz distributions. At the Stanford conference in 1961, Beurling [3] presented the foundations of a certain more general theory of distributions, partly based on ideas published already in [1] and [2]. The purpose of the present paper is to develop this generalized distribution theory, including the spaces $\mathcal{B}_{p,k}$ of [H] and to use the theory to generalize material from Chap. III, IV and VI of [H]. Thus we study questions of existence and approximation and interior regularity of solutions of equations with constant coefficients and also consider equations which have no solutions.

If φ is a continuous function with compact support, the condition that $\varphi \in C_0^\infty$ could be expressed on the Fourier transform side by demanding that $\hat{\varphi}(\xi) \exp (N \log (1 + |\xi|))$ is in L_1 for each natural number N. In Beurling's theory, $\log (1 + |\xi|)$ is replaced by another subadditive function ω, which we can think of as larger. Then the class of test functions will be smaller and the class of distributions larger. Although much of the classical theory goes through, we sometimes get complications from the fact that a general ω is not as closely related to differentiation and thus to *differential* operators (as opposed to general convolution operators) as is $\log (1 + |\xi|)$. Another kind of complication comes from the fact that we do not consider only those ω which give rise to the same class of test functions as $\boldsymbol{\omega}$ (with $\boldsymbol{\omega}(\xi) = \omega(-\xi)$).[1] A summary of the paper is formed by the introductions to the various chapters.

Since most of our theorems have easily recognizable counterparts in [H] and in many cases the proofs are virtually the same, it would not be practical to make our presentation self-contained. Thus the proofs often consist just of a remark that the proof in [H] works. Similarly, the bibliography and the introductions to the various chapters should be completed by the corresponding parts of [H]. To avoid confusion of theorems etc. in the present paper and in the references, we always use abbreviations in the latter case. Thus Theorem 1.7.4 is in the present paper, but Th. 1.7.4 (of [H]) is not.

The author is greatly indebted to Professor Beurling who has permitted the publishing of his distribution theory and to Professor Hörmander whose suggestions have led to many improvements of the manuscript. In particular, the author had originally obtained only partial results in connection with Theorems 1.5.12, 3.4.11, 4.1.5 and 5.1.2.

Chapter I. Generalized distributions

1.0. *Introduction*

The purpose of this chapter is to develop those parts of the generalized distribution theory created by Beurling [3] which will be required in the following chapters. We have made two changes in the notation of [3]. First, we have called the space of test functions $\mathcal{D}_\omega$ instead of $\mathcal{A}_\omega$. This is done to stress the fact that Schwartz's space $\mathcal{D}$ is a special case of $\mathcal{D}_\omega$ and to get a natural notation for the space of multipliers on $\mathcal{D}_\omega$, namely $\mathcal{E}_\omega$. Our notation also parallels that of Roumieu [17], [18]. Second, in our notation $\mathcal{D}'_\omega$ is not the dual space of $\mathcal{D}_\omega$ but that of $\mathcal{D}_{\boldsymbol{\omega}}$ (where $\boldsymbol{\omega}(\xi) = \omega(-\xi)$). For Schwartz distributions, $\boldsymbol{\omega} = \omega$, and then the question does not occur. Our choice is due to the feeling that the Fourier transform is so important that e.g. the conditions

[1] Throughout the paper, the author has written and the reader is asked to read $\check{\omega}$ instead of $\boldsymbol{\omega}$. The printed notation is due to unfortunate typografical circumstances.

in the Paley–Wiener theorems (Theorems 1.4.1 and 1.8.14) should be similar for test functions and distributions.

We use freely the Schwartz theory as given in [19] or [H], Chap. I. For functional analysis see [6] or [21]. For the theory of subadditive functions (Section 1.2), see [11]. For the Denjoy–Carleman theory of classes of functions (Section 1.5), see [15] and the references given there.

Other treatments of generalized distributions are given by Gelfand and Šilov [10], Friedman [9] and Roumieu [17], [18].

1.1. *Notation*

We denote the points of two dual R^n spaces by $x=(x_1, \dots, x_n)$ and $\xi=(\xi_1, \dots, \xi_n)$ respectively. The letter n always denotes the dimension.

The scalar product is denoted by $\langle x, \xi\rangle = \sum_1^n x_i\xi_i$, and $|x|$ denotes $\langle x, x\rangle^{\frac{1}{2}}$.

In C^n the points are denoted by $z=x+iy$ or $\zeta=\xi+i\eta$. In this case, $\langle z, \zeta\rangle$ denotes $\Sigma\, z_i\zeta_i$ and $|z|$ denotes $\langle z, \bar{z}\rangle^{\frac{1}{2}} = (|x|^2+|y|^2)^{\frac{1}{2}}$.

The open ball $\{x; |x|<r\}$ in R^n we will denote by B_r.

We will use the multi-index notation as in [H], p. 4. Thus α denotes n-tuples $(\alpha_1, \dots, \alpha_n)$ of non-negative integers. $|\alpha|$ denotes $\Sigma\, \alpha_i$, and $\alpha!$ denotes $\alpha_1! \cdot \ldots \cdot \alpha_n!$. We write $D_j = -i\partial/\partial x_j$ and $D^\alpha = D_1^{\alpha_1} \dots D_n^{\alpha_n}$ and finally $\xi^\alpha = \xi_1^{\alpha_1} \dots \xi_n^{\alpha_n}$.

L_p norms will be denoted $\|\cdot\|_{L_p}$ (since $\|\cdot\|_p$ is given another meaning).

The Fourier transform $\hat{\varphi}$ of an element φ in $L_1(R^n)$ is defined as in [19] and [H], i.e.

$$\varphi^{\wedge}(\xi) = \hat{\varphi}(\xi) = \int e^{-i\langle x, \xi\rangle}\varphi(x)\,dx. \quad \left(\int \text{ means } \int_{R^n}.\right)$$

The symbol $\check{}$ is used as follows: $\check{f}(x) = f(-x)$. (See also footnote on p. 352.)

The translation operator τ_y is defined by $(\tau_y\varphi)(x) = \varphi(x-y)$.

The letter C (without super- or subscript) will always denote a positive constant, not necessarily the same at each occurrence.

Set-theoretic union is denoted by $\cup$, whereas $+$ stands for Minkowski addition. Thus if A and $B \subset R^n$ and $c \in R^n$, then $A+B = \{a+b;\ a\in A \text{ and } b \in B\}$ and $c+B = \{c\}+B$. Similarly, $\{x; x\in A \text{ and } x \notin B\}$ is written $A \cap \complement B$, whereas $A-B$ denotes $\{a-b;\ a\in A \text{ and } b\in B\}$.

Finally we introduce the following convenient notation concerning the inclusion of subsets S of R^n. The relation $S_1 \subset\subset S_2$ shall mean that the closure of S_1 is compact and contained in the interior of S_2. If $\{S_j\}_{j=1}^\infty$ is a sequence of sets, the relation $S_j \nearrow\nearrow S$ shall mean that $S_j \subset\subset S_{j+1}$ $(j=1, 2, \dots)$ and that $S = \bigcup S_j$. In particular we note that if $S_j \nearrow\nearrow S$ and K is a compact subset of S, then $K\subset S_j$ for some j.

1.2. *Subadditive functions* ω

Let ω be a real-valued function on R^n, continuous at the origin and having the property

$$(\alpha) \qquad 0 = \omega(0) = \lim_{x\to 0} \omega(x) \leqslant \omega(\xi+\eta) \leqslant \omega(\xi)+\omega(\eta) \quad (\forall\, \xi, \eta \in R^n).$$

An important class of such *subadditive* functions consists of those arising from concave functions in a way described in the following proposition.

Proposition 1.2.1. *If $\Omega(t)$ is an increasing continuous concave function on $[0, +\infty)$ and $\Omega(0)=0$, then the function ω defined by $\omega(\xi)=\Omega(|\xi|)$ satisfies (α).*

The proof is left to the reader.

It is helpful in the sequel to think of the special kind of ω given by Proposition 1.2.1. It is natural to ask how general this special case is. An important result in this direction is given in Theorem 1.2.7 below. For the moment we limit ourselves to the following question: Must a general ω be as smooth as a concave function? The answer is negative:

Example 1.2.2. Van der Wærden's example of a bounded continuous nowhere differentiable function of one variable, [20], satisfies condition (α).

In fact, $\omega(\xi)=\sum_1^\infty \omega_\nu(\xi)$ with

$$\omega_\nu(\xi) = \min \{|\xi - m\cdot 10^{-\nu}|;\ m \text{ integer}\},$$

and it is easy to see that ω_ν satisfies (α).

We remark that by adding the function of the example to an ω which satisfies (α) and is large at infinity we can destroy differentiability properties without violating (α) and without changing the growth properties of ω.

However, some regularity is implied by (α):

Proposition 1.2.3. *If ω satisfies (α), then ω is uniformly continuous in R^n.*

Proof. We get $$-\omega(-h) \leqslant \omega(\xi+h)-\omega(\xi) \leqslant \omega(h)$$

by two obvious applications of (α).

In the sequel we will constantly use condition (α) in a similar way to estimate ω upwards and downwards without explicit reference to condition (α). Before leaving the subject of smoothness, we prove a simple approximation lemma which will be used when the lack of smoothness gives rise to technical difficulties.

Lemma 1.2.4. *Let $\varepsilon>0$ and ω be given and suppose that ω satisfies (α). Then there exist a function ω_1 satisfying (α) and a constant $M>0$ with the following properties: For fixed $\xi_2, \ldots, \xi_n$, $\omega_1(\xi)$ is a piecewise linear function of ξ_1. We have $\sup_\xi|\omega_1(\xi)-\omega(\xi)| \leqslant \varepsilon$, and finally $|\partial\omega_1/\partial\xi_1| \leqslant M$ whenever the derivative exists.*

Proof. We choose $\delta>0$ in such a way that

$$\sup_{|\xi|\leqslant\delta} \omega(\xi) \leqslant \varepsilon.$$

Let us write $\xi=(\xi_1, \xi')$ with $\xi'=(\xi_2, \ldots, \xi_n)$. Then, if $0\leqslant h\leqslant\delta$, we have

$$|\omega(\xi_1+h, \xi')-\omega(\xi_1, \xi')| \leqslant \varepsilon.$$

We now define $\omega_1(\xi)=\omega(\xi)$ when $\xi_1=j\cdot\delta$ for integer j and define ω_1 by linearity between these points, keeping ξ' fixed. Then the approximation property follows, and it is also clear that we may take $M=\varepsilon/\delta$. It remains to prove that ω_1 satisfies

$$\omega_1(s\delta, \xi') \leqslant \omega_1(t\delta, \eta')+\omega_1((s-t)\delta, \xi'-\eta') \tag{1.2.1}$$

for all real s and t, where ξ' and η' are arbitrary given elements in R^{n-1}. We have (1.2.1) when s and t both are integers. We next claim that (1.2.1) holds when s is an integer but t is not. Let us for fixed integer s denote the right-hand side of (1.2.1) by $r(t)$. Let $m<t<m+1$, where m is an integer. Since r is an affine function in $(m, m+1)$, we have either $r(t) \geqslant r(m)$ or $r(t) \geqslant r(m+1)$, which proves our claim. Thus

$$-\omega_1(t\delta, \eta') \leqslant -\omega_1(s\delta, \xi') + \omega_1((s-t)\delta, \xi'-\eta'), \tag{1.2.2}$$

if s is an integer and t is real. Letting s vary in (1.2.2) and applying the same argument, we see that (1.2.2) holds for all real s and t. The proof is complete.

We will now discuss some growth properties of subadditive functions. We will mainly be interested in those ω which do not grow too fast at infinity. The crucial property is as follows.

Definition 1.2.5. *By $\mathfrak{M}_0 = \mathfrak{M}_0(n)$ we denote the set of all continuous real-valued functions ω on R^n satisfying the conditions* (α) *and*

$$(\beta) \qquad J_n(\omega) = \int_{|\xi|\geqslant 1} \frac{\omega(\xi)}{|\xi|^{n+1}}\, d\xi < \infty.$$

We collect in a proposition some obvious properties of $\mathfrak{M}_0$. The proof is left to the reader.

Proposition 1.2.6. *If $\omega \in \mathfrak{M}_0$, then $\boldsymbol{\omega} \in \mathfrak{M}_0$. If ω_1 and ω_2 are in $\mathfrak{M}_0$, then so are $\omega_1+\omega_2$ and* $\max(\omega_1, \omega_2)$.

We note that $\omega \in \mathfrak{M}_0(n)$ if $\omega(\xi) = \Omega(|\xi|)$, where Ω is a *concave function of convergence type*, i.e. a function having all the properties required in Proposition 1.2.1 and in addition satisfying

$$J(\Omega) = \int_1^\infty \frac{\Omega(t)}{t^2}\, dt < \infty.$$

We now give Beurling's proof ([3], Lem. 1) of a result which in many cases makes it possible to work with concave instead of subadditive functions.

Theorem 1.2.7. *Let $\omega \in \mathfrak{M}_0(n)$. Then there exists a concave function Ω of convergence type such that*

$$\max_{|\xi|\leqslant r} \omega(\xi) \leqslant \Omega(r).$$

Proof. We first consider the case $n=1$. By Proposition 1.2.6, we may assume that $\boldsymbol{\omega} = \omega$. Define $\omega_1(x) = \max_{|\xi|\leqslant |x|} \omega(\xi)$. We claim that $\omega_1 \in \mathfrak{M}_0(1)$. The proof that ω_1 satisfies (α) is left to the reader. We shall prove that $J_1(\omega_1) < \infty$. Let (a, b), with $b > a \geqslant 1$, be one of the intervals that form the open set where $\omega < \omega_1$. Let I be the interval $(a, a+l)$, where $l = \min(a, b-a)$. We will consider the following three sets:

$$E = \{x \in I;\ \omega(x) < \omega(a)/3\},\ E' = I \cap \complement E, \text{ and } E'' = I \cap (a+E-E).$$

We claim that $E'' \subset E'$. In fact, if $x \in E''$, then since ω satisfies (α) and $\boldsymbol{\omega} = \omega$, we have for some x_1 and x_2 in E that

$$\omega(x) \geqslant \omega(a) - \omega(-x_1) - \omega(x_2) > \omega(a)/3.$$

Denoting Lebesgue measure by $|\cdot|$, we thus have $|E| \leqslant |E''| \leqslant |E'|$, which implies that $|E'| \geqslant \frac{1}{2}l$. We get

$$A = \int_a^b \frac{\omega(x)}{x^2}\,dx \geqslant \int_{E'} \frac{\omega(x)}{x^2}\,dx \geqslant \frac{\omega(a)}{3}\int_{a+\frac{1}{2}l}^{a+l} \frac{dx}{x^2} = \frac{l\omega(a)}{6(a+\frac{1}{2}l)\,(a+l)}.$$

On the other hand, $$A_1 = \int_a^b \frac{\omega_1(x)}{x^2}\,dx = \omega(a)\,\frac{b-a}{ab}.$$

Considering the cases $l=a$ and $l=b-a$ separately, we find that $A_1/A \leqslant 18$. Thus $J_1(\omega_1) \leqslant 18 J_1(\omega) < \infty$.

Next we define ω_2 as the least concave majorant of ω_1 over $[0, \infty)$. Let (c, d) with $d > c \geqslant 1$ be one of the intervals where $\omega_1 < \omega_2$, and let $k = (\omega_1(d) - \omega_1(c))/(d-c)$. Then

$$\omega_2(x) = \omega_1(c) + k(x-c) \quad (\forall x \in (c, d)),$$

and $$\omega_1(x) \geqslant \omega_1(d) - \omega_1(d-x) \geqslant \omega_2(d) - \omega_2(d-x) \geqslant kx \quad (\forall x \in (c, d). \tag{1.2.3}$$

We also have $$\omega_1(x) \geqslant \omega_1(c) \quad (\forall x \geqslant c). \tag{1.2.4}$$

Let $B_i = \int_c^d \omega_i(x) x^{-2} dx$ $(i=1, 2)$. We claim that $B_2 \leqslant eB_1$. Without losing generality we may normalize by assuming that $c = \omega_1(c) = 1$. Then we must have $0 \leqslant k \leqslant 1$. If $d \leqslant e$, we combine (1.2.4) with the fact that $\omega_2(x)/x$ is decreasing to deduce that $\omega_2(x) \leqslant x\omega_1(x)$ in $(1, d)$, and hence $B_2 \leqslant eB_1$. In the sequel we suppose that $d > e$.

We have $$B_2 = (1-k)\left(1 - \frac{1}{d}\right) + k \log d. \tag{1.2.5}$$

We now distinguish the two cases $kd > 1$ and $kd \leqslant 1$. In the first case we use (1.2.4) when $kx \leqslant 1$ and (1.2.3) when $kx \geqslant 1$. We get

$$B_1 \geqslant 1 - k + k \log d + k \log k,$$

and from (1.2.5) we get

$$B_2 \leqslant 1 - k + k \log d.$$

Since $k \log k \geqslant -e^{-1}$ and $k \log d - k \geqslant 0$, we get $B_2/B_1 \leqslant e/(e-1)$ in this case. On the other hand, if $kd \leqslant 1$, it follows from (1.2.4) that $B_1 \geqslant 1 - 1/d$. From (1.2.5) we get

$$B_2 \leqslant 1 - \frac{1}{d} + k \log d \leqslant 1 - \frac{1}{d} - k \log k.$$

Since $1 - 1/d > 1 - 1/e$, we get in this case also $B_2/B_1 \leqslant e/(e-1) < e$. Thus we have in all cases $B_2 \leqslant eB_1$, which proves the theorem when $n=1$.

If $\omega \in \mathfrak{M}_0(n)$ with $n > 1$, we define functions ω_ν on R^1 by $\omega_\nu(t) = \omega(t^{(\nu)})$, where $t^{(\nu)} \in R^n$ has all coordinates zero except that $t_\nu^{(\nu)} = t$. It is clear that $\omega_\nu \in \mathfrak{M}_0(1)$, and thus there exist concave functions Ω_ν of convergence type such that $\omega_\nu(t) \leqslant \Omega_\nu(r)$ when $-r \leqslant t \leqslant r$. Since by condition (α), $\omega(\xi) \leqslant \sum_1^n \omega_\nu(\xi_\nu)$, it follows that $\Omega = \sum_1^n \Omega_\nu$ will have the properties required in the theorem. This completes the proof.

Remark. In Theorem 1.2.7 we may also arrange that $\Omega(r) \geqslant r^{\frac{1}{2}}$ and that $\omega(\xi)/\Omega(|\xi|) \to 0$ when $|\xi| \to \infty$. In fact, let Ω_0 be the function given by the theorem and take $\Omega_1(r) = \Omega_0(r) + r^{\frac{1}{2}}$. Then define $\Omega(r) = \lim_{\nu\to\infty} \Omega_\nu(r)$ where the Ω_ν ($\nu = 2, 3, \ldots$) are defined recursively as follows. Let G_1 be the graph of $\Omega_{\nu-1}$ and G_2 the graph of $2\Omega_{\nu-1}$. We shall construct the graph of Ω_ν. Choose $r_1 > 0$ such that $\int_{r_1}^\infty \Omega_{\nu-1}(r) r^{-2} dr < 2^{-\nu}$. Let T be a tangent of G_1 at r_1. Then T must intersect G_2 at some $r_2 > r_1$. The graph of Ω_ν shall coincide with G_1 for $r \leqslant r_1$, with T for $r_1 \leqslant r \leqslant r_2$ and with G_2 for $r \geqslant r_2$. Then the result follows, since

$$\int_1^\infty \frac{\Omega_\nu(r)}{r^2} dr \leqslant \int_1^\infty \frac{\Omega_{\nu-1}(r)}{r^2} dr + 2^{-\nu}.$$

Corollary 1.2.8.[1] *If $\omega \in \mathfrak{M}_0$, then $\omega(\xi) = o(|\xi|/\log|\xi|)$ when $|\xi| \to \infty$.*

Proof. In view of Theorem 1.2.7 and the remark following it, we need only prove that if Ω is a concave function of convergence type then $\Omega(t) = O(t/\log t)$ when $t \to +\infty$. Replacing the graph of Ω by the straight line segment from the origin to the point $(t, \Omega(t))$, we get

$$+\infty > J(\Omega) \geqslant \int_1^t \frac{x\Omega(t)}{t}\frac{dx}{x^2} = \frac{\Omega(t)\log t}{t},$$

which proves the corollary.

We will now prove that the result of the corollary is best possible.

Theorem 1.2.9. *If $\{t_\nu\}_1^\infty$ and $\{a_\nu\}_1^\infty$ are two sequences of positive numbers such that $t_\nu \to \infty$ and $\sum_1^\infty a_\nu < \infty$, then there exists a concave function Ω of convergence type such that $\Omega(t_\nu) \geqslant a_\nu t_\nu/\log t_\nu$ $(\forall \nu)$.*

Proof. Let $\Omega_\nu(t) = a_\nu \min(t, t_\nu)/\log t_\nu$ and define $\Omega(t) = \sum_1^\infty \Omega_\nu(t)$. Then Ω has the required properties, since $J(\Omega_\nu) = a_\nu(1 + 1/\log t_\nu)$.

1.3. *Spaces $\mathcal{D}_\omega$ of test functions*

Let ω satisfy (α). If $\varphi \in L_1(R^n)$ and if λ is a real number, we write

$$\|\varphi\|_\lambda = \|\varphi\|_\lambda^{(\omega)} = \int |\hat{\varphi}(\xi)|\, e^{\lambda\omega(\xi)}\, d\xi,$$

which may be finite or infinite.

We can now following Beurling [3] give the definition of the spaces of test functions to be used in the sequel.

Definition 1.3.1. *$\mathcal{D}_\omega$ is the set of all φ in $L_1(R^n)$ such that φ has compact support and $\|\varphi\|_\lambda < \infty$ for all $\lambda > 0$. The elements of $\mathcal{D}_\omega$ will be called test functions.*

Definition 1.3.2. *If E is a subset of R^n, then*

$$\mathcal{D}_\omega(E) = \{\varphi \in \mathcal{D}_\omega;\ \operatorname{supp} \varphi \subset E\}.$$

[1] Corollary 1.2.8 and Theorem 1.2.9 have been communicated to the author by Professor Hörmander.

Proposition 1.3.3. *If K is compact, $\mathcal{D}_\omega(K)$ is a Fréchet space under the natural linear structure and the seminorms $\|\cdot\|_m (m=1, 2, \ldots)$.*

Proof. Only completeness has to be proved. Let $\{\varphi_\nu\}_1^\infty$ be a Cauchy sequence in $\mathcal{D}_\omega(K)$. Since L_1 (with respect to the measure $e^{m\omega(\xi)}d\xi$) is complete, $\hat{\varphi}_\nu$ converges in L_1. Denote the limit function by f. It is clear that f is independent of m and that $f=\hat{\varphi}$ for some φ with $\operatorname{supp}\varphi \subset K$. This proves that $\varphi_\nu \to \varphi$ in $\mathcal{D}_\omega(K)$.

Definition 1.3.4. *If Ω is an open subset of R^n and if $K_\nu \nearrow \nearrow \Omega$ we define $\mathcal{D}_\omega(\Omega)$ as the inductive limit of the Fréchet spaces $\mathcal{D}_\omega(K_\nu)$.*

When $\Omega = R^n$, we will sometimes write $\mathcal{D}_\omega$ instead of $\mathcal{D}_\omega(R^n)$.

We note that $\mathcal{D}_\omega$ is a fundamental space in the sense of Gelfand and Šilov [10], [9], namely

1) $\mathcal{D}_\omega$ is a countable inductive limit of Fréchet spaces, and
2) If $\varphi_j \to 0$ in $\mathcal{D}_\omega$, then $\varphi_j \to 0$ pointwise.

In fact, 2) follows from the estimate

$$|\varphi_j(x)| = (2\pi)^{-n}\left|\int \hat{\varphi}_j(\xi)\, e^{i\langle x,\xi\rangle}\, d\xi\right| \leqslant (2\pi)^{-n}\int |\hat{\varphi}_j(\xi)|\, d\xi.$$

So far there has been no indication why we have demanded that the function ω entering in the definition of $\mathcal{D}_\omega$ shall satisfy condition (α). To give the main motivation for this we prove (cf. [3])

Proposition 1.3.5. *Let ω satisfy (α). Under pointwise multiplication, $\mathcal{D}_\omega(\Omega)$ is an algebra, and for each $\lambda > 0$ we have*

$$\|\varphi\psi\|_\lambda \leqslant (2\pi)^{-n}\|\varphi\|_\lambda \|\psi\|_\lambda \quad (\forall \varphi, \psi \in \mathcal{D}_\omega(\Omega)).$$

Proof. Since φ and $\psi \in L_2$, we have $(\varphi\psi)\hat{}\,(\xi) = (2\pi)^{-n}\hat{\varphi} * \hat{\psi}(\xi)$, and thus all we have to prove is that for all $\lambda > 0$,

$$\int e^{\lambda\omega(\xi)}\, d\xi \left|\int \hat{\varphi}(\xi-\eta)\,\hat{\psi}(\eta)\, d\eta\right| \leqslant \int e^{\lambda\omega(\xi)}\,|\hat{\varphi}(\xi)|\, d\xi \int e^{\lambda\omega(\eta)}\,|\hat{\psi}(\eta)|\, d\eta.$$

But this estimate follows from the inequality

$$\omega(\xi) \leqslant \omega(\xi-\eta) + \omega(\eta),$$

which is a form of (α).

After giving the definitions and first properties of $\mathcal{D}_\omega$, it is now natural to ask if $\mathcal{D}_\omega$ is non-trivial, i.e. contains any other function than zero. If the answer is affirmative, we want to know if $\mathcal{D}_\omega$ is sufficiently rich to contain partitions of unity. These problems of quasi-analyticity were solved by Beurling in [3]. We first give an important example, where the answer to these questions is affirmative:

Proposition 1.3.6. *If $\omega(\xi) = \log(1+|\xi|)$, then ω satisfies (α), and $\mathcal{D}_\omega = C_0^\infty(R^n) = \mathcal{D}$ (in the notation of Schwartz).*

In the simple proof, which is left to the reader, Proposition 1.2.1 could be used. On the other hand, it is easy to see by the properties of entire functions (or, of course, by the next theorem) that if $\omega(\xi)=|\xi|$, then $\mathcal{D}_\omega$ is trivial.

We now give Beurling's result.

Theorem 1.3.7. *If ω satisfies (α), then the following three conditions are equivalent:*

(β) $J_n(\omega)<\infty$ *(cf. Definition 1.2.5).*

(β') *For each compact K in R^n and each neighborhood V of K there exists $\varphi\in\mathcal{D}_\omega(V)$ such that $\varphi=1$ on K and $0\leqslant\varphi\leqslant 1$ everywhere.*

(β'') *$\mathcal{D}_\omega(R^n)$ is non-trivial.*

We remark that condition (β') implies the existence of partitions of unity, for instance in the form stated in Th. 1.2.3 of [H]. For convenience requiring slightly more than in [3], we make the following definition.

Definition 1.3.8. *We call φ of condition (β') a local unit for K.*

For the proof that $(\beta'')\Rightarrow(\beta)$ in Theorem 1.3.7 we refer to [3]. We will prove that $(\beta)\Rightarrow(\beta')$ by proving the following two lemmas:

Lemma 1.3.9. *Let $\omega\in\mathfrak{M}_0(n)$. Then $\mathcal{D}_\omega(B_\varepsilon)$ is non-trivial for each $\varepsilon>0$.*

Lemma 1.3.10. *If $\mathcal{D}_\omega(B_\varepsilon)$ is non-trivial for each $\varepsilon>0$, then condition (β') holds.*

The proofs we give are essentially those of [3]. We will start by considering the properties of a Poisson integral which will be used in the proof of Lemma 1.3.9. Let P be the Poisson kernel for the upper half-plane in one variable:

$$P(\xi,\eta)=\frac{\eta}{\pi}\frac{1}{\xi^2+\eta^2}.$$

We define $$u(\xi,\eta)=\int_{-\infty}^{+\infty}P(t-\xi,\eta)\,\omega(t)\,dt=\int_{-\infty}^{+\infty}P(s,\eta)\,\omega(s+\xi)\,ds.$$

We now prove a lemma which implies that u is finite and that $u(\xi,\eta)-\omega(\xi)$ is uniformly $o(1)$ when $\eta\to 0$ and uniformly $o(|\eta|)$ when $|\eta|\to\infty$.

Lemma 1.3.11. *Let $\omega\in\mathfrak{M}_0(1)$. For each $\delta>0$ there exists C_δ such that $|u(\xi,\eta)-\omega(\xi)|\leqslant C_\delta+\delta|\eta|$ and also $|u(\xi,\eta)-\omega(\xi)|\leqslant\delta+C_\delta|\eta|$.*

Proof. By the subadditivity of ω we have

$$u(\xi,\eta)\leqslant\int P(s,\eta)\,\omega(\xi)\,ds+\int P(s,\eta)\,\omega(s)\,ds=\omega(\xi)+u(0,\eta).$$

Similarly we prove that

$$u(\xi,\eta)\geqslant\omega(\xi)-\int P(-s,\eta)\,\omega(s)\,ds,$$

and thus $|u(\xi,\eta)-\omega(\xi)|\leqslant u(0,\eta)$. It remains to find C_δ such that we have $u(0,\eta)\leqslant C_\delta+\delta|\eta|$ and a similar estimate. Let Ω be the function of Theorem 1.2.7. It is enough to find C_δ such that we have

$$\frac{2|\eta|}{\pi}\int_0^\infty \frac{\Omega(r)\,dr}{r^2+\eta^2}\leqslant C_\delta+\delta|\eta| \tag{1.3.1}$$

and another similar estimate. But for any $R>0$ we have

$$\frac{2|\eta|}{\pi}\int_0^\infty \frac{\Omega(r)\,dr}{r^2+\eta^2}\leqslant \Omega(R)+\frac{2|\eta|}{\pi}\int_R^\infty \frac{\Omega(r)}{r^2}\,dr.$$

Choosing R large we prove (1.3.1), and choosing R small we prove the other estimate. This completes the proof of Lemma 1.3.11.

The essential part of the construction in the proof of Lemma 1.3.9 is given in the following lemma (cf. [16], Sect. 8 and 10).

Lemma 1.3.12. *Let Ω be a concave function of convergence type and let $\delta>0$. Then there exists a non-trivial continuous function g on R^1 such that g has its support in the interval $(-\delta,\delta)$ and such that $\hat{g}\exp\Omega$ is bounded.*

Proof. Without losing generality we may assume that Ω is continuously differentiable except at the origin and that $\Omega(t)\geqslant t^{\frac{1}{2}}$. We define Ω for $\xi<0$ by $\Omega(\xi)=\Omega(-\xi)$ and consider for $\eta>0$ the Poisson integral

$$u(\xi,\eta)=\int_{-\infty}^{+\infty}P(\xi-\tau,\eta)\,\Omega(\tau)\,d\tau=\int_{-\infty}^{+\infty}P(\tau,\eta)\,\Omega(\xi-\tau)\,d\tau. \tag{1.3.2}$$

Let v be the conjugate harmonic function of u and let

$$F(\zeta)=\exp\left(-2u(\xi,\eta)-2iv(\xi,\eta)\right)\quad(\eta>0).$$

By Lemma 1.3.11, u is continuous for $\eta\geqslant 0$, if we define $u(\xi,0)=\Omega(\xi)$. From (1.3.2) it follows that $\partial u/\partial\xi$ is continuous for $\eta\geqslant 0$ except at the origin. Since $\partial v/\partial\eta=\partial u/\partial\xi$, we may thus define $F(\xi)=\lim_{\eta\to+0}F(\xi+i\eta)$ for real ξ, and we have $|F(\xi)|=\exp(-2\Omega(\xi))$. For real x, define $f(x)=(2\pi)^{-1}\int_{-\infty}^{+\infty}e^{ix\xi}F(\xi)\,d\xi$, so that $F=\hat{f}$. Thus f is non-trivial. However, we claim that $f(x)=0$ if $x>0$. In fact, for any $\delta>0$ we have by Lemma 1.3.11,

$$|F(\zeta)|\leqslant C'_\delta\, e^{\delta\eta-2\Omega(\xi)}\quad(\eta>0). \tag{1.3.3}$$

Thus by contour deformation we have for each $\eta>0$ that

$$f(x)=(2\pi)^{-1}\int_{-\infty}^{+\infty}e^{ix(\xi+i\eta)}\,F(\xi+i\eta)\,d\xi.$$

Hence, if $x=2\delta>0$, we have by (1.3.3),

$$|f(x)|\leqslant C\,e^{-\delta\eta}\int_{-\infty}^{+\infty}e^{-2\xi^{\frac{1}{2}}}\,d\xi,$$

and letting η tend to $+\infty$, we prove that $f(x)=0$. Thus for appropriately chosen x_0, the function g on R^1, defined by $g(x)=f(x-x_0)f(-x-x_0)$, is non-trivial and has its support in $(-\delta, \delta)$. Finally, we have

$$\hat{g}(\xi)=(2\pi)^{-1}e^{-ix_0\xi}\int e^{2ix_0\tau}F(\xi-\tau)F(-\tau)\,d\tau, \tag{1.3.4}$$

and thus $$|\hat{g}(\xi)|\leqslant(2\pi)^{-1}\int e^{-2\Omega(\xi-\tau)-2\Omega(\tau)}\,d\tau\leqslant(2\pi)^{-1}e^{-\Omega(\xi)}\int e^{-\tau^{\frac{1}{2}}}\,d\tau.$$

That $\hat{g}$ is entire follows from the fact that $g\in\mathcal{E}'$ but can also be proved by contour deformation in (1.3.4) and in the integral obtained from (1.3.4) by the change of variable $\xi-\tau\to\tau$. This completes the proof.

Proof of Lemma 1.3.9. Let Ω be as in Theorem 1.2.7 and the remark following it. Let g be the function constructed in Lemma 1.3.12 with $\delta=\varepsilon n^{-\frac{1}{2}}$. When $\xi\in R^n$, define $\varphi(\xi)=\prod_1^n g(\xi_i)$. Then $\operatorname{supp}\varphi\subset B_\varepsilon$, and with the notation used in the end of the proof of Theorem 1.2.7 (ω_i = restriction of ω to the ξ_i-axis), we have

$$\|\varphi\|_\lambda=\int|\hat{\varphi}(\xi)|\,e^{\lambda\omega(\xi)}\,d\xi\leqslant C\int\exp\sum_1^n(\lambda\omega_i(\xi_i)-\Omega(\xi_i))\,d\xi.$$

Since $\omega_i(\xi_i)/\Omega(\xi_i)\to 0$ when $|\xi_i|\to\infty$, we get $\|\varphi\|_\lambda<\infty$, and the proof is complete.

We will now consider regularization of functions and use regularization to prove Lemma 1.3.10. We start with the following result.

Proposition 1.3.13. *Let $\omega\in\mathfrak{M}_0(n)$. Let u be an integrable function with compact support and let $\varphi\in\mathcal{D}_\omega$. Then $u*\varphi\in\mathcal{D}_\omega$.*

Proof. Since $|\hat{u}|\leqslant\int|u(x)|\,dx$ and $(u*\varphi)\hat{}=\hat{u}\hat{\varphi}$, we get $\|u*\varphi\|_\lambda\leqslant\|\varphi\|_\lambda\int|u(x)|\,dx$.

Corollary 1.3.14. *Let $\omega\in\mathfrak{M}_0(n)$. If φ and $\psi\in\mathcal{D}_\omega$, then $\varphi*\psi\in\mathcal{D}_\omega$.*

A slight complication in dealing with regularization is that if $\varphi\in\mathcal{D}_\omega$ and $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$, then it is not a priori clear that $\varphi_\varepsilon\in\mathcal{D}_\omega$. However, we have the following result.

Proposition 1.3.15. *Let $\omega\in\mathfrak{M}_0(n)$ and define $\omega'(\xi)=\sup_{|x|\leqslant|\xi|}\omega(x)$. Let $\varphi\in\mathcal{D}_{\omega'}$ and define $\varphi_\varepsilon(x)=\varepsilon^{-n}\varphi(x/\varepsilon)$. Then $\varphi_\varepsilon\in\mathcal{D}_{\omega'}\subset\mathcal{D}_\omega$.*

Proof. Let N be an integer satisfying $\varepsilon^{-1}\leqslant N\leqslant 1+\varepsilon^{-1}$. Then by condition (α),

$$\omega'\left(\frac{\xi}{\varepsilon}\right)\leqslant\omega'(N\xi)\leqslant N\omega'(\xi)\leqslant(1+\varepsilon^{-1})\,\omega'(\xi).$$

Thus, since $\hat{\varphi}_\varepsilon(\xi)=\hat{\varphi}(\varepsilon\xi)$ we get $\|\varphi_\varepsilon\|_\lambda^{(\omega')}\leqslant\varepsilon^{-n}\|\varphi\|_{(1+\varepsilon^{-1})\lambda}^{(\omega')}$.

From Propositions 1.3.13 and 1.3.15 we get as in [H], Th. 1.2.1:

Theorem 1.3.16. *Let $\omega\in\mathfrak{M}_0(n)$ and let $\omega'(\xi)=\sup_{|x|\leqslant|\xi|}\omega(x)$. Let Ω be an open subset of R^n. Let $u\in L_p(\Omega)$ $(1\leqslant p<\infty)$ and let u have a compact support contained in Ω. If*

$\varphi \in \mathcal{D}_{\omega'}(B_1)$ is such that $\int \varphi(x)\,dx = 1$ and if $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, then $u * \varphi_\varepsilon \in \mathcal{D}_\omega(\Omega)$ if ε is smaller than the distance from K to $\complement\Omega$. When $\varepsilon \to 0$ we have $u * \varphi_\varepsilon \to u$ in $L_p(\Omega)$.

Proof of Lemma 1.3.10. Let K and V be as in condition (β') and choose $\varepsilon > 0$ so small that $K + \bar{B}_{4\varepsilon} \subset V$. If we can find $\varphi \in \mathcal{D}_\omega(B_{2\varepsilon})$ such that $\varphi \geqslant 0$ and $\int \varphi(x)\,dx = 1$, then the result follows from Proposition 1.3.13, if as u we take the characteristic function of $K + \bar{B}_{2\varepsilon}$. To find φ we start with a non-trivial $\psi \in \mathcal{D}_\omega(B_\varepsilon)$ and form $\chi = \psi * \check{\psi}$. Then it is well known that χ is non-trivial. By Proposition 1.3.13, we have $\chi \in \mathcal{D}_\omega(B_{2\varepsilon}) \cap \mathcal{D}_\omega(B_{2\varepsilon})$, which implies that $\bar{\chi} \in \mathcal{D}_\omega(B_{2\varepsilon})$. Then by Proposition 1.3.5 we have $\varphi \in \mathcal{D}_\omega(B_{2\varepsilon})$, if we define $\varphi = |\chi|^2 = \chi\bar{\chi}$. Since φ is non-negative and non-trivial, it only remains to multiply φ by a suitable positive constant. This completes the proof.

Using once more the idea of this proof we can find non-trivial non-negative elements of $\mathcal{D}_\omega$ with non-negative Fourier transforms:

Corollary 1.3.17. *Let $\omega \in \mathfrak{M}_0(n)$ and let Ω be a neighborhood of the origin in R^n. Then there exists a non-trivial $\varphi \in \mathcal{D}_\omega(\Omega)$ such that $\varphi(x) \geqslant 0$ $(\forall x \in \Omega)$ and $\hat{\varphi}(\xi) \geqslant 0$ $(\forall \xi \in R^n)$.*

Proof. If $B_{2\varepsilon} \subset \Omega$, we start with $\psi \in \mathcal{D}_\omega(B_\varepsilon)$ such that $\psi \geqslant 0$. Define $\varphi = \psi * \check{\psi}$. Since ψ is real, we have $\hat{\varphi} = \hat{\psi}\bar{\hat{\psi}} = |\hat{\psi}|^2$. Thus φ and $\hat{\varphi}$ are both non-negative.

If we are given two functions ω, we may ask under what conditions they give rise to the same space $\mathcal{D}_\omega$ and, more generally, under what conditions one space is included in the other. This is settled by the following theorem:

Theorem 1.3.18. *Let ω_1 and $\omega_2 \in \mathfrak{M}_0(n)$. If for some real A and positive C we have*

$$\omega_2(\xi) \leqslant A + C\omega_1(\xi) \quad (\forall \xi \in R^n), \tag{1.3.5}$$

then $\mathcal{D}_{\omega_1} \subset \mathcal{D}_{\omega_2}$ and $\mathcal{D}_{\omega_1}(\Omega)$ is dense in $\mathcal{D}_{\omega_2}(\Omega)$ for each open $\Omega \subset R^n$. Conversely, if for some $E \subset R^n$ with non-empty interior, $\mathcal{D}_{\omega_1}(E) \subset \mathcal{D}_{\omega_2}(E)$, then (1.3.5) holds for some A and C.

Proof (cf. [H], Th. 2.2.2). In the first part of the theorem, the inclusion is trivial. To prove that $\mathcal{D}_{\omega_1}(\Omega)$ is dense in $\mathcal{D}_{\omega_2}(\Omega)$, let $u \in \mathcal{D}_{\omega_2}(\Omega)$ and let $u * \varphi_\varepsilon$ be as in Theorem 1.3.16 with $\varphi \in \mathcal{D}_{\omega_1}(\Omega)$. We get

$$\|u - (u * \varphi_\varepsilon)\|_\lambda^{(\omega_2)} = \int |\hat{u}(\xi)|\, e^{\lambda\omega_2(\xi)}\, |1 - \hat{\varphi}(\varepsilon\xi)|\, d\xi,$$

which tends to zero by the dominated convergence theorem. To prove the converse, choose K compact with non-empty interior and contained in E. We claim that the inclusion map of $\mathcal{D}_{\omega_1}(K)$ into $\mathcal{D}_{\omega_2}(K)$ is closed. In fact, if $\varphi_\nu \to f_i$ in $\mathcal{D}_{\omega_i}$ $(i = 1, 2)$, then $\hat{\varphi}_\nu \to \hat{f}_i$ in $L_1(R^n)$ and so $\hat{f}_1 = \hat{f}_2$ which implies $f_1 = f_2$. Then the closed graph theorem gives the existence of positive constants C' and C such that

$$\|\varphi\|_1^{(\omega_2)} \leqslant C' \|\varphi\|_C^{(\omega_1)} \quad (\forall \varphi \in \mathcal{D}_{\omega_1}(K)). \tag{1.3.6}$$

Let us choose a non-trivial $\psi \in \mathcal{D}_{\omega_1}(K)$. Let $\xi_0 \in R^n$ and define $\varphi(x) = \psi(x)\, e^{i\langle x, \xi_0\rangle}$. Then $\hat{\varphi}(\xi) = \hat{\psi}(\xi - \xi_0)$. We get

$$\|\varphi\|_C^{(\omega_1)} = \int |\hat{\varphi}(\xi)| \, e^{C\omega_1(\xi+\xi_0)} \, d\xi \leqslant e^{C\omega_1(\xi_0)} \|\psi\|_C^{(\omega_1)}$$

and $$\|\varphi\|_1^{(\omega_2)} = \int |\hat{\varphi}(\xi)| \, e^{\omega_2(\xi+\xi_0)} \, d\xi \geqslant e^{\omega_2(\xi_0)} \int |\hat{\psi}(\xi)| \, e^{-\omega_2(-\xi)} \, d\xi = e^{\omega_2(\xi_0)} \|\psi\|_{-1}^{(\omega_2)}.$$

Hence from (1.3.6) we derive (1.3.5) with

$$A = \log\,(C' \|\psi\|_C^{(\omega_1)}) - \log \|\psi\|_{-1}^{(\omega_2)}.$$

This completes the proof of the theorem.

Definition 1.3.19. *If ω_2 and ω_1 are related as in Theorem 1.3.18 we will write $\omega_2 \prec \omega_1$.*

Corollary 1.3.20. *Let $\omega \in \mathfrak{M}_0(n)$. Then $\mathcal{D}_\omega(\Omega) = \mathcal{D}_{\check{\omega}}(\Omega)$ for every open Ω in R^n (or for some non-trivial such Ω) if and only if $\check{\omega} \prec \omega$.*

Corollary 1.3.21. *Let $\omega \in \mathfrak{M}_0(n)$. Then $\mathcal{D}_\omega(\Omega) \subset C_0^\infty(\Omega)$ for every open Ω in R^n (or for some non-trivial such Ω) if and only if for some real a and positive b we have*

$$(\gamma) \qquad \omega(\xi) \geqslant a + b \log\,(1 + |\xi|) \quad (\forall \xi \in R^n).$$

In the sequel we will mainly consider spaces $\mathcal{D}_\omega$ consisting entirely of infinitely differentiable functions. Thus we are lead to the following definition:

Definition 1.3.22. *We denote by $\mathfrak{M}$ the set of all continuous real-valued functions ω on R^n, satisfying conditions (α), (β) and (γ):*

$$(\alpha) \qquad 0 = \omega(0) \leqslant \omega(\xi + \eta) \leqslant \omega(\xi) + \omega(\eta) \quad (\forall \xi, \eta \in R^n),$$

$$(\beta) \qquad \int \frac{\omega(\xi)\, d\xi}{(1 + |\xi|)^{n+1}} < \infty,$$

$$(\gamma) \qquad \omega(\xi) \geqslant a + b \log\,(1 + |\xi|) \quad (\forall \xi \in R^n)$$

(for some real a and positive b).

Occasionally we must limit ourselves to the "symmetric case" described in Corollary 1.3.20 or even to the case where ω is given by a concave function of convergence type. For convenience we therefore also make the following definitions:

Definition 1.3.23. *We denote by $\mathfrak{M}_s$ the set of all $\omega \in \mathfrak{M}$ satisfying $\check{\omega} \prec \omega$ and by $\mathfrak{M}_c$ the set of all $\omega \in \mathfrak{M}$ such that $\omega(\xi) = \Omega(|\xi|)$ with Ω concave on $[0, +\infty)$.*

Definition 1.3.24. *If $\omega \in \mathfrak{M}$, we denote by ω^c the element of $\mathfrak{M}_c$ given by $\omega(\xi) = \Omega(|\xi|)$, where Ω is the function constructed in Theorem 1.2.7.*

A consequence of condition (γ) is that supremum norms can be used instead of integral norms as follows:

Definition 1.3.25. *Let $\omega \in \mathfrak{M}$. If $\varphi \in L_1(R^n)$ and if λ is a real number we define*

$$|||\varphi|||_\lambda = |||\varphi|||_\lambda^{(\omega)} = \sup_{\xi \in R^n} |\hat{\varphi}(\xi)| \, e^{\lambda\omega(\xi)}.$$

Proposition 1.3.26. *Let $\omega \in \mathfrak{M}$. Then there exists a positive constant Λ such that $C_\Lambda = \int \exp(-\Lambda\omega(\xi))\, d\xi < \infty$ and*

$$\|\varphi\|_\lambda \leqslant C_\Lambda |||\varphi|||_{\lambda+\Lambda} \quad (\forall \lambda, \forall \varphi \in L_1(R^n)),$$

(*with the natural interpretation if $\|\varphi\|_\lambda = +\infty$*).

Proof. Clearly, we may take $\Lambda = (n+1)/b$, where b is the constant of condition (γ).

Another consequence of condition (γ) is that $\mathcal{D}_\omega$ is closed under differentiation:

Theorem 1.3.27. *Let $\omega \in \mathfrak{M}$. Then if $\varphi \in \mathcal{D}_\omega$ and α is any multi-index, we have $D^\alpha\varphi \in \mathcal{D}_\omega$ and the mapping $\varphi \to D^\alpha\varphi$ is continuous.*

Proof. Since $(D^\alpha\varphi)^\wedge(\xi) = \xi^\alpha\hat{\varphi}(\xi)$, we get $\|D^\alpha\varphi\|_\lambda \leqslant C\|\varphi\|_{\lambda+|\alpha|/b}$ where b is the constant of condition (γ).

Apart from differentiation, we will consider two other continuous mappings of $\mathcal{D}_\omega$ into itself. One is multiplication by an analytic function (Theorem 1.5.16). The other is translation:

Proposition 1.3.28. *Let $\omega \in \mathfrak{M}$ and let $y \in R^n$ be given. Then the mapping τ_y from $\mathcal{D}_\omega$ into $\mathcal{D}_\omega$ defined by*

$$\tau_y\varphi(x) = \varphi(x-y)$$

is continuous and in fact an isometry.

Proof. Since $(\tau_y\varphi)^\wedge(\xi) = e^{-i\langle y,\xi\rangle}\hat{\varphi}(\xi)$, the result follows from the fact that by definition $\|\varphi\|_\lambda$ depends only on the modulus of $\hat{\varphi}$.

If φ is fixed and y varies we also get a continuous mapping:

Proposition 1.3.29. *Let $\omega \in \mathfrak{M}$ and let $\varphi \in \mathcal{D}_\omega$ be given. Then the (non-linear) mapping from R^n into $\mathcal{D}_\omega$ defined by*

$$y \to \tau_y(\varphi)$$

is continuous.

Proof. We have

$$\|\tau_x(\varphi) - \tau_y(\varphi)\|_\lambda = \int |(e^{-i\langle x,\xi\rangle} - e^{-i\langle y,\xi\rangle})\, \hat{\varphi}(\xi)| \, e^{\lambda\omega(\xi)}\, d\xi,$$

which clearly tends to zero when $x \to y$.

We conclude this section by giving some examples. First, by Proposition 1.2.1 it is clear that if $\omega(\xi) = |\xi|^{1/\gamma}$ with $\gamma > 1$, then $\omega \in \mathfrak{M}_c$. Then $\mathcal{D}_\omega$ is closely related to the Gevrey class with index γ, as stated in Example 1.5.7. Our next example was studied by Domar [8], p. 18.

Definition 1.3.30. *We denote by E the set of all sequences $\{a_k\}_0^\infty$ such that $a_0 = 1$ and $a_k \geqslant 0$ and*

$$(k+l)!\, a_{k+l} \leqslant k!\, a_k\, l!\, a_l. \tag{1.3.7}$$

Definition 1.3.31. *We denote by $\mathcal{M}_E$ the set of all non-negative functions ω on R^n for which $e^{\omega(\xi)} = \sum_0^\infty a_k |\xi|^k$ with $\{a_k\} \in E$ and*

$$\int_0^\infty \frac{\log (\sum_0^\infty a_k t^k)}{1+t^2}\, dt < \infty.$$

Proposition 1.3.32. *If $\omega \in \mathcal{M}_E$ and $\omega \neq 0$ then $\omega \in \mathcal{M}_s$.*

Proof. Condition (α) follows easily from (1.3.7), condition (β) follows from Definition 1.3.31, and condition (γ) follows from the fact that $\omega \neq 0$.

Finally we give an example showing that $\mathcal{M}_s \neq \mathcal{M}$. Let $\xi = (\xi_1, \ldots, \xi_n)$ and define $\omega(\xi) = \log(1+|\xi|)$ when $\xi_1 < 0$ and $\omega(\xi) = \log(1+|\xi|) + \xi_1^{\frac{1}{2}}$ when $\xi_1 \geqslant 0$. We leave it to the reader to verify that $\omega \in \mathcal{M}$ but $\omega \notin \mathcal{M}_s$.

1.4. *The Paley–Wiener theorem for test functions*

We will now relate the support of a test function to the behavior in the complex plane of its Fourier–Laplace transform. Thus Theorem 1.4.1 will generalize part of the Paley–Wiener theorem as given in [H], Th. 1.7.7. The remaining part, dealing with distributions, will be considered in Theorem 1.8.14. We will also complete the study of the equivalence of the sets of semi-norms $\{\|\cdot\|_\lambda\}_{\lambda>0}$ and $\{|||\cdot|||_\lambda\}_{\lambda>0}$, initiated in Proposition 1.3.26.

Theorem 1.4.1. *Let $\omega \in \mathcal{M}$ and let K be a compact convex set in R^n with support function H. If U is an entire function of n complex variables $\zeta = \xi + i\eta = (\zeta_1, \ldots, \zeta_n)$, the following three conditions are equivalent:*

(i) *For each $\lambda > 0$ and each $\varepsilon > 0$ there exists a constant $C_{\lambda,\varepsilon}$ such that*

$$\int_{R^n} |U(\xi+i\eta)|\, e^{\lambda\omega(\xi)}\, d\xi \leqslant C_{\lambda,\varepsilon}\, e^{H(\eta)+\varepsilon|\eta|} \quad (\forall \eta \in R^n).$$

(ii) *For each $\lambda > 0$ and each $\varepsilon > 0$ there exists a constant $C'_{\lambda,\varepsilon}$ such that*

$$|U(\xi+i\eta)| \leqslant C'_{\lambda,\varepsilon}\, e^{H(\eta)+\varepsilon|\eta|-\lambda\omega(\xi)} \quad (\forall (\xi+i\eta) \in C^n).$$

(iii) $U(\zeta) = \int e^{-i\langle x,\zeta\rangle} \varphi(x)\, dx$ *with some $\varphi \in \mathcal{D}_\omega(K)$.*

Proof. We first prove that (ii) implies (iii). By condition (γ), the classical Paley–Wiener result shows that we have $U = \hat{\varphi}$ for some $\varphi \in C_0^\infty$ with $\operatorname{supp} \varphi \subset K$. It remains to prove that $\varphi \in \mathcal{D}_\omega$. By Proposition 1.3.26 this follows from (ii) with $\eta = 0$. To prove that (i) implies (ii) we use Cauchy's integral formula to get an estimate of the form

$$|U(\zeta)| \leqslant C \int_{\substack{|\xi'|\leqslant 1\\ |\eta'|\leqslant 1}} |U(\zeta + \xi' + i\eta')|\, d\xi'\, d\eta'.$$

Hence, using condition (α) we get from (i)

$$|U(\zeta)\,e^{\lambda\omega(\xi)}| \leqslant C\cdot 2^n \sup_{|\eta'|\leqslant 1}\int_{|\xi'|\leqslant 1}|U(\zeta+\xi'+i\eta')|\,e^{\lambda\omega(\xi+\xi')+\lambda\omega(-\xi')}\,d\xi'$$

$$\leqslant C\cdot 2^n \sup_{|\xi'|\leqslant 1} e^{\lambda\omega(-\xi')}\cdot \sup_{|\eta'|\leqslant 1}\int_{R^n}|U(\xi+\xi'+i(\eta+\eta'))|\,e^{\lambda\omega(\xi+\xi')}\,d\xi'$$

$$\leqslant C\cdot 2^n C_{\lambda,\varepsilon} \sup_{|\xi'|\leqslant 1} e^{\lambda\omega(-\xi')} \sup_{|\eta'|\leqslant 1} e^{H(\eta+\eta')+\varepsilon|\eta+\eta'|} \leqslant C'_{\lambda,\varepsilon}\,e^{H(\eta)+\varepsilon|\eta|}$$

with
$$C'_{\lambda,\varepsilon} = C\,2^n C_{\lambda,\varepsilon} \sup_{|\xi'|\leqslant 1} e^{\lambda\omega(-\xi')} \sup_{|\eta'|\leqslant 1} e^{H(\eta')+\varepsilon|\eta'|}.$$

This proves (ii).

It now remains to prove that (iii) implies (i). In the proof we will use the following well-known result, where P is the Poisson kernel considered in Lemma 1.3.11:

Lemma 1.4.2. *Let g be a function of one complex variable $z=x+iy$, analytic for $y>0$ and continuous for $y\geqslant 0$, and suppose that $|g(z)|\leqslant C e^{Ay}$ for $y>0$. Then for all such y we have*

$$\log|g(z)| \leqslant \int_{-\infty}^{+\infty}\log|g(t)|\,P(x-t,y)\,dt + Ay.$$

Proof. Consider the function $f(z)=\log|g(z)e^{iAz}/C|$. Since f is subharmonic and non-positive for non-negative y, it follows that f is not greater than its Poisson integral. (We can for instance map the half-plane $y>0$ conformally onto the unit disk.)

End of proof of Theorem 1.4.1. Let $\lambda>0$ and $\varepsilon>0$ be given. Clearly the result will follow if we can find a constant $C''_{\lambda,\varepsilon}$ (independent of φ) such that for every choice of (orthonormal) coordinate system (with the given origin) and for every real A and each $\eta_1>0$ we have (with $\xi'=(\xi_2,\ldots,\xi_n)$ etc.)

$$\int_{R^n}|\hat\varphi(\xi_1+i\eta_1,\xi')|\,e^{\lambda\omega(\xi_1,\xi')}\,d\xi \leqslant C''_{\lambda,\varepsilon}\|\varphi\|_\lambda\,e^{(A+\varepsilon)\eta_1} \quad (\forall\varphi\in\mathcal{D}_\omega(\{x;x_1\leqslant A\})). \tag{1.4.1}$$

Let $\varphi\in\mathcal{D}_\omega(\{x;x_1\leqslant A\})$. Then by the classical Paley–Wiener result we have

$$|\hat\varphi(\xi_1+i\eta_1,\xi')|\leqslant C\,e^{A\eta_1},$$

and thus we can use Lemma 1.4.2 to get

$$\log|\hat\varphi(\xi_1+i\eta_1,\xi')| \leqslant \int_{-\infty}^{+\infty}P(\xi_1-t,\eta_1)\log|\hat\varphi(t,\xi')|\,dt + A\eta_1 \quad (\forall\eta_1>0). \tag{1.4.2}$$

For fixed ξ' we will also consider the Poisson integral of ω and write

$$u(\xi,\eta_1)=\int_{-\infty}^{+\infty}P(\xi_1-t,\eta_1)\,\omega(t,\xi')\,dt.$$

To estimate $|u-\omega|$ we repeat the calculations in the proof of Lemma 1.3.11, using the inequality $\omega(s+\xi_1,\xi')\leqslant\omega(\xi_1,\xi')+\omega(s,0)$, so that $\omega(s)$ will be replaced by $\omega(s,0)$.

Since the same Ω can be used for all $\omega(s, 0)$, we find a constant $C''_{\lambda,\varepsilon}$ (with the required independence) such that

$$\lambda\,|u(\xi,\eta_1)-\omega(\xi)|\leqslant \log C''_{\lambda,\varepsilon}+\varepsilon\eta_1 \quad (\forall\,\xi\in R^n,\ \forall\,\eta_1>0). \tag{1.4.3}$$

Using Jensen's inequality we get from (1.4.2),

$$|\hat{\varphi}(\xi_1+i\eta_1,\xi')|\,e^{\lambda u(\xi,\eta_1)}\leqslant e^{A\eta_1}\exp\left(\int_{-\infty}^{+\infty}(\log|\hat{\varphi}(t,\xi')|+\lambda\omega(t,\xi'))\,P(\xi_1-t,\eta_1)\,dt\right)$$

$$\leqslant e^{A\eta_1}\int_{-\infty}^{+\infty}|\hat{\varphi}(t,\xi')|\,e^{\lambda\omega(t,\xi')}\,P(\xi_1-t,\eta_1)\,dt.$$

Integrating over $R^1\times R^{n-1}$ and using the fact that $\int_{-\infty}^{+\infty}P(\xi_1-t,\eta_1)\,d\xi_1=1$, we get

$$\int|\hat{\varphi}(\xi_1+i\eta_1,\xi')|\,e^{\lambda u(\xi,\eta_1)}\,d\xi\leqslant e^{A\eta_1}\|\varphi\|_\lambda. \tag{1.4.4}$$

Since (1.4.1) follows from (1.4.3) and (1.4.4), the proof of the theorem is complete.

From the proof of Theorem 1.4.1, combined with Proposition 1.3.26, we get the following result:

Corollary 1.4.3. *Let $\omega\in\mathcal{M}$ and let K be a compact subset of R^n. Then the family of semi-norms $\{\varphi\to|||\varphi|||_\lambda\}_{\lambda>0}$ on $\mathcal{D}_\omega(K)$ is equivalent to the family $\{\varphi\to\|\varphi\|_\lambda\}_{\lambda>0}$. Still another equivalent family of semi-norms is*

$$\{\varphi\to\sup_{\zeta\in C^n}|\hat{\varphi}(\zeta)|\exp(\lambda\omega(\xi)-H(\eta)-|\eta|)\}_{\lambda>0}.$$

1.5. *Spaces $\mathcal{E}_\omega$ and Denjoy–Carleman classes*

Starting from the space $\mathcal{D}_\omega$, we will define $\mathcal{E}_\omega$, $\mathcal{D}'_\omega$ and $\mathcal{E}'_\omega$ as generalisations of $\mathcal{E}$, $\mathcal{D}'$ and $\mathcal{E}'$. The distribution spaces will be considered in Section 1.6. We will now first define $\mathcal{E}_\omega$. Then we will discuss some relations between spaces $\mathcal{D}_\omega$ and $\mathcal{E}_\omega$, on one hand, and D.–C. classes, on the other hand. Here "D.–C. classes" stands for classes of infinitely differentiable functions of the kind studied by Denjoy and Carleman.

Definition 1.5.1. *$\mathcal{E}_\omega(\Omega)$ is the set of all complex-valued functions φ in Ω such that for each compact subset K of Ω the restrictions to K of φ and of some $\psi\in\mathcal{D}_\omega(\Omega)$ agree. The topology of $\mathcal{E}_\omega(\Omega)$ is given by the semi-norms*

$$\varphi\to\inf_{\psi=\varphi\ \text{in}\ K}\|\psi\|_\lambda \quad (\forall\,\lambda>0,\forall\,K).$$

From Proposition 1.3.5 and the existence of local units it is clear that we may also consider $\mathcal{E}_\omega$ as the set of multipliers on $\mathcal{D}_\omega$. We formulate this fact as a proposition.

Proposition 1.5.2. *$\mathcal{E}_\omega(\Omega)$ is the set of all complex-valued functions φ in Ω such that if $\psi\in\mathcal{D}_\omega(\Omega)$, then $\psi\varphi\in\mathcal{D}_\omega(\Omega)$. The topology in $\mathcal{E}_\omega(\Omega)$ is given by the semi-norms $\varphi\to\|\psi\varphi\|_\lambda$ $(\forall\,\lambda>0,\ \forall\,\psi\in\mathcal{D}_\omega(\Omega))$.*

A more general interpretation of $\mathcal{E}_\omega(\Omega)$ is given in Proposition 2.3.3.

We now collect some definitions and results from the theory of D.–C. classes.

Definition 1.5.3. *Let $L=\{L_k\}_{k=0}^\infty$ be an* increasing *sequence of positive numbers and let Ω be an open subset of R^n. Then $C^L(\Omega)$ is the set of all u in $C^\infty(\Omega)$ such that to each compact subset K of Ω there exists a constant C such that*

$$\sup_K |D^\alpha u| \leqslant C^{k+1} L_k^k \quad (\forall\, \alpha \text{ with } |\alpha|=k;\ k=0,1,\ldots),$$

and $c^L(\Omega)$ is the set of all u in $C^\infty(\Omega)$ such that to each compact subset K of Ω and each $\varepsilon>0$ there exists a constant C such that

$$\sup_K |D^\alpha u| \leqslant C\varepsilon^k L_k^k \quad (\forall \alpha \text{ with } |\alpha|=k;\ k=0,1,\ldots).$$

We call $C^L(\Omega)$ a D.–C. class. For $C^L(R^n)$ we sometimes write C^L.

Definition 1.5.4. *A D.–C. class $C^L(\Omega)$ is said to be non-quasianalytic (n.q.a.) if it contains a non-trivial function with compact support contained in Ω.*

Theorem 1.5.5. *(Denjoy–Carleman) The class $C^L(\Omega)$ is non-quasianalytic if and only if*

$$\int_0^\infty \log\left(\sum_{k=0}^\infty \left(\frac{t}{L_k}\right)^k\right)\frac{dt}{1+t^2} < \infty,$$

or equivalently, $\sum_0^\infty L_k^{-1} < \infty$.

Example 1.5.6. If we denote by $\mathcal{A}(\Omega)$ the class of functions analytic in Ω, then we have $\mathcal{A}(\Omega)=C^{\{k\}}$. (Here and in the sequel we agree to replace L_k by 1 if it is 0 or undefined).

Example 1.5.7. If $\omega(\xi)=|\xi|^{1/\gamma}$ and $L_k=k^\gamma$ with $\gamma>1$, then $c^L(\Omega)\cap \mathcal{D}(\Omega)=\mathcal{D}_\omega(\Omega)$. This follows from Lem. 5.7.2 of [H].

Theorem 1.5.8. *The intersection of all n.q.a. classes C^L, where $L_k/(k!)^{1/k}$ is increasing, is equal to the class $C^{\{k\log k\}}$.*

Proof. This follows from Th. 2 of [5] (cf. also Th. 7 of [5]).

Let us write $q_L(\xi)=\sum_{k=0}^\infty(|\xi|/L_k)^k$. Then we have:

Proposition 1.5.9. *If $u\in\mathcal{D}(\Omega)$ and $|\hat u(\xi)|\leqslant C/q_L(a\xi)(1+|\xi|)^{n+1}$ where C and $a>0$ are constants, then $u\in C^L(\Omega)$.*

Proof. From the formula $D^\alpha u(x)=(2\pi)^{-n}\int \xi^\alpha \hat u(\xi)\, e^{i\langle x,\xi\rangle}\,d\xi$ and the hypothesis we get

$$\max_{|\alpha|=k}\sup_{x\in\Omega}|D^\alpha u(x)|\leqslant C\max_{|\alpha|=k}\int\frac{|\xi^\alpha|\,d\xi}{q_L(a\xi)\,(1+|\xi|)^{n+1}}\leqslant Ca^{-k}L_k^k\int\frac{d\xi}{(1+|\xi|)^{n+1}}\leqslant C_1a^{-k}L_k^k$$
$$(k=0,1,2,\ldots).$$

Finally we will use the following lemma as a replacement for local units in a quasi-analytic case:

Lemma 1.5.10. *Let Ω be an open set in R^n and let $K \subset\subset U \subset\subset \Omega$. Then there exists a sequence $\{g_k\}_0^\infty$ of functions in $C_0^\infty(U)$ such that all $g_k = 1$ in K, and all $|\hat{g}_k| \leqslant |\hat{g}_0|$, and such that for every increasing sequence $\{L_k\}$ with $L_k \geqslant k$ $(\forall k)$ and every $u \in C^L(\Omega)$ there exists a constant C such that*

$$\sup_{x \in U} |D^\alpha(g_k u)| \leqslant C^{k+1} L_k^k \quad (\forall\, \alpha, k \text{ with } |\alpha| \leqslant k). \tag{1.5.1}$$

Proof. This follows from Lem. 1 and Lem. 2 of [4], except for the inequalities

$$|\hat{g}_k| \leqslant |\hat{g}_0| \quad (\forall k). \tag{1.5.2}$$

But (1.5.2) follows from the proof of Lem. 1 of [4]. (Let $g_0 \in C_0^\infty(U)$ be a local unit for K. Choose $\varepsilon > 0$ such that $B_\varepsilon + \operatorname{supp} g_0 \subset\subset U$, and let $\varphi \in C_0^\infty(B_\varepsilon)$ be such that $|\hat{\varphi}(\xi)| \leqslant \hat{\varphi}(0) = 1$. Then g_k is defined by $\hat{g}_k(\xi) = \hat{g}_0(\xi)(\hat{\varphi}(\xi/k))^k$.)

The following result connects the D.–C. classes considered in Theorem 1.5.8 with the subclass $\mathfrak{M}_E$ of $\mathfrak{M}$, considered in Proposition 1.3.32:

Theorem 1.5.11. *Let C^L be n.q.a. and such that $L_k/(k!)^{1/k}$ is increasing. Define $\omega_L = \log q_L$. Then $\omega_L \in \mathfrak{M}_E$ and $\mathcal{D}_{\omega_L} \subset C^L$.*

Proof. Since $A_k = L_k/(k!)^{1/k}$ is increasing, we have $A_{k+l}^{k+l} \geqslant A_k^k A_l^l$ or

$$(k+l)!\, L_{k+l}^{-(k+l)} \leqslant k!\, L_k^{-k}\, l!\, L_l^{-l}. \tag{1.5.3}$$

Thus $\{L_k^{-k}\} \in E$. By Theorem 1.5.5 and Proposition 1.3.32, we have $\omega_L \in \mathfrak{M}_E \subset \mathfrak{M}$. Finally it follows from Proposition 1.5.9 and condition (γ) that $\mathcal{D}_{\omega_L} \subset C^L$. This completes the proof.

We now state the main result of this section.

Theorem 1.5.12. $\bigcap_{\omega \in \mathfrak{M}} \mathcal{E}_\omega(\Omega) = C^{\{k \log k\}}(\Omega)$.

Proof. We first prove that $\bigcap \mathcal{E}_\omega(\Omega) \subset C^{\{k \log k\}}(\Omega)$. By Theorem 1.5.8, it is clearly enough to prove that each n.q.a. $C^L(\Omega)$ such that $L_k/(k!)^{1/k}$ is increasing, contains $\mathcal{E}_\omega(\Omega)$ for some $\omega \in \mathfrak{M}$. Let ω_L be as in Theorem 1.5.11. Then $\mathcal{D}_{\omega_L}(\Omega) \subset C^L(\Omega)$. Let $\psi \in \mathcal{E}_{\omega_L}(\Omega)$ and consider a local unit φ in $\mathcal{D}_{\omega_L}(\Omega)$. Then $\varphi\psi \in \mathcal{D}_{\omega_L}(\Omega) \subset C^L(\Omega)$. Since the property $\psi \in C^L$ is a local one, the result follows.

Since $\bigcap_{\omega \in \mathfrak{M}} \mathcal{E}_\omega(\Omega) = \bigcap_{\omega \in \mathfrak{M}_c} \mathcal{E}_\omega(\Omega)$, the proof of the theorem will be complete, if we prove the following result:

Lemma 1.5.13.[1] *Let $\omega \in \mathfrak{M}_c$. Then $C^{\{k \log k\}}(\Omega) \subset \mathcal{E}_\omega(\Omega)$.*

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[1] Lemma 1.5.13 and its proof have been communicated to the author by Professor Hörmander, who has also pointed out that a self-contained proof of the first inclusion in Theorem 1.5.12 (not using Theorem 1.5.8) can be obtained by characterizing the functions in $\mathcal{E}_\omega(\Omega)$, with $\omega \in \mathfrak{M}_c$, as those u for which, when $t \to +\infty$,

$$\inf_f \sup_{x \in K} |f(x) - u(x)| = O(\exp(-\lambda\omega(t)) \quad (\forall \lambda > 0,\ \forall K \subset\subset \Omega),$$

where f ranges over all entire functions of exponential type t with an appropriate a priori bound, and using Theorem 1.2.9.

Proof. Let $u \in C^{\{k \log k\}}$, $\varphi \in \mathcal{D}_\omega(\Omega)$ and $\lambda > 0$ be given. We have to prove that $|||u\varphi|||_\lambda < \infty$. Let $K = \operatorname{supp} \varphi$ and choose U such that $K \subset\subset U \subset\subset \Omega$. Let g_k be as in Lemma 1.5.10 and write $u_k = g_k u$. Then $u\varphi = u_k \varphi$ $(\forall k)$, and it suffices to find C such that for each $\xi \in R^n$ there exists k with

$$\left| \int \hat{\varphi}(\xi - \eta)\, \hat{u}_k(\eta)\, d\eta \right| \leqslant C\, e^{-\lambda\omega(\xi)}. \tag{1.5.4}$$

From (1.5.1) with $L_k = k \log k$ and $|\alpha| = k$ we derive the existence of a constant C such that

$$|\hat{u}_k(\eta)| \leqslant C \left(\frac{Ck \log k}{|\eta|} \right)^k \quad (\forall \eta \in R^n,\ \forall k \geqslant 2). \tag{1.5.5}$$

We claim that for some C we also have

$$\int |\hat{u}_k(\eta)|\, d\eta \leqslant C \quad (\forall k). \tag{1.5.6}$$

In fact, if we choose $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ in U, then $u_k = g_k v$ with $v = \psi u$. Using (1.5.2) and inverting the order of integration, we have proved (1.5.6) with

$$C = \int |\hat{g}_0(\eta)|\, d\eta \int |\hat{v}(\xi - \eta)|\, d\xi.$$

Using (1.5.5) when $|\eta| \geqslant \frac{1}{2}|\xi|$ and (1.5.6) when $|\xi - \eta| \geqslant \frac{1}{2}|\xi|$, and writing $\omega(\xi) = \omega(|\xi|)$, we get

$$\left| \int \hat{\varphi}(\xi - \eta)\, \hat{u}_k(\eta)\, d\eta \right| \leqslant C\, \|\varphi\|_0 \left(\frac{2Ck \log k}{|\xi|} \right)^k + C\, |||\varphi|||_{2\lambda}\, e^{-2\lambda\omega(\frac{1}{2}|\xi|)} \quad (\forall k).$$

Taking $k \approx |\xi| / (2Ce \log |\xi|)$ and using Corollary 1.2.8, we have proved 1.5.4 with a new constant. This completes the proof of Lemma 1.5.13 and Theorem 1.5.12.

Since $C^{\{k \log k\}}$ is quasianalytic (Theorem 1.5.5) and contains the analytic class, we have the following two results:

Corollary 1.5.14. $\bigcap_{\omega \in \mathfrak{M}} \mathcal{D}_\omega$ *is trivial.*

Corollary 1.5.15. *Let* $\omega \in \mathfrak{M}$. *Then* $\mathcal{A}(\Omega) \subset \mathcal{E}_\omega(\Omega)$.

We will finally prove a quantitative form of Corollary 1.5.15, needed in Chapter V, namely:

Theorem 1.5.16. *Let* $O \subset C^n$ *and* $\Omega \subset R^n$ *be given open sets such that* $\Omega \subset\subset O \cap R^n$ *and let* $\omega \in \mathfrak{M}$ *be given. Then for each* $\lambda > 0$ *there is a constant* K_λ *such that for each* f *analytic in* O *and each* $\varphi \in \mathcal{D}_\omega(\Omega)$ *we have*

$$|||f\varphi|||_\lambda \leqslant K_\lambda\, \|\varphi\|_\lambda \sup_O |f|.$$

The essential part of the proof is the case where Ω is a cube and O is a polycylinder with the same center. We formulate this case as a lemma:

Lemma 1.5.17. *Let the polycylinder* $P=\{z\in C^n;\ |z_j-x_j^0|\leqslant 3nR\ (j=1,2,\ldots,n)\}$ *and the cube* $Q=\{x\in R^n;\ |x_j-x_j^0|<R\ (j=1,2,\ldots,n)\}$ *be given. Then the conclusion of Theorem* 1.5.16 *holds with* Ω *replaced by* Q *and* O *replaced by* P.

Proof of Lemma 1.5.17. Without restriction we may assume that $x^0=0$. Let $f(z)=\sum_\alpha a_\alpha z^\alpha$ in P and let $\varphi\in\mathcal{D}_\omega(Q)$. Then

$$(f\varphi)^\wedge(\xi)=\sum a_\alpha\int_Q e^{-i\langle x,\xi\rangle}\varphi(x)\,x^\alpha\,dx,$$

since the series we integrate is absolutely and uniformly convergent in Q. Thus

$$(f\varphi)^\wedge(\xi)=\sum a_\alpha(-1)^{|\alpha|}D^\alpha\hat{\varphi}(\xi). \tag{1.5.7}$$

By Theorem 1.4.1 and Corollary 1.4.3 we have

$$|\hat{\varphi}(\xi+i\eta)|\leqslant C\|\varphi\|_\lambda\exp\left(-\lambda\omega(\xi)+2R\sum_{j=1}^n|\eta_j|\right).$$

Thus by Cauchy's inequalities we get

$$|D^\alpha\hat{\varphi}(\xi)|\leqslant C\|\varphi\|_\lambda\,k!\,r^{-k}\exp(-\lambda\omega(\xi)+\lambda\omega_1(r)+2nRr)\quad(\forall\xi\in R^n)$$

for $k=0,1,\ldots$ and any α with $|\alpha|=k$ and any $r>0$, where we have written $\omega_1(r)=\sup_{|\xi_j|\leqslant r}\omega(\xi)$. Choosing in particular $r=k/2nR$ and using Stirling's formula we get with a new constant

$$|D^\alpha\hat{\varphi}(\xi)|\leqslant C\|\varphi\|_\lambda(k+1)(2nR)^k\exp(-\lambda\omega(\xi)+\lambda\omega_1(k/2nR)).$$

Applying Cauchy's inequalities to f we get on the other hand

$$|a_\alpha|\leqslant(3nR)^{-k}\sup_P|f|\quad(|\alpha|=k).$$

Introducing our estimates in (1.5.7) and using the inequality $\sum_{|\alpha|=k}1\leqslant(k+1)^{n-1}$, we get

$$|(f\varphi)^\wedge(\xi)|\,e^{\lambda\omega(\xi)}\leqslant C\|\varphi\|_\lambda\sup_P|f|\sum_{k=0}^\infty(k+1)^n\left(\tfrac{2}{3}\right)^k\exp(\lambda\omega_1(k/2nR)),$$

and it only remains to prove that the series is convergent. But since $\omega(\xi)/|\xi|\to 0$ when $|\xi|\to\infty$, we have for every $\varepsilon>0$ that $\lambda\omega_1(k/2nR)<\varepsilon k$ if k is sufficiently large. Choosing ε so small that $\frac{2}{3}e^\varepsilon<1$, we have proved the lemma.

Proof of Theorem 1.5.16. Since $\bar{\Omega}$ is a compact subset of $O\cap R^n$, we may cover $\bar{\Omega}$ with a finite number of open cubes $Q_j\subset R^n$ such that the corresponding closed polycylinders $P_j\subset C^n$ with the same centers and $3n$ times the "sides" are contained in O. Let $\{\chi_j\}$ be a partition of unity for $\bar{\Omega}$ such that $\chi_j\in\mathcal{D}_\omega(Q_j)$. If $\varphi\in\mathcal{D}_\omega(\Omega)$ and $f\in\mathcal{A}(O)$ we apply Lemma 1.5.17 to $\varphi\chi_j$ and observe that $\varphi=\sum_j\varphi\chi_j$. This proves the theorem.

1.6. Spaces $\mathcal{D}'_\omega$ and $\mathcal{E}'_\omega$ of generalized distributions

The reason for the apparent awkwardness of the following definition is given in Section 1.0.

We recall that $\boldsymbol{\omega}(\xi)$ stands for $\omega(-\xi)$ and note that if $\omega \in \mathcal{M}$, then $\boldsymbol{\omega} \in \mathcal{M}$.

Definition 1.6.1. *Let Ω be an open subset of R^n and let $\omega \in \mathcal{M}$. Then $\mathcal{D}'_\omega(\Omega)$ is the space of all continuous linear functionals on $\mathcal{D}_{\boldsymbol{\omega}}(\Omega)$.*

An equivalent definition is: $\mathcal{D}'_\omega(\Omega)$ is the space of all linear functionals u on $\mathcal{D}_{\boldsymbol{\omega}}(\Omega)$ such that for each compact $K \subset \Omega$ there exist $\lambda > 0$ and C such that

$$|u(\varphi)| \leqslant C \|\varphi\|_\lambda^{(\boldsymbol{\omega})} \quad (\forall \varphi \in \mathcal{D}_{\boldsymbol{\omega}}(K)). \tag{1.6.1}$$

$\mathcal{D}'_\omega(\Omega)$ is given the weak topology, that is the topology given by the semi-norms $u \to |u(\varphi)|$, where φ is any element of $\mathcal{D}_{\boldsymbol{\omega}}$.

Following Beurling [3] we remark that two important properties of $\mathcal{D}_\omega$ make it possible to take over much of the Schwartz theory. The first property is that of being an algebra (Proposition 1.3.5). This property gives sense to the usual definition of the product of a test function and a distribution, although at this point we must pay for the choice we made in Definition 1.6.1:

Definition 1.6.2. *If $\varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega)$ and $u \in \mathcal{D}'_\omega(\Omega)$, we define $\varphi u \in \mathcal{D}'_\omega(\Omega)$ by*

$$(\varphi u)(\psi) = u(\varphi\psi) \quad (\forall \psi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega)).$$

The second property is the existence of partitions of unity. This property makes it possible to prove that if two elements of $\mathcal{D}'_\omega(\Omega)$ agree locally, they agree globally (cf. [H], Th. 1.3.3). Thus we may make the usual definition of support:

Definition 1.6.3. *Let $\omega \in \mathcal{M}$. If $u \in \mathcal{D}'_\omega(\Omega)$, the support of u (denoted* supp u*) is defined as the smallest closed set K such that $u = 0$ in $\Omega \cap \complement K$.*

In analogy with this we generalize the notion of singular support ([H], Def. 1.3.3) to the present situation:

Definition 1.6.4. *Let ω_1 and $\omega \in \mathcal{M}$. If $u \in \mathcal{D}'_{\omega_1}(\Omega)$, the ω-singular support of u (denoted* $\operatorname{sing}_\omega$ supp u*) is defined as the smallest closed set K such that $u \in \mathcal{E}_\omega(\Omega \cap \complement K)$.*

Another property of $\mathcal{D}'_\omega$, essential when dealing with differential operators, is closedness under the usual differentiation operators. Theorem 1.3.27 gives sense to the following definition:

Definition 1.6.5. *If $u \in \mathcal{D}'_\omega(\Omega)$ and α is any multi-index we define $D^\alpha u \in \mathcal{D}'_\omega(\Omega)$ by*

$$D^\alpha u(\varphi) = (-1)^{|\alpha|} u(D^\alpha \varphi) \quad (\forall \varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega)).$$

We will also consider the space of generalized distributions with compact support:

Definition 1.6.6. *Let $\omega \in \mathcal{M}$. Then $\mathcal{E}'_\omega(\Omega)$ is defined as the space of all continuous linear functionals on $\mathcal{E}_{\boldsymbol{\omega}}(\Omega)$.*

Just as in the classical case (cf. [H], Th. 1.5.1 and Th. 1.5.2) we have

Theorem 1.6.7. *$\mathcal{E}'_\omega(\Omega)$ can be identified with the set of all elements of $\mathcal{D}'_\omega(\Omega)$ which have compact supports contained in Ω.*

Clearly we have $\mathcal{E}'_\omega(\Omega)\subset\mathcal{D}'_{\omega,F}(\Omega)$, if we define the space $\mathcal{D}'_{\omega,F}(\Omega)$ of generalized distributions of finite order in the natural way:

Definition 1.6.8. *$\mathcal{D}'_{\omega,F}(\Omega)$ is the set of all $u\in\mathcal{D}'_\omega(\Omega)$ with the property that λ can be chosen independent of K in* (1.6.1).

Next we note that Theorem 1.3.18 and condition (γ) imply the following theorem and corollary:

Theorem 1.6.9. *If ω_1 and $\omega_2\in\mathcal{M}$ and $\omega_2\prec\omega_1$, then $\mathcal{D}'_{\omega_2}\subset\mathcal{D}'_{\omega_1}$ algebraically and topologically.*

Corollary 1.6.10. *If $w\in\mathcal{M}$, then $\mathcal{D}'\subset\mathcal{D}'_\omega$.*

Finally, since locally integrable functions can be identified with certain elements in $\mathcal{D}'$, it follows from Corollary 1.6.10 that the following definition makes sense:

Definition 1.6.11. *If $u\in L_1^{\mathrm{loc}}(\Omega)$, then we identify u with the element u in $\mathcal{D}'_\omega(\Omega)$ which is defined by*

$$u(\varphi)=\int u(x)\,\varphi(x)\,dx=u*\check{\varphi}(0)\quad(\forall\varphi\in\mathcal{D}_\omega(\Omega)).$$

1.7. *Convolutions of generalized distributions*

We will start by defining the convolution of a test function and a distribution and proving two theorems generalizing Th. 1.6.2 and Th. 1.6.1 of [H].

Definition 1.7.1. *Let $\omega\in\mathcal{M}$. If $u\in\mathcal{D}'_\omega$ and $\varphi\in\mathcal{D}_\omega$ we define the convolution $u*\varphi$ as the function given by*

$$(u*\varphi)(x)=u_y(\varphi(x-y))=u(\tau_x\check{\varphi}).$$

Theorem 1.7.2. *Let $\omega\in\mathcal{M}$. If φ and $\psi\in\mathcal{D}_\omega$ and $u\in\mathcal{D}'_\omega$, then $(u*\varphi)*\psi=u*(\varphi*\psi)$.*

Proof. For $\varepsilon>0$ we form the Riemann sum

$$f_\varepsilon(x)=\varepsilon^n\sum_t\varphi(x-\varepsilon t)\,\psi(\varepsilon t),$$

where t runs through all points with integer coordinates. We claim that $f_\varepsilon\to\varphi*\psi$ in $\mathcal{D}_\omega$ when $\varepsilon\to 0$. In fact,

$$\|f_\varepsilon-\varphi*\psi\|_\lambda=\int|\hat{\varphi}(\xi)|\,e^{\lambda\omega(\xi)}\,|\hat{\psi}(\xi)-R_\varepsilon(\xi)|\,d\xi,$$

where $R_\varepsilon(\xi)$ is a Riemann sum for the integral defining $\hat{\psi}$. Since $\hat{\psi}(\xi)\to 0$ when $|\xi|\to\infty$, the claimed convergence follows from the dominated convergence theorem.

Consequently,

$$(u * (\varphi * \psi))(x) = u_y((\varphi * \psi)(x-y)) = \lim_{\varepsilon \to 0} u_y(f_\varepsilon(x-y))$$

$$= \lim \varepsilon^n \sum_t u_y(\varphi(x-y-\varepsilon t))\,\psi(\varepsilon t) = ((u*\varphi)*\psi)(x),$$

which proves the theorem.

Theorem 1.7.3. *Let $\omega \in \mathcal{M}$. If $u \in \mathcal{D}'_\omega$ and $\varphi \in \mathcal{D}_\omega$, then $u*\varphi \in \mathcal{E}_\omega$ and* supp $(u*\varphi) \subset$ supp $u +$ supp φ.

Proof. The last result is trivial. We now choose $\psi \in \mathcal{D}_\omega$ and have to prove that $v \in \mathcal{D}_\omega$ if $v(x) = \psi(x)\,u(\tau_x \check{\varphi})$. Let ξ be fixed. By Proposition 1.3.29, $u(\tau_x\check{\varphi})$ is a continuous function of x. Hence, since v has compact support,

$$\hat{v}(\xi) = \int \psi(x)\, e^{-i\langle x,\xi\rangle}(u*\varphi)(x)\,dx = ((u*\varphi)*\check{\psi}_\xi)(0),$$

where we have written $\psi_\xi(x) = \psi(x)\,e^{-i\langle x,\xi\rangle}$. Using Theorem 1.7.2 we get

$$\hat{v}(\xi) = (u*(\varphi*\check{\psi}_\xi))(0) = u(\check{\varphi}*\psi_\xi).$$

Since supp $(\check{\varphi}*\psi_\xi)$ is contained in a fixed compact when ξ varies, we then have

$$|\hat{v}(\xi)| \leqslant C\,|||\check{\varphi}*\psi_\xi|||_\lambda^{(\omega)} \tag{1.7.1}$$

for some $\lambda > 0$ and C independent of ξ. But $(\check{\varphi}*\psi_\xi)^\wedge(\eta) = \hat{\varphi}(-\eta)\,\hat{\psi}(\xi+\eta)$, and consequently (1.7.1) gives

$$|\hat{v}(\xi)| \leqslant C\,(\sup_\eta |\hat{\varphi}(-\eta)\,\hat{\psi}(\xi+\eta)|\,e^{\lambda\omega(\eta)}) \leqslant C\,|||\varphi|||_{2\lambda}^{(\omega)}\,|||\psi|||_\lambda^{(\omega)}\,e^{-\lambda\omega(\xi)}. \tag{1.7.2}$$

This completes the proof of the theorem.

In view of the complication mentioned in connection with Proposition 1.3.15, we give the regularization of distributions (cf. [H], Th. 1.6.3) the following form, for convenience using Definition 1.3.24:

Theorem 1.7.4. *Let $\omega \in \mathcal{M}$. If $u \in \mathcal{D}'_\omega$ and $\varphi \in \mathcal{D}_{\omega^c}$ and $\int\varphi(x)\,dx = 1$ and $\varphi_\varepsilon(x) = \varepsilon^{-n}\varphi(x/\varepsilon)$, then $u*\varphi_\varepsilon \to u$ in $\mathcal{D}'_\omega$.*

The proof is the same as in [H]. Similarly, all remaining material of Sect. 1.6 of [H] can be taken over without difficulty. We will here only mention that letting $|||\varphi|||_{2\lambda}^{(\omega)} \to 0$ in (1.7.2) we see that the mapping $\varphi \to u*\varphi$ from $\mathcal{D}_\omega$ into $\mathcal{E}_\omega$ is continuous. This is the starting point of the argument which gives sense to the following definition.

Definition 1.7.5. *Let $\omega \in \mathcal{M}$. If $u_1 \in \mathcal{D}'_\omega$ and $u_2 \in \mathcal{E}'_\omega$ or conversely, then $u_1 * u_2$ is defined as the unique element u of $\mathcal{D}'_\omega$ satisfying $u_1 * (u_2 * \varphi) = u * \varphi$ $(\forall \varphi \in \mathcal{D}_\omega)$.*

1.8. *The Fourier transform and the spaces S_ω, S'_ω and $\mathfrak{F}_\omega$*

We note that $\mathcal{D}_\omega$ does not necessarily satisfy Cond. Φ_4 of [9], p. 101, or in our notation, that we have $\mathfrak{M}_s \neq \mathfrak{M}$. In spite of this we could of course define the Fourier transform of any u in $\mathcal{D}_\omega$ by the formula $\hat{u}(\psi) = u(\hat{\psi})$ $(\forall \psi \in \hat{\mathcal{D}}_\omega)$, or equivalently, $u(\varphi) = (2\pi)^{-n}\hat{u}(\check{\hat{\varphi}})$ $(\forall \varphi \in \mathcal{D}_\omega)$. Then $\hat{u}$ would be a "generalized function" over the test-space $\hat{\mathcal{D}}_\omega$ of Fourier transforms of the elements of $\mathcal{D}_\omega$. But we will avoid this generality and only define the Fourier transform $\hat{u}$ when $u \in S'_\omega$, where S'_ω generalizes the space S' of tempered distributions. To prepare for this definition we will first study a generalization of the space S (cf. [H], Sect. 1.7). If $\omega \notin \mathfrak{M}_c$ we may by Theorem 1.6.9 consider $\mathcal{D}'_\omega$ as a subspace of $\mathcal{D}'_{\omega^c}$. Thus it is no restriction that we will define S'_ω only for the case $\omega \in \mathfrak{M}_c$.

Definition 1.8.1. *Let $\omega \in \mathfrak{M}_c$. We denote by S_ω the set of all functions $\varphi \in L_1(R^n)$ with the property that (φ and $\hat{\varphi} \in C^\infty$ and) for each multi-index α and each non-negative number λ we have*

$$p_{\alpha,\lambda}(\varphi) = \sup_{x \in R^n} e^{\lambda\omega(x)} |D^\alpha \varphi(x)| < \infty$$

and

$$\pi_{\alpha,\lambda}(\varphi) = \sup_{\xi \in R^n} e^{\lambda\omega(\xi)} |D^\alpha \hat{\varphi}(\xi)| < \infty.$$

The topology of S_ω is defined by the semi-norms $p_{\alpha,\lambda}$ and $\pi_{\alpha,\lambda}$.

We recall that the Fourier transform is an automorphism of S. Using this fact and the symmetry of the definition of S_ω and applying condition (γ) we get the following result.

Proposition 1.8.2. *If $\omega(\xi) = \log(1 + |\xi|)$, then S_ω is identical with S. For any $\omega \in \mathfrak{M}_c$ we have $S_\omega \subset S$, and the Fourier transform is a continuous automorphism of S_ω.*

Just as in the classical case we also have

Proposition 1.8.3. *If $\omega \in \mathfrak{M}_c$, then S_ω is a topological algebra under point-wise multiplication and also under convolution.*

Proof. It suffices to prove the first result. Thus let φ and $\psi \in S_\omega$. Fix α and λ. Since e.g. all $p_{\beta,\lambda}(\varphi) < \infty$ and all $p_{\beta,0}(\psi) < \infty$, Leibniz' formula proves that $p_{\alpha,\lambda}(\varphi\psi) < \infty$. On the other hand,

$$D^\alpha \widehat{\psi\varphi} = (2\pi)^{-n} D^\alpha \hat{\varphi} * \hat{\psi},$$

and thus

$$\pi_{\alpha,\lambda}(\varphi\psi) \leq (2\pi)^{-n} \sup_\xi \int |D^\alpha \hat{\varphi}(\xi - \eta)| \, e^{\lambda\omega(\xi-\eta)} \cdot |\hat{\psi}(\eta)| \, e^{\lambda\omega(\eta)} \, d\eta$$

$$\leq (2\pi)^{-n} \pi_{\alpha,\lambda}(\varphi) \, \pi_{0,l}(\psi) \int e^{(\lambda - l)\omega(\eta)} \, d\eta,$$

which by condition (γ) is $< \infty$, if l is chosen sufficiently large. This completes the proof.

We leave it to the reader to check that we also have the following two results:

Proposition 1.8.4. *If $\omega \in \mathcal{M}_c$, then differentiation and multiplication by x^α are continuous operators in S_ω.*

Proposition 1.8.5. *If $\omega \in \mathcal{M}_c$, then the translation operator τ_a and the multiplication by* $\exp(-i\langle \cdot, a\rangle)$, *where $a \in R^n$, are continuous operators in S_ω.*

Next we relate S_ω to $\mathcal{D}_\omega$ and $\mathcal{E}_\omega$.

Proposition 1.8.6. *If $\omega \in \mathcal{M}_c$, then $\mathcal{D}_\omega \subset S_\omega \subset \mathcal{E}_\omega$.*

Proof. Let $\varphi \in \mathcal{D}_\omega$ and let α and λ be given. Then $p_{\alpha,\lambda}(\varphi) < \infty$, since φ has compact support. That $\pi_{\alpha,\lambda}(\varphi) < \infty$ follows from Theorem 1.5.16 with $f(x) = x^\alpha$ (or directly from Theorem 1.4.1 and Cauchy's integral formula). Thus $\varphi \in S_\omega$. Next we suppose that $\varphi \in S_\omega$ and choose $\psi \in \mathcal{D}_\omega$. We have to prove that $\varphi\psi \in \mathcal{D}_\omega$. By what we have just proved, $\psi \in S_\omega$. Thus by Proposition 1.8.3, $\varphi\psi \in S_\omega$. But since $\varphi\psi$ also has compact support, the result follows.

Finally S_ω has the following important property (cf. [H], Lem. 1.7.2):

Theorem 1.8.7. *If $\omega \in \mathcal{M}_c$, then $\mathcal{D}_\omega$ is dense in S_ω.*

Proof. Let us write $\omega(\xi) = \Omega(|\xi|)$. Let $\varphi \in S_\omega$. Choose $\psi \in \mathcal{D}_\omega$ such that ψ is a local unit for $\bar{B}_1$ and $0 \leq \psi(x) \leq \psi(0) = 1$ $(\forall x \in R^n)$. Define $\varphi_{(\varepsilon)}(x) = \varphi(x)\cdot\psi(\varepsilon x)$. Since $S_\omega \subset \mathcal{E}_\omega$, we have $\varphi_{(\varepsilon)} \in \mathcal{D}_\omega$. Thus it suffices to fix α and λ and prove that when $\varepsilon \to 0$ we have

$$p_{\alpha,\lambda}(\varphi_{(\varepsilon)} - \varphi) \to 0 \tag{1.8.1}$$

and

$$\pi_{\alpha,\lambda}(\varphi_{(\varepsilon)} - \varphi) \to 0. \tag{1.8.2}$$

Expanding $D^\alpha\varphi_{(\varepsilon)}$ by Leibniz' formula and using the boundedness of each derivative of ψ we see that due to the ε-factors it is enough to prove that

$$\sup_{x \in R^n} \left| e^{\lambda\omega(x)} (\psi(\varepsilon x) - 1)\, D^\alpha \varphi(x) \right| \to 0 \tag{1.8.3}$$

in order to have (1.8.1). But since $\psi(\varepsilon x) - 1 = 0$ when $|x| \leq 1/\varepsilon$, the left-hand side of (1.8.3) is

$$\leq p_{\alpha,\lambda+1}(\varphi) \cdot \sup_{|x| \geq 1/\varepsilon} e^{-\omega(x)},$$

which implies (1.8.3) and thus (1.8.1).

On the other hand, we have

$$|D^\alpha \hat{\varphi}_{(\varepsilon)}(\xi) - D^\alpha \hat{\varphi}(\xi)| \leq (2\pi)^{-n} \int |\hat{\psi}(\eta)| \cdot |D^\alpha \hat{\varphi}(\xi - \varepsilon\eta) - D^\alpha \hat{\varphi}(\xi)|\, d\eta = (2\pi)^{-n} (I_B + I_U),$$

where I_B denotes the integral over $B = \{\eta;\, |\eta| \leq |\xi| + \varepsilon^{-1/(n+2)}\}$ and I_U denotes the integral over $U = \complement B$. Evidently we have for any $l > 0$,

$$\begin{aligned} I_U &\leq 2\pi_{\alpha,0}(\varphi) \cdot \int_U |\hat{\psi}(\eta)|\, d\eta \leq 2\pi_{\alpha,0}(\varphi) \cdot |||\psi|||_{\lambda+l} \int_U e^{-(\lambda+l)\omega(\eta)}\, d\eta \\ &\leq C \sup_{\eta \in U} e^{-\lambda\omega(\eta)} \cdot \int_U e^{-l\omega(\eta)}\, d\eta \leq C\, e^{-\lambda\omega(\xi)} \int_{|\eta| \geq \varepsilon^{-1/(n+2)}} e^{-l\omega(\eta)}\, d\eta. \end{aligned}$$

From condition (γ) then it follows that if l is sufficiently large, we have $I_U \cdot e^{\lambda\omega(\xi)} \to 0$ when $\varepsilon \to 0$. To prove (1.8.2) and the theorem it thus remains to prove that $\sup_{\xi \in R^n} I_B e^{\lambda\omega(\xi)} \to 0$.

By the theorem of dominated convergence we see that $I_B \to 0$ when $\varepsilon \to 0$. Thus it suffices to prove for instance that

$$\sup_{|\xi| \geqslant 2} I_B e^{\lambda\omega(\varepsilon)} \to 0. \tag{1.8.4}$$

In the rest of the proof we suppose that $|\xi| \geqslant 2$ and that $\varepsilon < \frac{1}{2}$. Let us denote the line element vector in R^n by dr. Since $\hat{\varphi}$ is bounded, we have

$$\begin{aligned} |I_B| &\leqslant C(|\xi| + \varepsilon^{-1/(n+2)})^n \sup_{\eta \in B} |D^\alpha \hat{\varphi}(\xi - \varepsilon\eta) - D^\alpha \hat{\varphi}(\xi)| \\ &\leqslant C(|\xi| + \varepsilon^{-1/(n+2)})^n \sup_{\eta \in B} \left| \int_{\xi}^{\xi - \varepsilon\eta} \langle \operatorname{grad} D^\alpha \hat{\varphi}, dr \rangle \right| \\ &\leqslant C\varepsilon(|\xi| + \varepsilon^{-1/(n+2)})^{n+1} \sup_{\tau \in B} |\operatorname{grad} D^\alpha \hat{\varphi}(\xi - \varepsilon\tau)| \\ &\leqslant C\varepsilon^{1/(n+2)}(|\xi| + 1)^{n+1} \sum_{|\beta| = |\alpha| + 1} \pi_{\beta, l}(\varphi) e^{-l\Omega(|\xi| - \varepsilon|\xi| - 1)}. \end{aligned}$$

Since $\Omega(|\xi| - \varepsilon|\xi| - 1) > \frac{1}{2}\Omega(|\xi|) - \Omega(1)$, we get with a new constant,

$$|I_B| \leqslant C\varepsilon^{1/(n+2)} e^{-\lambda\omega(\xi)}, \tag{1.8.5}$$

if we choose $l = 2\lambda + (n+1)/b$, where b is the constant of condition (γ). This proves (1.8.4). The proof of the theorem is complete.

We can now define S'_ω and the Fourier transform in S'_ω.

Definition 1.8.8. *Let $\omega \in \mathfrak{M}_c$. A continuous linear form on $S_\omega = S_\omega$ is called an ω-temperate distribution. The space of all ω-temperate distributions is given the weak topology and is called S'_ω.*

In view of Theorem 1.8.7 we may identify S'_ω with a subspace of $\mathcal{D}'_\omega$. It is obvious that $\mathcal{E}'_\omega \subset S'_\omega$. Another important subset of S'_ω will be considered in Definition 1.8.10.

Definition 1.8.9. *If $\omega \in \mathfrak{M}_c$ and $u \in S'_\omega$ we define the Fourier transform $\hat{u} \in S'_\omega$ by*

$$\hat{u}(\varphi) = u(\hat{\varphi}) \quad (\forall \varphi \in S_\omega = S_\omega),$$

or equivalently,

$$\hat{u}(\hat{\varphi}) = (2\pi)^{-n} u(\check{\varphi}) \quad (\forall \varphi \in S_\omega).$$

As in [H] it follows that the Fourier transform is a continuous automorphism of S'_ω.

In Chapter II we will work with those u in S'_ω for which

$$|u(\varphi)| \leqslant C\pi_{0,\lambda}(\varphi) \quad (\forall \varphi \in S_\omega).$$

We prefer to define $\hat{u}$ explicitly in this case even when $\omega \notin \mathfrak{M}_c$ so that we have not defined S'_ω:

Definition 1.8.10. *Let $\omega \in \mathfrak{M}$. We denote by $\mathfrak{F}_\omega$ the set of all elements u in $\mathcal{D}'_\omega$ such that for some measurable function U with*

$$\int |U(\xi)| e^{-\lambda\omega(\xi)} d\xi < \infty \quad \text{for some } \lambda > 0$$

we have
$$u(\varphi) = (2\pi)^{-n} \int U(\xi)\hat{\varphi}(-\xi)\, d\xi \quad (\forall \varphi \in \mathcal{D}_\omega).$$

If $u \in \mathfrak{F}_\omega$ we define its Fourier transform $\hat{u}$ as [the equivalence class of] the function U:

$$\hat{u}(\xi) = U(\xi) \quad \text{(a.e.)}.$$

We remark that if functions are identified with certain distributions as in Definition 1.6.11, then u and $\hat{u} \in \mathcal{S}'_{\omega^c}$ and the new definition of the Fourier transform agrees with Definition 1.8.9. We also remark that if $u_1 \in \mathfrak{F}_{\omega_1}$ and $u_2 \in \mathfrak{F}_{\omega_2}$ and if $u_1 = u_2$ as elements of $\mathcal{D}'_{\omega_1+\omega_2}$, then the corresponding functions U_1 and U_2 are equal (almost everywhere).

The following theorem generalizes Th. 1.7.5 of [H] and is similarly proved.

Theorem 1.8.11. *The Fourier transform of $u \in \mathcal{E}'_\omega$ is the function*

$$\hat{u}(\xi) = u_x(e^{-i\langle x, \xi\rangle}).$$

The right-hand side is also defined for every complex vector $\xi \in C^n$ and is an entire function of ξ, called the Fourier–Laplace transform of u.

The following theorem and corollary connect convolution and Fourier transform and partly correspond to Th. 1.7.6 of [H]. Another related result is given in Theorem 1.8.15.

Theorem 1.8.12. *Let $\omega \in \mathfrak{M}_c$. If $\varphi \in \mathcal{S}_\omega$, and $u \in \mathcal{S}'_\omega$, then $\varphi * u \in \mathcal{S}'_\omega$ and $(\varphi * u)^\wedge = \hat{\varphi} \cdot \hat{u}$.*

Proof. If $\psi \in \mathcal{D}_\omega = \mathcal{D}_\omega$ we have (using Definition 1.6.11 and results from Section 1.7):

$$(\varphi * u)(\psi) = \varphi * u * \check{\psi}(0) = u(\check{\varphi} * \psi),$$

and thus there exist constants C, λ and k such that

$$|(\varphi * u)(\psi)| \leq C \sum_{|\alpha| \leq k} (p_{\alpha,\lambda}(\check{\varphi} * \psi) + \pi_{\alpha,\lambda}(\check{\varphi} * \psi)) \quad (\forall \psi \in \mathcal{D}_\omega).$$

Then it follows from Proposition 1.8.3 that there exist constants C, l and k such that

$$|(\varphi * u)(\psi)| \leq C \sum_{|\alpha| \leq k} (p_{\alpha,l}(\psi) + \pi_{\alpha,l}(\psi)) \quad (\forall \psi \in \mathcal{D}_\omega).$$

Thus $\varphi * u$, considered as an element of $\mathcal{D}'_\omega$, is in fact [extendable to] an element of $\mathcal{S}'_\omega$, defined by

$$(\varphi * u)(\psi) = u(\check{\varphi} * \psi) \quad (\forall \psi \in \mathcal{S}_\omega).$$

Thus, using Definition 1.8.9,

$$(\varphi * u)^\wedge(\psi) = (\varphi * u)(\hat{\psi}) = u(\check{\varphi} * \hat{\psi}).$$

Since $\mathcal{S}_\omega \subset \mathcal{S}$, we have $(\hat{\varphi}\cdot\psi)^\wedge = \check{\varphi} * \hat{\psi}$, and thus we get

$$(\varphi * u)^\wedge(\psi) = \hat{u}(\hat{\varphi}\psi) = (\hat{\varphi}\hat{u})(\psi),$$

which completes the proof.

From the proof follows (when only $\pi_{0,\lambda}$ or only $p_{0,\lambda}$ is used):

Corollary 1.8.13. *Let $\omega \in \mathfrak{M}$. If $u \in \mathcal{F}_\omega$ and $\varphi \in \mathcal{D}_\omega$ and $\psi \in \mathcal{D}_\omega$, then $\varphi * u$ and $\psi u \in \mathcal{F}_\omega$ and we have*

$$(\varphi * u)^\wedge = \hat{\varphi}\cdot\hat{u}$$

and

$$(\psi u)^\wedge = (2\pi)^{-n}\hat{\psi} * \hat{u}.$$

We can now state and prove the Paley–Wiener theorem for generalized distributions with compact support.

Theorem 1.8.14. *Let $\omega \in \mathfrak{M}$ and let K be a compact convex set in R^n and let H be the support function of K. If U is an entire function of n complex variables $\zeta = \xi + i\eta = (\zeta_1, \ldots, \zeta_n)$, the following three conditions are equivalent:*

(*a*) *For some real λ and all positive ε there exists a constant $C_{\lambda,\varepsilon}$ such that*

$$\int_{R^n} |U(\xi + i\eta)|\, e^{-\lambda\omega(\xi)}\, d\xi \leqslant C_{\lambda,\varepsilon}\, e^{H(\eta)+\varepsilon|\eta|}.$$

(*b*) *For some real λ and all positive ε there exists a constant $C'_{\lambda,\varepsilon}$ such that*

$$|U(\xi + i\eta)| \leqslant C'_{\lambda,\varepsilon}\, e^{H(\eta)+\varepsilon|\eta|+\lambda\omega(\xi)} \quad (\forall \xi + i\eta \in C^n).$$

(*c*) *U is the Fourier–Laplace transform of some $u \in \mathcal{E}'_\omega$ with* supp $u \subset K$.

Proof. That (*b*) implies (*a*) is clear (cf. the proof of Proposition 1.3.26). To prove that (*a*) implies (*b*) we may suppose that λ is positive. We can then use the inequality

$$-\lambda\omega(\xi) \leqslant -\lambda\omega(\xi + \xi') + \lambda\omega(\xi')$$

and proceed as in the proof that (i) implies (ii) in Theorem 1.4.1.

To prove that (*b*) implies (*c*), we derive from (*b*) with $\eta = 0$ that if $\psi \in \mathcal{D}_\omega$ then

$$\left|\int U(\xi)\hat{\psi}(-\xi)\, d\xi\right| \leqslant C\int e^{\lambda\omega(\xi)}|\hat{\psi}(-\xi)|\, d\xi \leqslant C\|\psi\|_\lambda^{(\omega)}.$$

Hence the linear functional

$$\psi \to (2\pi)^{-n}\int U(\xi)\hat{\psi}(-\xi)\, d\xi$$

is an element u in $\mathcal{D}'_\omega$. Thus $u \in \mathcal{F}_\omega$ and $\hat{u} = U$.

Let φ be in $\mathcal{D}_{\omega^c}(B_1)$ and $\int\varphi(x)dx = 1$ and let $\varphi_\delta(x) = \delta^{-n}\varphi(x/\delta)$ and $u_\delta = u * \varphi_\delta$. Then by Corollary 1.8.13 we have $u_\delta \in \mathcal{F}_\omega$ and $\hat{u}_\delta = U\cdot\hat{\varphi}_\delta$. From conditions (ii) of Theorem 1.4.1 and (*b*) (with λ replaced by l) we then derive for any $\lambda > 0$ and any $\varepsilon > 0$ the existence of a constant $C''_{\lambda,\varepsilon}$ such that

$$|\hat{u}_\delta(\xi + i\eta)| \leqslant C''_{\lambda,\varepsilon} \exp\,(l\omega(\xi) + H(\eta) + \varepsilon|\eta| - \lambda\omega(\xi) + \delta|\eta| + \varepsilon|\eta|).$$

Thus, by Theorem 1.4.1, $u_\delta \in \mathcal{D}_\omega(K+B_\delta)$. Hence when $\delta \to 0$, we get $\operatorname{supp} u \subset K$ (Theorem 1.7.4), which completes the proof of (*c*).

Finally, we prove that (*c*) implies (*b*). The meaning of (*c*) is that $U(\zeta) = u_x(e^{-i\langle x, \zeta\rangle})$ for some $u \in \mathcal{E}'_\omega$ with $\operatorname{supp} u \subset K$. Clearly, for some $l \geq 0$ we have

$$|u(\varphi)| \leq C \, |||\varphi|||_l^{(\omega)} \quad (\forall \varphi \in \mathcal{D}_\omega). \tag{1.8.6}$$

Let $\varrho \in \mathcal{D}_\omega(K+B_{\frac{1}{2}\varepsilon})$ be a local unit for K. Then $U(\zeta) = u(\varphi)$ with $\varphi(x) = \varrho(x)\, e^{-i\langle x, \zeta\rangle}$ and $\hat{\varphi}(\tau) = \hat{\varrho}(\tau+\zeta)$. Thus, if we apply to ϱ condition (ii) of Theorem 1.4.1 with ε replaced by $\frac{1}{2}\varepsilon$, we get

$$\begin{aligned} |U(\xi+i\eta)| &\leq C \sup_\tau |\hat{\varrho}(\tau+\xi+i\eta) \exp(l\boldsymbol{\omega}(\tau))| \\ &\leq C \exp(l\omega(\xi)) \sup_\tau |\hat{\varrho}(\tau+\xi+i\eta) \exp(l\boldsymbol{\omega}(\tau+\xi))| \\ &\leq C\, C'_{l,\frac{1}{2}\varepsilon} \exp(l\omega(\xi) + H(\eta) + \varepsilon|\eta|). \end{aligned}$$

This proves (*b*) and completes the proof of the theorem.

Remark. From the proof it follows that as λ in condition (*b*) we can use any l satisfying (1.8.6).

We note that Theorem 1.8.14 implies that $\mathcal{E}'_\omega \subset \mathcal{F}_\omega$. This now permits us to prove the following result:

Theorem 1.8.15. *Let $\omega \in \mathfrak{M}$. If $u_1 \in \mathcal{E}'_\omega$ and $u_2 \in \mathcal{F}_\omega$, then $u_1 * u_2 \in \mathcal{F}_\omega$ and $(u_1 * u_2)^\wedge = \hat{u}_1 \cdot \hat{u}_2$.*

Proof. We have if $\psi \in \mathcal{D}_\omega$

$$(u_1 * u_2)(\psi) = (u_1 * u_2 * \check{\psi})(0) = u_2(\check{u}_1 * \psi).$$

Since by Theorem 1.7.3, $\check{u}_1 * \psi \in \mathcal{D}_\omega$ we have by Definition 1.8.10 and Corollary 1.8.13 (note that $u_1 \in \mathcal{F}_\omega$):

$$(u_1 * u_2)(\psi) = (2\pi)^{-n} \int \hat{u}_2(\xi)\, (u_1 * \check{\psi})^\wedge(\xi)\, d\xi = (2\pi)^{-n} \int \hat{u}_2(\xi)\, \hat{u}_1(\xi)\, \hat{\psi}(-\xi)\, d\xi,$$

and the result follows, since $\hat{u}_2 \cdot \hat{u}_1$ is a function of the kind considered in Definition 1.8.10.

We will now prove an analogue of the Paley–Wiener theorem for the ω-singular support (cf. Definition 1.6.4 and [H], Th. 1.7.8).

Theorem 1.8.16. *Suppose that ω and $\omega_1 \in \mathfrak{M}$ and that $\omega_1 \prec \omega$ and $\omega_1 \prec \boldsymbol{\omega}$, and let $u \in \mathcal{E}'_{\omega_1}$. Let K be a compact convex set in R^n with support function H. In order that $\operatorname{sing}_\omega \operatorname{supp} u \subset K$, it is necessary and sufficient that there exist a constant λ and a sequence of constants C_m $(m = 1, 2, \ldots)$ such that*

$$|\eta| \leq m\omega(\xi) \text{ implies } |\hat{u}(\xi+i\eta)| \leq C_m\, e^{\lambda\omega_1(\xi) + H(\eta) + |\eta|/m}. \tag{1.8.7}$$

Proof. To show that (1.8.7) is *necessary*, we choose $\lambda > 1$ such that

$$|u(\varphi)| \leqslant C\, |||\varphi|||_\lambda^{(\omega_1)} \quad (\forall \varphi \in \mathcal{D}_{\omega_1}). \tag{1.8.8}$$

For each m we can by hypothesis write $u = u_1 + u_2$ where $\operatorname{supp} u_1 \subset K + B_{1/2m}$ and $u_2 \in \mathcal{D}_\omega(R^n)$. Since

$$|u_2(\varphi)| \leqslant (2\pi)^{-n}\, |||\varphi|||_0 \int |\hat{u}_2(\xi)|\, d\xi,$$

(1.8.8) implies that
$$|u_1(\varphi)| \leqslant C'\, |||\varphi|||_\lambda^{(\omega_1)} \quad (\forall \varphi \in \mathcal{D}_{\omega_1}). \tag{1.8.9}$$

Thus by Theorem 1.8.14 (with $\varepsilon = 1/2m$) and the remark following it, we have

$$|\hat{u}_1(\xi + i\eta)| \leqslant C_m' \exp\left(\lambda\omega_1(\xi) + H(\eta) + |\eta|/m\right) \quad (\forall \xi + i\eta \in C^n). \tag{1.8.10}$$

On the other hand, if $\operatorname{supp} u_2 \subset B_{N-1}$, we may apply Theorem 1.4.1 to u_2, taking $\varepsilon = 1$ and $\lambda = Nm + l$. We get

$$|\hat{u}_2(\xi + i\eta)| \leqslant C_m'' \exp\left(N|\eta| - (Nm + l)\omega(\xi)\right). \tag{1.8.11}$$

Now (1.8.7) follows from (1.8.10) and (1.8.11) if l is sufficiently large.

To prove the *sufficiency* of (1.8.7), we make an orthogonal transformation and reduce the problem to the following one. Suppose that

$$|\hat{u}(\xi + i\eta)| \leqslant C_m \exp\left(\lambda\omega_1(\xi) + A\eta_1 + |\eta|/m\right) \quad \text{if} \quad |\eta| \leqslant m\omega(\xi) \quad \text{and} \quad \eta_1 > 0. \tag{1.8.12}$$

Prove that if $\psi \in \mathcal{D}_\omega(\{x;\, x_1 > A\})$, then $\psi u \in \mathcal{D}_\omega$. By Corollary 1.8.13 we thus have to prove that for each (sufficiently large) l there is a constant C_l such that

$$\left| \int \hat{\psi}(\tau - \xi)\, \hat{u}(\xi)\, d\xi \right| \leqslant C_l\, e^{-l\omega(\tau)} \quad (\forall \tau \in R^n). \tag{1.8.13}$$

We want to deform the integration contour in this integral. By Lemma 1.2.4 and Theorem 1.3.18 we may suppose that ω is so smooth that the following integral is well defined:

$$I_m = \int d\xi' \int_{\Gamma_m} \hat{\psi}(\tau_1 - \zeta_1, \tau' - \xi')\, \hat{u}(\zeta_1, \xi')\, d\zeta_1, \tag{1.8.14}$$

where $\xi' = (\xi_2, \ldots, \xi_n) \in R^{n-1}$ and the integration with respect to $\zeta_1 = \xi_1 + i\eta_1$ is over the contour Γ_m defined by $\eta_1 = m\omega(\xi)$.

Since the support of ψ is compact and contained in $\{x;\, x_1 > A\}$, it is in fact contained in $\{x;\, x_1 > A + 3\delta\}$ for some $\delta > 0$. Thus, taking $\varepsilon = \delta$ in Theorem 1.4.1, we see that for each $l > 0$ there is C_l' such that for $\eta_1 \geqslant 0$:

$$|\hat{\psi}(\tau_1 - \zeta_1, \tau' - \xi')| \leqslant C_l' \exp\left(-l\omega(\tau - \xi) - (A + 2\delta)\eta_1\right). \tag{1.8.15}$$

From (1.8.12) and (1.8.15) it follows that if $m > 1/\delta$, then the modulus of the integrand of (1.8.13) and (1.8.14) is

$$\leqslant C \exp\left(\lambda\omega_1(\xi) - l\omega(\tau - \xi) - \delta\eta_1\right)$$

when $0 \leqslant \eta_1 \leqslant m\omega(\xi)$. Hence, since $\omega_1 \prec \boldsymbol{\omega}$ and

$$-l\omega(\tau-\xi) \leqslant -l\omega(-\xi)+l\omega(-\tau)$$

(if $l>0$), the integrals in (1.8.13) and (1.8.14) are equal by deformation if l is sufficiently large. (A similar argument is carried through in more detail in the proof of Lemma 4.1.3.) On the other hand, since $\omega_1 \prec \omega$ and

$$-l\omega(\tau-\xi) \leqslant l\omega(\xi)-l\omega(\tau),$$

the modulus of the considered integrand on Γ_m is

$$\leqslant C \exp\,(C'\omega(\xi)-\delta\eta_1-l\omega(\tau)) = C \exp\,((C'-\delta m)\omega(\xi)-l\omega(\tau)).$$

Further, by Lemma 1.2.4, $|d\zeta_1|/|d\xi_1|$ is bounded on Γ_m. Thus, if we take m sufficiently large, the result follows.

Specializing ω_1, we get the two following results (with K, H, λ and C_m as in Theorem 1.8.16).

Corollary 1.8.17. *Suppose that* $u \in \mathcal{E}'$ *and* $\omega \in \mathcal{M}$. *In order that* $\operatorname{sing}_\omega \operatorname{supp} u \subset K$ *it is necessary and sufficient that*

$$|\eta| \leqslant m\omega(\xi) \ \text{ implies } \ |\hat{u}(\xi+i\eta)| \leqslant C_m(1+|\xi|)^\lambda\, e^{H(\eta)+|\eta|/m}.$$

Proof. Since ω and $\boldsymbol{\omega}$ satisfy condition (γ) we may choose $\omega_1(\xi)=\log\,(1+|\xi|)$ in Theorem 1.8.16.

Corollary 1.8.18. *Suppose that* $\omega \in \mathcal{M}_s$ *and* $u \in \mathcal{E}'_\omega$. *In order that* $\operatorname{sing}_\omega \operatorname{supp} u \subset K$ *it is necessary and sufficient that*

$$|\eta| \leqslant m\omega(\xi) \ \text{ implies } \ |\hat{u}(\xi+i\eta)| \leqslant C_m\, e^{\lambda\omega(\xi)+H(\eta)}. \tag{1.8.16}$$

Proof. Since $\omega_1=\omega$, (1.8.16) is equivalent to (1.8.7), if we replace λ by $\lambda+1$ when necessary.

Chapter II. Some special spaces of generalized distributions

2.0. *Introduction*

In this chapter we generalize the spaces $\mathcal{B}_{p,k}$ and $\mathcal{B}^{\text{loc}}_{p,k}$ considered in Chap. II of [H]. This generalization will be done by considering weight functions k with more rapid growth than those considered in [H]. To each $\omega \in \mathcal{M}$ we will define a class $\mathcal{K}_\omega$ of admissible weight functions k. Then $\mathcal{B}_{p,k}$ will be the set of generalized distributions u whose Fourier transforms $\hat{u}$ are such that $\hat{u}\cdot k \in L_p$. It turns out that $\mathcal{B}_{p,k}$ does not depend on ω as long as $k \in \mathcal{K}_\omega$. Since the conditions defining $\mathcal{D}_\omega$ are given on the Fourier transform side, spaces of type $\mathcal{B}_{p,k}$ are particularly well adapted to our situation. On the other hand, when replacing $\log\,(1+|\xi|)$ by a general ω, we lose the close connection between ω and differentiation, as pointed out in the preface.

2.1. *Weight functions k*

We start with the following two definitions and note that the first one reduces to the definition of $\mathcal{K}$ in [H], when $\omega(\xi)=\log(1+|\xi|)$, since the constant that occurs there could be left out without any change.

Definition 2.1.1. *Let $\omega\in\mathfrak{M}$ be given. Then $\mathcal{K}_\omega$ is the set of all positive functions k in R^n with the following property. There exists $\lambda>0$ such that*

$$k(\xi+\eta)\leqslant e^{\lambda\omega(-\xi)}\,k(\eta) \text{ for all } \xi \text{ and } \eta \text{ in } R^n.$$

Definition 2.1.2. *If k is a positive function on R^n, we write*

$$M_k(\xi)=\sup_{\eta\in R^n}\frac{k(\xi+\eta)}{k(\eta)}.$$

We note that by condition (γ) we have $\mathcal{K}\subset\mathcal{K}_\omega$ for every $\omega\in\mathfrak{M}$. We also note that by condition (α) we have $\exp\boldsymbol{\omega}\in\mathcal{K}_\omega$. Next we list as Theorem 2.1.3 some results which are proved just as are the corresponding ones in Sect. 2.1 of [H].

Theorem 2.1.3. *Let $\omega\in\mathfrak{M}$. If $k\in\mathcal{K}_\omega$, then $\log k$ is uniformly continuous, $M_k\in\mathcal{K}_\omega$ and the following inequalities hold for all ξ and $\eta\in R^n$ (with the λ of Definition 2.1.1):*

$$e^{-\lambda\omega(\xi)}\leqslant\frac{k(\xi+\eta)}{k(\eta)}\leqslant e^{\lambda\omega(-\xi)}, \tag{2.1.1}$$

$$k(0)\,e^{-\lambda\omega(\xi)}\leqslant k(\xi)\leqslant k(0)\,e^{\lambda\omega(-\xi)}, \tag{2.1.2}$$

$$M_k(\xi+\eta)\leqslant M_k(\xi)\,M_k(\eta), \tag{2.1.3}$$

$$1=M_k(0)\leqslant M_k(\xi). \tag{2.1.4}$$

As in [H] we immediately get the following result from Definition 2.1.2 and Theorem 2.1.3. However, the situation is complicated by the fact that we do not assume $\omega\in\mathfrak{M}_s$.

Theorem 2.1.4. *Let $\omega\in\mathfrak{M}$. If k_1 and k_2 are in $\mathcal{K}_\omega$, it follows that k_1+k_2, k_1k_2, $\sup(k_1,k_2)$ and $\inf(k_1,k_2)$ are also in $\mathcal{K}_\omega$. If $k\in\mathcal{K}_\omega$ we have $k^s\in\mathcal{K}_\omega$ for every non-negative s but $k^s\in\mathcal{K}_{\boldsymbol{\omega}}$ for every non-positive s.*

We note that in particular $k\cdot\tilde{P}$ and $k/\tilde{P}\in\mathcal{K}_\omega$ if $k\in\mathcal{K}_\omega$, where we have defined $\tilde{P}(\xi)\geqslant 0$ by $\tilde{P}(\xi)^2=\sum_{|\alpha|\geqslant 0}|D^\alpha P(\xi)|^2$, when P is a polynomial.

Since we work with a whole family of classes $\mathcal{K}_\omega$, the question naturally arises: Given k, when does there exist $\omega\in\mathfrak{M}$ such that $k\in\mathcal{K}_\omega$? The following theorem answers that question.

Theorem 2.1.5. *Let k be a positive function on R^n. Then a necessary and sufficient condition that there exists $\omega\in\mathfrak{M}$ with $k\in\mathcal{K}_\omega$ is that $\log k$ is uniformly continuous and*

$$\int_{R^n}\sup_{\eta\in R^n}\frac{|\log k(\xi+\eta)-\log k(\eta)|}{(1+|\xi|)^{n+1}}\,d\xi<+\infty. \tag{2.1.5}$$

Proof. By Theorem 2.1.3 and condition (β), the condition is necessary. To prove that it is sufficient, we define

$$\omega(\xi) = \max(\log M_k(-\xi), \log(1+|\xi|)),$$

where M_k is given by Definition 2.1.2. Now (2.1.3) has nothing to do with the fact that $\omega \in \mathcal{M}$ in Theorem 2.1.3, but follows directly from Definition 2.1.2. Thus (2.1.3) holds also in the present case, and hence ω satisfies condition (α). By hypothesis, ω satisfies condition (β). By the uniform continuity, $\lim_{\xi\to 0}\omega(\xi)=0$, and hence the subadditivity of ω implies that ω is continuous. Finally, ω satisfies condition (γ), and the theorem is proved.

Definition 2.1.6. *We denote by $\mathcal{K}_{(\mathcal{M})}$ the class of functions $\bigcup_{\omega\in\mathcal{M}}\mathcal{K}_\omega$, that is the class of all positive functions k with uniformly continuous logarithm on R^n and satisfying* (2.1.5).

2.2. *The spaces $\mathcal{B}_{p,k}$*

We will now generalize the spaces $\mathcal{B}_{p,k}$ of [H] to the case when $k\in\mathcal{K}_{(\mathcal{M})}$. We will first give a definition of a space $\mathcal{B}^\omega_{p,k}$, apparently depending on ω.

Definition 2.2.1. *Let $\omega\in\mathcal{M}$ and $k\in\mathcal{K}_\omega$ and let $1\leqslant p\leqslant\infty$. We denote by $\mathcal{B}^\omega_{p,k}$ the set of all u in $\mathcal{F}_\omega$ for which*

$$\|u\|_{p,k} = \left((2\pi)^{-n}\int |k(\xi)\,\hat{u}(\xi)|^p\,d\xi\right)^{1/p} < \infty, \tag{2.2.1}$$

where, of course, $\|u\|_{\infty,k}$ means ess. sup. $k(\xi)|\hat{u}(\xi)|$.

When $\omega\in\mathcal{M}_c$ we need not *assume* that $u\in\mathcal{F}_\omega$:

Proposition 2.2.2. *Let $\omega\in\mathcal{M}_c$ and $k\in\mathcal{K}_\omega$ and let $1\leqslant p\leqslant\infty$. Then $\mathcal{B}^\omega_{p,k}$ is identical with the set of all $u\in\mathcal{S}'_\omega$ such that $\hat{u}$ is a function and* (2.2.1) *holds.*

Proof. From Hölder's inequality and condition (γ) follows that (2.2.1) and (2.1.2) imply that for some λ,

$$\int|\hat{u}(\xi)|\,e^{-\lambda\omega(\xi)}\,d\xi \leqslant C\|u\|_{p,k} < \infty. \tag{2.2.2}$$

Thus $u\in\mathcal{F}_\omega$.

We now prove the counterpart of Th. 2.2.1 of [H].

Theorem 2.2.3. *Let $\omega\in\mathcal{M}_c$. Then $\mathcal{B}^\omega_{p,k}$ is a Banach space with the norm $\|\cdot\|_{p,k}$. We have*

$$\mathcal{S}_\omega \subset \mathcal{B}^\omega_{p,k} \subset \mathcal{S}'_\omega$$

also in the topological sense. $\mathcal{D}_\omega$ is dense in $\mathcal{B}^\omega_{p,k}$ if $p<\infty$.

Proof. Just as in [H] we have $\mathcal{S}_\omega\subset L_p(R^n, k(\xi)^p d\xi)\subset\mathcal{S}'_\omega$ algebraically and topologically, and we take the Fourier transforms of these three spaces. We do not repeat the details.

We will now prepare for an invariant definition of $\mathcal{B}_{p,k}$.

Theorem 2.2.4. *If $\omega_r \in \mathcal{M}$ and $k \in \mathcal{K}_{\omega_r}$ $(r=1,2)$ and $\omega_2 \prec \omega_1$, so that we have a continuous injection $i: \mathcal{D}'_{\omega_2} \to \mathcal{D}'_{\omega_1}$, then i restricted to $\mathcal{B}^{\omega_2}_{p,k}$ is an isometry of $\mathcal{B}^{\omega_2}_{p,k}$ onto $\mathcal{B}^{\omega_1}_{p,k}$.*

Proof. Since ω does not appear in the definition of the norm $\|\cdot\|_{p,k}$, we only have to prove that the restricted mapping is onto. Let us therefore choose $u \in \mathcal{B}^{\omega_1}_{p,k}$. Then since $u \in \mathcal{F}_{\omega_1}$, we have

$$u(\varphi) = (2\pi)^{-n} \int \hat{u}(\xi)\, \hat{\varphi}(-\xi)\, d\xi \quad (\forall \varphi \in \mathcal{D}_{\omega_1}). \tag{2.2.3}$$

Repeating the proof of (2.2.2) but using ω_2 instead of ω, we get

$$\int |\hat{u}(\xi)|\, e^{-\lambda\omega_2(\xi)}\, d\xi < \infty.$$

Thus, since $\mathcal{D}_{\omega_1}$ is dense in $\mathcal{D}_{\omega_2}$, we can define $u(\varphi)$ by (2.2.3) for every $\varphi \in \mathcal{D}_{\omega_2}$, which means that $u \in \mathcal{F}_{\omega_2}$ and thus $u \in \mathcal{B}^{\omega_2}_{p,k}$. The proof is complete.

From Theorem 2.2.4 it follows that if $k \in \mathcal{K}_{(\mathcal{M})}$ is given, the choice of ω in the definition of $\mathcal{B}^{\omega}_{p,k}$ is irrelevant as long as $k \in \mathcal{K}_{\omega}$. We prefer to express this fact in the following slightly inexact way:

Definition 2.2.5. *Let $k \in \mathcal{K}_{(\mathcal{M})}$ and let $1 \leqslant p \leqslant \infty$. Then we identify all $\mathcal{B}^{\omega}_{p,k}$ for which $\omega \in \mathcal{M}$ and $k \in \mathcal{K}_{\omega}$. We denote the result of the identification by $\mathcal{B}_{p,k}$. If Ω is an open subset of R^n we also define*

$$\mathcal{B}^{c}_{p,k}(\Omega) = \mathcal{B}_{p,k} \cap \mathcal{E}'_{\omega}(\Omega).$$

We may now summarize part of our results as follows:

Theorem 2.2.6. *$\mathcal{B}_{p,k}$ is a Banach space with the norm $\|\cdot\|_{p,k}$. For any ω such that $k \in \mathcal{K}_{\omega}$, we have*

$$\mathcal{D}_{\omega} \subset \mathcal{B}_{p,k} \subset \mathcal{D}'_{\omega}$$

algebraically and topologically. Further, if $p < \infty$, then $\mathcal{D}_{\omega}$ is dense in $\mathcal{B}_{p,k}$.

The results of [H], Sect. 2.2, on relations between $\mathcal{B}_{p,k}$-spaces could now be proved for general $k \in \mathcal{K}_{(\mathcal{M})}$. (In the statements, $\mathcal{B}_{p,k} \cap \mathcal{E}'(\Omega)$ should be replaced by $\mathcal{B}^{c}_{p,k}(\Omega)$.) As an example we give the following theorem, which we will use several times.

Theorem 2.2.7. *Let $\omega \in \mathcal{M}$ and $k \in \mathcal{K}_{\omega}$. If $u \in \mathcal{B}_{p,k}$ and $\varphi \in \mathcal{D}_{\omega}$, then $\varphi u \in \mathcal{B}_{p,k}$ and*

$$\|\varphi u\|_{p,k} \leqslant \|\varphi\|_{1,M_k} \|u\|_{p,k}.$$

If $\omega \in \mathcal{M}_c$, then the same result is true when $\varphi \in \mathcal{S}_{\omega}$.

Proof. By Corollary 1.8.13 (and Theorem 1.8.7), we have

$$(\varphi u)\hat{\ }(\xi) = (2\pi)^{-n} \int \hat{\varphi}(\xi - \eta)\, \hat{u}(\eta)\, d\eta,$$

and the proof proceeds as in [H], Th. 2.2.5.

Finally we prove the following theorem, which is more general than the obvious counterpart of Th. 2.2.7 of [H], and which is also related to Th. 2.2.2 of [H]:

Theorem 2.2.8. *Suppose that k_1 and $k_2 \in \mathcal{K}_{(\mathcal{M})}$ and that $1 \leqslant p \leqslant \infty$. Then, if*

$$k_1/k_2 \in L_{p'} \quad \textit{with} \quad \frac{1}{p}+\frac{1}{p'}=1, \tag{2.2.4}$$

it follows that $\mathcal{B}_{p,k_2} \subset \mathcal{B}_{1,k_1}$ algebraically and topologically. Conversely, if $\mathcal{B}^c_{p,k_2}(\Omega) \subset \mathcal{B}_{1,k_1}$ for some open non-void Ω, then (2.2.4) *holds.*

Proof. If (2.2.4) holds, then by Hölder's inequality,

$$\|u\|_{1,k_1} \leqslant (2\pi)^{-n/p'} \|k_1/k_2\|_{L^{p'}} \|u\|_{p,k_2} \quad (\forall u \in \mathcal{B}_{p,k_2}),$$

which proves the first part. Conversely, we may assume that $0 \in \Omega$. Choose $\varphi \in \mathcal{D}_\omega(\Omega)$ such that $\hat{\varphi} \geqslant 0$ (Corollary 1.3.17). Then if $u \in \mathcal{B}_{p,k_2}$ we have $\varphi u \in \mathcal{B}^c_{p,k_2}(\Omega)$ by Theorem 2.2.7. Hence, by hypothesis, $\varphi u \in \mathcal{B}_{1,k_1}$. We claim that the mapping $u \to \varphi u$ is a closed mapping of $\mathcal{B}_{p,k_2}$ into $\mathcal{B}_{1,k_1}$. In fact, if $u_\nu \to u$ in $\mathcal{B}_{p,k_2}$ and $\varphi u_\nu \to v$ in $\mathcal{B}_{1,k_1}$, then for a suitable $\omega \in \mathcal{M}_c$ we have on one hand that $u_\nu \to u$ in $\mathcal{S}'_\omega$ which implies that $\varphi u_\nu \to \varphi u$ in $\mathcal{S}'_\omega$, and on the other hand that $\varphi u_\nu \to v$ in $\mathcal{S}'_\omega$. Hence $\varphi u = v$. Thus by the closed graph theorem,

$$\int k_1(\xi)\,|(\varphi u)^\wedge(\xi)|\,d\xi \leqslant C_\varphi \|u\|_{p,k_2} \quad (\forall u \in \mathcal{B}_{p,k_2}). \tag{2.2.5}$$

Now $\hat{\varphi}$ is non-negative. If also $\hat{u}$ were non-negative, we would have (using Definition 2.1.1)

$$(2\pi)^n k_1(\xi)\,|(\varphi u)^\wedge(\xi)| = k_1(\xi)\int \hat{u}(\xi-\eta)\,\hat{\varphi}(\eta)\,d\eta \geqslant \int \hat{u}(\xi-\eta)\,k_1(\xi-\eta)\,\hat{\varphi}(\eta)\,e^{-\lambda\omega(\eta)}\,d\eta.$$

Hence, inverting the order of integration, we would get

$$(2\pi)^n \int k_1(\xi)\,|(\varphi u)^\wedge(\xi)|\,d\xi \geqslant \|\varphi\|_{-\lambda} \int \hat{u}(\xi)\,k_1(\xi)\,d\xi.$$

Combining this with (2.2.5), we have proved that

$$\int \frac{k_1(\xi)}{k_2(\xi)}\,g(\xi)\,d\xi \leqslant C\|g\|_{L_p}$$

for every $g \geqslant 0$ such that $g = k_2 \cdot \hat{u}$ for some $u \in \mathcal{B}_{p,k_2}$. Since k_2 is bounded away from zero on each compact set, every measurable non-negative g with compact support trivially is of this form. Thus the result follows from the inverse of Hölder's inequality.

2.3. *Local spaces*

In this section we will define spaces $\mathcal{B}^{\mathrm{loc}}_{p,k}$ when $k \in \mathcal{K}_{(\mathcal{M})}$, thus generalizing Sect. 2.3 of [H].

Definition 2.3.1. *If $\mathcal{G}$ is a linear subspace of $\mathcal{D}'_\omega(\Omega)$, we define*

$$\mathcal{G}^{\omega\,\mathrm{loc}} = \{u \in \mathcal{D}'_\omega;\ \varphi \in \mathcal{D}_\omega(\Omega) \Rightarrow \varphi u \in \mathcal{G}\}.$$

Definition 2.3.2. *If $\mathcal{G}\subset\mathcal{G}^{\omega\,\mathrm{loc}}$, we say that $\mathcal{G}$ is ω-semilocal, and if $\mathcal{G}=\mathcal{G}^{\omega\,\mathrm{loc}}$, we say that $\mathcal{G}$ is ω-local.*

It is clear that the properties of local and semilocal spaces, given in the beginning of Sect. 2.3 of [H], generalize to our situation. In particular, by Theorem 2.2.7 we see that $\mathcal{B}_{p,k}$ is ω-semilocal if $k\in\mathcal{K}_\omega$. We leave to the reader the proof of the following result.

Proposition 2.3.3. $\mathcal{E}_\omega(\Omega)=(\mathcal{D}_\omega(\Omega))^{\omega\,\mathrm{loc}}$.

We will now define spaces $\mathcal{B}^{\mathrm{loc}}_{p,k}$, and just as in Section 2.2 we first consider spaces which apparently depend on ω.

Definition 2.3.4. *If $\omega\in\mathcal{M}$ and $k\in\mathcal{K}_\omega$ and $1\leqslant p\leqslant\infty$, we define $\mathcal{B}^{\omega\,\mathrm{loc}}_{p,k}(\Omega)=\mathcal{G}^{\omega\,\mathrm{loc}}$ where $\mathcal{G}$ is the set of all restrictions to Ω of elements of $\mathcal{B}^{\omega}_{p,k}$. The topology is given by the semi-norms $u\to\|\varphi u\|_{p,k}$ $(\varphi\in\mathcal{D}_\omega(\Omega))$.*

Corresponding to Theorem 2.2.6 we have the following theorem, which is proved like Th. 2.3.8 of [H].

Theorem 2.3.5. *$\mathcal{B}^{\omega\,\mathrm{loc}}_{p,k}$ is a Fréchet space and*

$$\mathcal{E}_\omega(\Omega)\subset\mathcal{B}^{\omega\,\mathrm{loc}}_{p,k}(\Omega)\subset\mathcal{D}'_\omega(\Omega)$$

algebraically and topologically.

Corresponding to Theorem 2.2.4 we have

Theorem 2.3.6. *If ω_1, ω_2 and i are as in Theorem 2.2.4, then i restricted to $\mathcal{F}_2$ is an isomorphism of $\mathcal{F}_2$ onto $\mathcal{F}_1$, where $\mathcal{F}_r=\mathcal{B}^{\omega_r\,\mathrm{loc}}_{p,k}(\Omega)$ $(r=1,2)$.*

Proof. Since $\mathcal{B}^{\omega_1}_{p,k}$ and $\mathcal{B}^{\omega_2}_{p,k}$ can be identified and $\mathcal{D}_{\omega_1}\subset\mathcal{D}_{\omega_2}$, it is clear that the restriction of i to $\mathcal{F}_2$ is a linear injection j of $\mathcal{F}_2$ into $\mathcal{F}_1$. Since every semi-norm $u\to\|u\varphi\|_{p,k}$ in $\mathcal{F}_1$, given by a function φ in $\mathcal{D}_{\omega_1}$, can be considered as a semi-norm in $\mathcal{F}_2$, we see that j is continuous. Then by Theorem 2.3.5 and Banach's theorem it suffices to prove that j is onto. Thus let $u\in\mathcal{F}_1$ and let $\varphi\in\mathcal{D}_{\omega_1}$. Let K_ν be compact subsets of Ω such that $K_\nu\nearrow\nearrow\Omega$ and let $\varphi_\nu\in\mathcal{D}_{\omega_1}(\Omega)$ be a local unit for K_ν. If ν is so large that $\operatorname{supp}\varphi\subset K_\nu$, we have $\varphi u=\varphi\varphi_\nu u$, so that by Theorem 2.2.7,

$$\|\varphi u\|_{p,k}\leqslant\|\varphi\|_{1,M_k}\|\varphi_\nu u\|_{p,k}.$$

Since $k\in\mathcal{K}_{\omega_2}$, it then follows from Theorem 2.1.3 that there exist constants λ and C_ν such that

$$\|\varphi u\|_{p,k}\leqslant C_\nu\|\varphi\|^{(\omega_2)}_\lambda\quad(\forall\varphi\in\mathcal{D}_{\omega_1}(K_\nu)).\tag{2.3.1}$$

Then we may extend u so that (2.3.1) holds for all $\varphi\in\mathcal{D}_{\omega_2}(K_\nu)$. Clearly, the extended u is in $\mathcal{F}_2$, which completes the proof.

We can now define $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$.

Definition 2.3.7. *Let $k\in\mathcal{K}_{(\mathcal{M})}$ and let $1\leqslant p\leqslant\infty$. We identify all $\mathcal{B}^{\omega\,\mathrm{loc}}_{p,k}(\Omega)$ for which $k\in\mathcal{K}_\omega$ and call the result of the identification $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$. We give $\mathcal{B}^{\mathrm{loc}}_{p,k}(\Omega)$ the natural topology.*

We could now prove for general $k \in \mathcal{K}_{(m)}$ the results of [H], Sect. 2.3, on relations between spaces $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega)$ and on their interplay with $P(D)$. We leave this to the reader and only remark that the proof of Th. 2.3.6 of [H] actually gives the following more precise result:

Theorem 2.3.8. *Let U and W be bounded open sets such that $\overline{U}-\overline{W}\subset\Omega$, and let $u_1 \in \mathcal{B}_{p,k_1}^{c}(W)$ and $u_2 \in \mathcal{B}_{\infty,k_2}^{\mathrm{loc}}(\Omega)$ with k_1 and $k_2 \in \mathcal{K}_{(m)}$. Then $u_1 \ast u_2 \in \mathcal{B}_{p,k_1k_2}^{\mathrm{loc}}(U)$.*

We now prove a theorem which generalizes formula (2.3.2) of [H].

Theorem 2.3.9. *Let $\omega \in \mathfrak{M}$ and $k_\mu \in \mathcal{K}_\omega$ and let $1 \leqslant p_\mu \leqslant \infty$ $(\mu=1, 2, \ldots)$. If the space $\mathcal{F}=\bigcap_\mu \mathcal{B}_{p_\mu,k_\mu}^{\mathrm{loc}}(\Omega)$ is equipped with the topology given by all the semi-norms $u \to \|\varphi u\|_{p_\mu,k_\mu}$ $(\varphi \in \mathcal{D}_\omega(\Omega))$, then $\mathcal{F}$ is a Fréchet space. If in particular $k_\mu = \exp(\mu\omega)$, then $\mathcal{F}$ is naturally isomorphic to $\mathcal{E}_\omega(\Omega)$.*

Proof. To prove the first result we only have to prove that the topology is metrizable. Choose compact sets $K_\nu \nearrow \nearrow \Omega$ and local units φ_ν for K_ν with $\varphi_\nu \in \mathcal{D}_\omega(\Omega)$. It then suffices to use the semi-norms $u \to \|\varphi_\nu u\|_{p_\mu,k_\mu}$. The last result follows from the local version of Theorem 2.2.8, since we have $k_{\mu_1}/k_{\mu_2} \in L_{p'}$ for every $p'(1 \leqslant p' \leqslant \infty)$, if $\mu_2-\mu_1$ is sufficiently large.

As an application of Theorem 2.3.9 we prove the following result, which will be used in Chapter IV, and which could easily have been proved in Chapter I.

Theorem 2.3.10. *Let $\omega \in \mathfrak{M}$. Let U and W be bounded open sets such that $\overline{U}-\overline{W}\subset\Omega$ and let $u \in \mathcal{E}_\omega'(W)$ and $\varphi \in \mathcal{E}_\omega(\Omega)$. Then $u \ast \varphi \in \mathcal{E}_\omega(U)$.*

Proof. By Theorem 2.3.9, we have that $\varphi \in \mathcal{B}_{\infty,k_\mu}^{\mathrm{loc}}(\Omega)$ with $k_\mu = \exp(\mu\omega)$ $(\mu=1, 2, \ldots)$. By definition, $u \in \mathcal{B}_{1,k_0}^{c}(W)$ with $k_0 = \exp(-l\omega(\xi))$ for some $l>0$. Hence, by Theorem 2.3.8, $\varphi \ast u \in \mathcal{B}_{1,k_\mu'}^{\mathrm{loc}}(U)$ with $k_\mu' = \exp(\mu-l)\omega$ and hence also with $k_\mu' = \exp[\mu-l]\omega$ $(\mu=1, 2, \ldots)$. Applying again Theorem 2.3.9, we get the desired result.

In Section 3.3 we will consider the space $\mathcal{D}_{\omega,F}'(\Omega)$ (cf. Definition 1.6.8). Then we will need the following result:

Theorem 2.3.11. *Let $\omega \in \mathfrak{M}$. For every p with $1 \leqslant p \leqslant \infty$ we have*

$$\bigcup_{k \in \mathcal{K}_\omega} \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega) = \mathcal{D}_{\omega F}'(\Omega).$$

Proof. If $u \in \mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega)$ and $\varphi \in \mathcal{D}_\omega(\Omega)$, then from (2.2.2) it follows that

$$\int |(\varphi u)^{\wedge}(\xi)|\, e^{-\lambda\omega(\xi)}\, d\xi < \infty,$$

with λ depending only on k. Thus

$$|\varphi u(\psi)| \leqslant C\, |||\psi|||_\lambda^{(\omega)} \quad (\forall \psi \in \mathcal{D}_\omega).$$

Let K be a compact subset of Ω. Choosing φ as a local unit for K we get

$$|u(\psi)| \leqslant C_K |||\psi|||_\lambda^{(\omega)} \quad (\forall \psi \in \mathcal{D}_\omega(K)). \tag{2.3.2}$$

Conversely, suppose that (2.3.2) holds and choose $\varphi \in \mathcal{D}_\omega(K)$. Then $\varphi u \in \mathcal{E}'_\omega$, and we have

$$(\varphi u)^\wedge(\xi) = (\varphi u)_x(e^{-i\langle x,\xi\rangle}) = u_x(\varphi(x)\, e^{-i\langle x,\xi\rangle}).$$

Thus
$$|(\varphi u)^\wedge(\xi)| \leqslant C_K \sup_{\tau \in R^n} |e^{\lambda\omega(\tau)} \hat{\varphi}(\tau+\xi)| \leqslant C_K |||\varphi|||_\lambda^{(\omega)} e^{\lambda\omega(\xi)},$$

and hence $\varphi u \in \mathcal{B}_{p,k}$, if k is a suitable negative power of $\exp \omega$. This completes the proof.

Chapter III. Existence and approximation of solutions of differential equations

3.0. *Introduction*

This chapter corresponds to Chap. III of [H]. For various distribution spaces $\mathcal{G}(\Omega)$ we consider the question of finding necessary and sufficient conditions on Ω to have $P(D)\,\mathcal{G}(\Omega) = \mathcal{G}(\Omega)$.

It turns out that the classical fundamental solutions will suffice to treat the case $\mathcal{G} = \mathcal{E}'_\omega$ (Section 3.1), and that P-convexity (cf. [H], Sect. 3.5) still is the relevant property if $\mathcal{G}$ is related to spaces $\mathcal{B}_{p,k}^{\mathrm{loc}}$ (Section 3.3). We also find that the results in [H], Sect. 3.4, on approximation of solutions of homogeneous equations, generalize in a natural way (Section 3.2). In the final section we prove that an analogue of strong P-convexity (cf. [H], Sect. 3.6, and [14]) is necessary and sufficient when $\mathcal{G} = \mathcal{D}'_\omega$. Our result, which is given in Theorem 3.4.12, in particular implies that convexity is always sufficient.

3.1. *The equation $P(D)\,u = f$ when $f \in \mathcal{E}'_\omega$*

We recall the definition of a fundamental solution:

Definition 3.1.1. *A distribution $E \in \mathcal{D}'(R^n)$ is called a fundamental solution for the differential operator $P(D)$ with constant coefficients if*

$$P(D)\,E = \delta,$$

where δ is the Dirac measure at 0.

If $\omega \in \mathcal{M}$ we could of course define a "fundamental ω-solution" in a similar way but with $E \in \mathcal{D}'_\omega(R^n)$. But we avoid this generality for the following reasons. First, just as in [H], p. 64, it follows that if for some p and some $k \in \mathcal{K}_\omega$ there exists a "fundamental ω-solution" E for $P(D)$ such that

$$E \in \mathcal{B}_{p,k}^{\mathrm{loc}}(R^n),$$

then $\mathcal{B}_{\infty,\tilde{P}}^{\mathrm{loc}} \subset \mathcal{B}_{p,k}^{\mathrm{loc}}$. Second, we know from [H], Th. 3.1.1, that to every differential operator $P(D)$ there exists a fundamental solution $E \in \mathcal{B}_{\infty,\tilde{P}}^{\mathrm{loc}}(R^n)$.

Let now E be a fundamental solution for $P(D)$ and let $\omega \in \mathcal{M}$. Then

$$P(D)(E * f) = f \quad (\forall f \in \mathcal{E}'_\omega), \tag{3.1.1}$$

$$E * (P(D)\,u) = u \quad (\forall u \in \mathcal{E}'_\omega). \tag{3.1.2}$$

Starting from (3.1.1) and (3.1.2) and using results from Sections 2.2 and 2.3, we can now take over the results from [H], Sect. 3.2, on spaces $\mathcal{B}_{p,k}$ and $\mathcal{B}_{p,k}^{\mathrm{loc}}$ and their relations to $P(D)$. We leave the details to the reader.

3.2. *Approximation of solutions of homogeneous differential equations*

This section generalizes Sect. 3.4 of [H]. Thus we deal with questions concerning on one hand the approximation of arbitrary solutions of the differential equation

$$P(D)u = 0 \tag{3.2.1}$$

by sums of exponential solutions, on the other hand the approximation of solutions in one open set by solutions in a larger open set. We start by recalling the definition of an exponential solution:

Definition 3.2.1. *A solution u of the differential equation* (3.2.1) *in R^n is called an exponential solution if it can be written in the form*

$$u(x) = f(x)\, e^{i\langle x, \zeta\rangle},$$

where $\zeta \in C^n$ and f is a polynomial.

Since every exponential solution is analytic, Corollary 1.5.15 gives

Proposition 3.2.2. *If u is an exponential solution, then $u \in \mathcal{E}_\omega$ for any $\omega \in \mathfrak{M}$.*

In the rest of section 3.2 we use the following set-up. Let Ω be an open subset of R^n. Let $\omega \in \mathfrak{M}$ be given and let I be an arbitrary index set. Let $k_\iota \in \mathcal{K}_\omega$ and p_ι be given for each $\iota \in \mathrm{I}$. *We suppose that* $1 \leqslant p_\iota < \infty$ $(\forall \iota \in \mathrm{I})$. We define

$$\mathcal{F}(\Omega) = \bigcap_{\iota \in \mathrm{I}} \mathcal{B}_{p_\iota, k_\iota}^{\mathrm{loc}}(\Omega)$$

with the topology given in Theorem 2.3.9. In particular, we may have $\mathcal{F}(\Omega) = \mathcal{E}_\omega(\Omega)$. Then all theorems, lemmas etc. of [H], Sect. 3.4, remain true if $\mathcal{E}'$ is everywhere replaced by $\mathcal{E}'_\omega$. The verification of this is left to the reader.

3.3. *The equation $P(D)u = f$ when f is in a local space $\subset \mathcal{D}'_{\omega,F}$*

In this section we will study the equation $P(D)u = f$ when f belongs to some space $\mathcal{B}_{p,k}^{\mathrm{loc}}(\Omega)$. We start by recalling the definition of P-convexity, which is the key concept in the corresponding Sect. 3.5 of [H].

Definition 3.3.1. *An open set Ω is called P-convex if to every compact set $K \subset \Omega$ there exists another compact set $K' \subset \Omega$ such that $\varphi \in C_0^\infty(\Omega)$ and* $\operatorname{supp} P(-D)\varphi \subset K$ *implies* $\operatorname{supp} \varphi \subset K'$.

The following procedure may now seem natural. In Definition 3.3.1 we could replace the condition "$\varphi \in C_0^\infty(\Omega)$" by "$\varphi \in \mathcal{D}_\omega(\Omega)$" and thus define an apparently weaker property of Ω, which might be called "(P, ω)-convexity". However, it is clear by regularization (Theorem 1.7.4) that this property does not depend on ω

and is thus identical with P-convexity and also with the corresponding property where the condition "$\varphi \in \mathcal{D}_\omega(\Omega)$" is replaced by "$\varphi \in \mathcal{E}'_\omega(\Omega)$". Having made this observation, we will now prove:

Theorem 3.3.2. *Let $\omega \in \mathfrak{M}$. Suppose that the equation*

$$P(D)u = f \tag{3.3.1}$$

has a solution $u \in \mathcal{D}'_\omega(\Omega)$ for every $f \in \mathcal{E}_\omega(\Omega)$. Then Ω is P-convex.

Proof. Let K be a fixed compact set $\subset \Omega$. Consider the bilinear form

$$B: \ (\varphi, f) \to \int \varphi f \, dx,$$

defined when f is in the Fréchet space $\mathcal{E}_\omega(\Omega)$ and $\varphi \in \Phi$, which is a metrizable space, defined as follows. Φ consists of all functions $\varphi \in \mathcal{D}_\omega(\Omega)$ with $\operatorname{supp} P(-D)\varphi \subset K$. The topology is defined by all semi-norms

$$\varphi \to \|P(-D)\,\varphi\|_\lambda^{(\omega)}.$$

B is continuous in f for fixed φ, since φ has compact support. If $f \in \mathcal{E}_\omega$ we have by hypothesis $P(D)u = f$ for some $u \in \mathcal{D}'_\omega(\Omega)$. Thus $\int \varphi f \, dx = u(P(-D)\varphi)$, which proves continuity in φ for fixed f. Thus B is continuous ([6], Chap. III, § 4, Prop. 2), which means that there exist $\psi \in \mathcal{D}_\omega(\Omega)$ and constants λ_1, λ_2 and C such that

$$\left| \int \varphi f \, dx \right| \leqslant C \, \|P(-D)\,\varphi\|_{\lambda_1}^{(\omega)} \, \|\psi f\|_{\lambda_2}^{(\omega)} \quad (\forall \varphi \in \Phi, \ \forall f \in \mathcal{E}_\omega(\Omega)).$$

In particular, $\operatorname{supp} \varphi \subset \operatorname{supp} \psi$ if $\varphi \in \Phi$, and taking $K' = \operatorname{supp} \psi$ in Definition 3.3.1 we have proved the theorem.

Conversely, we have

Theorem 3.3.3. *Let $\omega \in \mathfrak{M}$. Let Ω be P-convex and let*

$$f \in \bigcap_{j=1}^{\infty} \mathcal{B}^{\mathrm{loc}}_{p_j, k_j}(\Omega), \ \textit{where} \ k_j \in \mathcal{K}_\omega \ \textit{and} \ 1 \leqslant p_j < \infty.$$

Then equation (3.3.1) *has a solution*

$$u \in \bigcap_{j=1}^{\infty} \mathcal{B}^{\mathrm{loc}}_{p_j, \tilde{P} k_j}(\Omega).$$

The proof is the same as in [H], Th. 3.5.5. Using Theorems 2.3.9 and 2.3.11 we now get the following two results:

Corollary 3.3.4. *If Ω is P-convex, the equation* (3.3.1) *has a solution $u \in \mathcal{E}_\omega(\Omega)$ for each $f \in \mathcal{E}_\omega(\Omega)$.*

Corollary 3.3.5. *If Ω is P-convex, the equation* (3.3.1) *has a solution $u \in \mathcal{D}'_{\omega,F}(\Omega)$ for each $f \in \mathcal{D}'_{\omega,F}(\Omega)$.*

3.4. *The equation $P(D)u=f$ for general $f\in\mathcal{D}'_\omega$*

We start with the following definition (note that we consider Schwartz's space $\mathcal{E}'$):

Definition 3.4.1. *Let $\omega\in\mathfrak{M}$. An open set Ω is called strongly (P,ω)-convex if it is P-convex and to every compact set $K\subset\Omega$ there exists another compact set $K'\subset\Omega$ such that for all $\mu\in\mathcal{E}'(\Omega)$ we have*

$$\operatorname{sing}_{\boldsymbol{\omega}}\operatorname{supp} P(-D)\mu\subset K \;\Rightarrow\; \operatorname{sing}_{\boldsymbol{\omega}}\operatorname{supp}\mu\subset K'. \tag{3.4.1}$$

The following theorem is proved just as Th. 3.6.1 of [H], twice using Corollary 1.8.17.

Theorem 3.4.2. *If $\mu\in\mathcal{E}'(R^n)$, the convex hull of $\operatorname{sing}_\omega\operatorname{supp}\mu$ is identical with that of $\operatorname{sing}_\omega\operatorname{supp}P(-D)\mu$.*

Replacing ω by $\boldsymbol{\omega}$ we thus get

Corollary 3.4.3. *Every open convex set Ω is strongly (P,ω)-convex for every $\omega\in\mathfrak{M}$, and as K' we may take the convex hull of K.*

Our next theorem gives an equivalent definition of strong (P,ω)-convexity (cf. [H], p. 84, and [14]).

Theorem 3.4.4. *Let $\omega\in\mathfrak{M}$ and let Ω be a P-convex subset of R^n. In order for Ω to be strongly (P,ω)-convex it is necessary and sufficient that for each $\mu\in\mathcal{E}'(\Omega)$ the distances from $\complement\Omega$ to $\operatorname{sing}_{\boldsymbol{\omega}}\operatorname{supp}\mu$ and to $\operatorname{sing}_{\boldsymbol{\omega}}\operatorname{supp}P(-D)\mu$ are equal.*

Proof. The sufficiency is proved in the following way. Let $\Omega_\varepsilon=\{x\in\Omega;\, d(x,\complement\Omega)>\varepsilon\}$. If K is a compact subset of Ω, we have $K\subset\Omega_\varepsilon$ for some $\varepsilon>0$. Then if $\mu\in\mathcal{E}'(\Omega)$ and $\operatorname{sing}_{\boldsymbol{\omega}}\operatorname{supp}P(-D)\mu\subset K$ we have by hypothesis $\operatorname{sing}_{\boldsymbol{\omega}}\operatorname{supp}\mu\subset\Omega_\varepsilon$. On the other hand, considering μ as an element of $\mathcal{E}'(R^n)$, we have by Corollary 3.4.3 that $\operatorname{sing}_{\boldsymbol{\omega}}\operatorname{supp}\mu\subset H$, if H is the convex hull of K. Thus we may take $K'=\overline{\Omega}_\varepsilon\cap H$ in (3.4.1). The necessity is proved as in [H], Th. 3.5.2.

We also have the following two results, which can be proved as the corresponding ones in [H].

Theorem 3.4.5. *If I is any index set and Ω_ι is strongly (P,ω)-convex for every $\iota\in\mathrm{I}$, then the interior of $\bigcap_{\iota\in\mathrm{I}}\Omega_\iota$ is strongly (P,ω)-convex.*

Corollary 3.4.6. *To every open set Ω there is a smallest strongly (P,ω)-convex open set containing Ω.*

We will now prove the first main result of this section, using the proof of Th. 3.6.4 of [H].

Theorem 3.4.7. *Let $\omega\in\mathfrak{M}$ and suppose that Ω is strongly (P,ω)-convex. Then the equation $P(D)u=f$ has a solution $u\in\mathcal{D}'_\omega(\Omega)$ for each $f\in\mathcal{D}'_\omega(\Omega)$.*

Proof. It is sufficient to prove that for given $f\in\mathcal{D}'_\omega(\Omega)$ there exists a continuous semi-norm q on $\mathcal{D}_\omega(\Omega)$ such that

$$|f(\varphi)|\leqslant q(P(-D)\varphi)\quad(\forall\varphi\in\mathcal{D}_\omega(\Omega)). \tag{3.4.2}$$

For then it will follow from the Hahn–Banach theorem that the linear form $P(-D)\varphi \to f(\varphi)$, defined when $\varphi \in \mathcal{D}_\omega(\Omega)$, can be extended to a linear form $u \in \mathcal{D}'_\omega(\Omega)$, which is a solution.

To construct q we choose compact sets $K_1, K_2, \ldots$ such that $K_j \nearrow \nearrow \Omega$. We take $K_1 = K_2 = \emptyset$. Since Ω is strongly (P, ω)-convex, we may choose compact sets $K'_j \nearrow \nearrow \Omega$ (with $K'_1 = K'_2 = \emptyset$) such that

$$\varphi \in \mathcal{E}'(\Omega),\ \operatorname{supp} P(-D)\varphi \subset K_j \Rightarrow \operatorname{supp} \varphi \subset K'_j, \tag{3.4.3}$$

and
$$\varphi \in \mathcal{E}'(\Omega),\ \operatorname{sing}_\omega \operatorname{supp} P(-D)\varphi \subset K_j \Rightarrow \operatorname{sing}_\omega \operatorname{supp} \varphi \subset K'_j. \tag{3.4.4}$$

The construction of q will be made in an infinite number of steps, each using the following lemma.

Lemma 3.4.8. *Let q be a semi-norm on $\mathcal{D}_\omega(\Omega)$ which is stronger than the L_2 norm and assume that*

$$|f(\varphi)| \leqslant q(P(-D)\varphi) \quad \textit{if} \quad \varphi \in \mathcal{D}_\omega(K'_j). \tag{3.4.5}$$

For every $\varepsilon > 0$ we can then find another semi-norm q' on $\mathcal{D}_\omega(\Omega)$ such that $q' \geqslant q$,

$$q'(\psi) = (1+\varepsilon)\, q(\psi) \qquad (\forall \psi \in \mathcal{D}_\omega(K_{j-1})), \tag{3.4.6}$$

and
$$|f(\varphi)| \leqslant q'(P(-D)\varphi) \quad (\forall \varphi \in \mathcal{D}_\omega(K'_{j+1})).$$

Proof of Lemma 3.4.8. Let Φ be the completion of $\mathcal{D}_\omega(K'_{j+1})$ with respect to the metrizable locally convex topology defined by the semi-norms $\varphi \to q(P(-D)\varphi)$ and $\varphi \to \|\psi P(-D)\varphi\|_\lambda^{(\omega)}$ where λ is any real number and ψ is any element of $\mathcal{D}_\omega(\complement K_{j-1})$, with $\complement$ denoting complement relative to Ω. Since q is stronger than the L_2 norm it follows from Th. 3.2.5 of [H] that $\Phi \subset L_2 \cap \mathcal{E}'(K'_{j+1})$. If $\varphi \in \Phi$ we have $P(-D)\varphi \in \mathcal{E}_\omega(\complement K_{j-1})$ and hence by (3.4.4), $\varphi \in \mathcal{E}_\omega(\complement K'_{j-1})$. Since Φ is a Fréchet space, it follows from the closed graph theorem that the natural mapping of Φ into $\mathcal{E}_\omega(\complement K'_{j-1})$ is continuous.

Let us now consider the Fréchet space $\mathcal{E}_\omega(\complement K_{j-1})$ and choose a special sequence $\{p_\nu\}_1^\infty$ of semi-norms giving its topology. The semi-norm p_ν shall have the form $p_\nu(\varphi) = \|\psi_\nu \varphi\|_{\lambda_\nu}^{(\omega)}$ with $\lambda_\nu > 0$ and $\psi_\nu \in \mathcal{D}_\omega(\complement K_{j-1})$, and the sequence shall have the property that for every pair $(\varkappa, N)$ of natural numbers there exists M such that

$$p_\nu \geqslant N p_\varkappa \quad (\forall \nu \geqslant M). \tag{3.4.7}$$

Clearly, this is always possible. For each ν we now define a continuous semi-norm q'_ν on $\mathcal{D}_\omega(\Omega)$ by

$$q'_\nu = (1+\varepsilon)\, q + p_\nu \tag{3.4.8}$$

(where $p_\nu(\varphi) = \|\psi_\nu \varphi\|_{\lambda_\nu}^{(\omega)}$ for any $\varphi \in \mathcal{D}_\omega(\Omega)$).

Suppose now that the lemma is false. Then, since (3.4.6) is satisfied if we take $q' = q'_\nu$, there exists $\varphi_\nu \in \mathcal{D}_\omega(K'_{j+1})$ such that

$$|f(\varphi_\nu)| \geqslant 1 + \varepsilon \tag{3.4.9}$$

and
$$q'_\nu(P(-D)\varphi_\nu) < 1 + \varepsilon. \tag{3.4.10}$$

From (3.4.7), (3.4.8) and (3.4.10) we get

$$p_\varkappa(P(-D)\varphi_\nu) < \frac{1+\varepsilon}{N} \quad (\forall \nu \geq M).$$

This implies that $$P(-D)\varphi_\nu \to 0 \text{ in } \mathcal{E}_\omega(\complement K_{j-1}). \tag{3.4.11}$$

Combining (3.4.8) and (3.4.10) we also have

$$q(P(-D)\varphi_\nu) < 1. \tag{3.4.12}$$

Thus the sequence $\{\varphi_\nu\}$ is bounded in Φ, and hence by the continuity proved above it is bounded in $\mathcal{E}_\omega(\complement K'_{j-1})$. Hence by Theorem 2.3.5 the sequence is bounded in $\mathcal{B}^{\text{loc}}_{2,k_\mu}(\complement K'_{j-1})$ with $k_\mu = \exp(\mu\boldsymbol{\omega})$. Applying a generalized Th. 2.3.9 of [H] to two different μ we then see that the sequence is precompact in each $\mathcal{B}^{\text{loc}}_{2,k_\mu}(\complement K'_{j-1})$. Hence by a diagonal process it is precompact in $\mathcal{E}_\omega(\complement K'_{j-1})$ (Theorem 2.3.9). We want to prove that $\varphi_\nu \to 0$ in $\mathcal{E}_\omega(\complement K'_{j-1})$. Replace $\{\varphi_\nu\}$ by any subsequence converging in $\mathcal{E}_\omega(\complement K'_{j-1})$ and let the limit be φ. We claim that $\varphi = 0$ in $\complement K'_{j-1}$. By (3.4.3) it is enough to prove that $P(-D)\psi = 0$ in $\complement K_{j-1}$ for some $\psi \in \mathcal{E}'(\Omega)$ such that $\psi = \varphi$ in $\complement K'_{j-1}$. We will now construct such a ψ. Since q is stronger than the L_2 norm and $q(P(-D)\varphi_\nu) < 1$, the sequence $\{P(-D)\varphi_\nu\}$ is bounded in L_2. Thus by Th. 3.2.5 of [H], the sequence $\{\varphi_\nu\}$ is bounded in $L_2 = \mathcal{B}_{2,1}$. Hence by Th. 2.2.3 of [H], $\{\varphi_\nu\}$ is precompact in $\mathcal{B}_{2,k}$ if $k \in \mathcal{K}$ is such that $k(\xi) \to 0$ when $|\xi| \to \infty$. Choose such a k. Take a subsubsequence, this time converging in $\mathcal{B}_{2,k}$ and call its limit ψ. By (3.4.11) we have $P(-D)\psi = 0$ in $\complement K_{j-1}$. But we also have $\psi = 0$ in $\complement K'_{j+1}$ (by the definition of Φ) and $\psi = \varphi$ in $\complement K'_{j-1}$ (by Theorem 2.3.5). Thus we have found a suitable ψ. This proves that $\varphi_\nu \to 0$ in $\mathcal{E}_\omega(\complement K'_{j-1})$.

To complete the proof of the lemma we choose $\chi \in \mathcal{D}_\omega(K'_j)$, a local unit for K'_{j-1}. We get $\varphi'_\nu = (1-\chi)\varphi_\nu \to 0$ in $\mathcal{D}_\omega(\Omega)$ and thus $P(-D)\varphi'_\nu \to 0$ in $\mathcal{D}_\omega(\Omega)$ by Theorem 1.3.27. Taking $\varphi''_\nu = \chi\varphi_\nu$ we then get from (3.4.9) and (3.4.12) that for sufficiently large ν,

$$|f(\varphi''_\nu)| > 1 + 2\varepsilon/3 \quad \text{and} \quad q(P(-D)\varphi''_\nu) < 1 + \varepsilon/3.$$

Since $\operatorname{supp} \varphi''_\nu \subset K'_j$, this contradicts (3.4.5). The proof of the lemma is complete.

End of proof of Theorem 3.4.7. Choose $\varepsilon_j > 0$ such that $\sum_1^\infty \varepsilon_j < \infty$. Let q_1 be the L_2 norm. Using the lemma we successively construct semi-norms q_j in $\mathcal{D}_\omega(\Omega)$ such that

$$|f(\varphi)| \leq q_j(P(-D)\varphi) \quad \text{if} \quad \varphi \in \mathcal{D}_\omega(K'_j), \tag{3.4.13}$$

and $$q_{j+1}(\psi) = (1+\varepsilon_j)q_j(\psi) \quad \text{if} \quad \psi \in \mathcal{D}_\omega(K_{j-1}).$$

Then $q(\psi) = \lim q_j(\psi)$ exists, and q is a continuous semi-norm in $\mathcal{D}_\omega(\Omega)$, since

$$q(\psi) = q_j(\psi) \prod_j^\infty (1+\varepsilon_k) \quad \text{if} \quad \psi \in \mathcal{D}_\omega(K_{j-1}).$$

From (3.4.13) it follows that (3.4.2) holds. This completes the proof of Theorem 3.4.7.

From Theorem 3.4.7 and Corollary 3.4.3 we get:

Corollary 3.4.9. *Let $\omega \in \mathcal{M}$. If Ω is an open convex set in R^n, the equation $P(D)u = f$ has a solution $u \in \mathcal{D}'_\omega(\Omega)$ for every $f \in \mathcal{D}'_\omega(\Omega)$.*

We will now prove the counterpart of Th. 3.6.3 of [H] (necessity of strong P-convexity for the existence of solutions). The proof of that theorem depends on Lem. 3.6.1, which roughly states that if $v \in \mathcal{E}'$ and there is a fixed "degree of local regularity" shared by every derivative of v, then $v \in \mathcal{D}$. Instead of derivatives we will consider convolutions with distributions having their supports near the origin:

Lemma 3.4.10. *Let $\omega \in \mathcal{M}$ and let $r > 0$, $\lambda > 0$ and $a \in R^n$ be given. Let $\mu \in \mathcal{D}'_\omega(a + B_{5r})$ and suppose that*

$$u * \mu \in \mathcal{B}^{\mathrm{loc}}_{\infty, \exp(-\lambda\omega)}(a + B_{4r}) \quad (\forall u \in \mathcal{E}'_\omega(B_r)).$$

Then $\mu \in \mathcal{E}_\omega(a + B_r)$.

Proof. We may assume that $a = 0$. Let $\psi \in \mathcal{D}_{\omega^c}(B_{2r})$ be a local unit for $\bar{B}_r$ and let $\varphi \in \mathcal{D}_\omega(B_{4r})$. Then with u as in the hypothesis, we have $\varphi(u * \psi\mu) \in \mathcal{B}_{\infty, \exp(-\lambda\omega)}$.[1] Thus we may in the rest of the proof assume that $\mu \in \mathcal{E}'_\omega(B_{2r})$. Then if $\chi \in \mathcal{D}_\omega(B_{4r})$ is a local unit for $\bar{B}_{3r}$, the hypothesis implies that

$$u * \mu = \chi(u * \mu) \in \mathcal{B}_{\infty, \exp(-\lambda\omega)} \quad (\forall u \in \mathcal{E}'_\omega(B_r)).$$

Thus $|\hat{u}\hat{\mu}| \leq C_u \exp(\lambda\omega)$, and the lemma follows, if for each $l \geq 0$ we can find u such that $\inf |\hat{u} \exp(-l\omega)| > 0$. This can be done by choosing $v \in \mathcal{D}_\omega(B_r)$ with $\hat{v} \geq 0$ (Corollary 1.3.17) and defining $u \in \mathcal{E}'_\omega(B_r)$ by $\hat{u} = \hat{v} * \exp(l\omega)$.

Theorem 3.4.11. *Let $\omega \in \mathcal{M}$. If $P(D)u = f$ has a solution $u \in \mathcal{D}'_\omega(\Omega)$ for each $f \in \mathcal{D}'_\omega(\Omega)$, it follows that Ω is strongly (P, ω)-convex.*

Proof. Suppose that Ω is not strongly (P, ω)-convex. Let K be a compact subset of Ω and choose compact sets K_j $(j = 1, 2, \ldots)$ such that $K_j \nearrow \nearrow \Omega$. Using Theorem 3.3.2 we construct (as in the proof of Th. 3.6.3. of [H]) a sequence $\{x_j\}_1^\infty$ of points in Ω, a sequence $\{\mu_j\}_1^\infty$ of elements in $\mathcal{E}'(\Omega) \subset \mathcal{E}'_\omega(\Omega)$ and a decreasing sequence $\{\Omega_j\}_1^\infty$ of open balls $\Omega_j = B_{4r_j}$ (with center origin and radius $4r_j$) such that the compact sets $\bar{\Omega}_j + \operatorname{supp} \mu_j$ are contained in Ω and the following four relations hold:

$$\operatorname{sing}_\omega \operatorname{supp} P(-D)\mu_j \subset K \quad (\forall j),$$

$$x_j \in \operatorname{sing}_\omega \operatorname{supp} \mu_j \quad (\forall j), \tag{3.4.14}$$

$$x_j \notin K_j \quad (\forall j), \tag{3.4.15}$$

and

$$x_j \notin \bar{\Omega}_k + \operatorname{supp} \mu_k \quad (j > k). \tag{3.4.16}$$

By succesively shrinking the Ω_j for $j = 1, 2, \ldots$, we strengthen (3.4.15) and (3.4.16), respectively, as follows:

[1] *Note added in proof.* This is not evident. We may use functional analysis to give the hypothesis a quantitative form and then apply Fourier's inversion formula to ψ.

$$(x_j+\overline{\Omega}_j)\cap K_j=\emptyset \quad (\forall j), \tag{3.4.17}$$

and
$$(x_j+\overline{\Omega}_j)\cap(\overline{\Omega}_k+\operatorname{supp}\mu_k)=\emptyset \quad (j>k). \tag{3.4.18}$$

To simplify notation we will in the rest of the proof write $\|\cdot\|_\lambda$ for $\|\cdot\|_\lambda^{(\boldsymbol{\omega})}$ and $|||\cdot|||_\lambda$ for $\|\cdot\|_{\infty,k}$ with $k=\exp\lambda\boldsymbol{\omega}$. By Theorem 1.8.14 we may choose $s_k>0$ in such a way that

$$|||\mu_k|||_{-s_k}<\infty. \tag{3.4.19}$$

We will now choose the elements of two sequences $\{l_k\}_0^\infty$ and $\{\lambda_k\}_1^\infty$ of positive numbers and a sequence $\{u_k\}_1^\infty$ with $u_k\in\mathcal{E}'_{\boldsymbol{\omega}}(\Omega)$ in the following order: $l_0, \lambda_1, u_1, l_1, \lambda_2, u_2, \ldots,$ and in the following way: We start with $l_0=0$. We define $\lambda_k=s_k+l_{k-1}+1$. Since $x_k\in\operatorname{sing}_{\boldsymbol{\omega}}\operatorname{supp}\mu_k$ we may by Lemma 3.4.10 (with ω replaced by $\boldsymbol{\omega}$) choose $u_k\in\mathcal{E}'_{\boldsymbol{\omega}}(B_{r_k})$ such that

$$u_k*\mu_k\notin\mathcal{B}^{\mathrm{loc}}_{\infty,\exp(-\lambda_k\boldsymbol{\omega})}(x_k+\Omega_k). \tag{3.4.20}$$

Finally, by Theorem 1.8.14 we may choose l_k in such a way that

$$|||u_k|||_{-l_k}<\infty. \tag{3.4.21}$$

We now claim that $l_k\to+\infty$. In fact, $|||u_k*\mu_k|||_{-l_k-s_k}<\infty$, and thus by (3.4.20) we have $\lambda_k<l_k+s_k$, which means that $l_k>1+l_{k-1}$.

We now define $\hat{f}(\xi)=\sum_1^\infty\exp(-i\langle x_k,\xi\rangle)\,\hat{u}_k(-\xi)$, that is,

$$f(\psi)=\sum_1^\infty\check{u}_k(\tau_{-x_k}\psi)\quad(\forall\psi\in\mathcal{D}_{\boldsymbol{\omega}}(\Omega)).$$

The series converges in $\mathcal{D}'_\omega(\Omega)$ since by (3.4.17) only a finite number of the sets $x_k+\overline{\Omega}_k$ meet the compact set $\operatorname{supp}\psi$. Now suppose that $P(D)u=f$ for some $u\in\mathcal{D}'_\omega(\Omega)$. This means that

$$u(P(-D)\psi)=f(\psi)=\sum_1^\infty\check{u}_k(\tau_{-x_k}\psi)\quad(\forall\psi\in\mathcal{D}_{\boldsymbol{\omega}}(\Omega)). \tag{3.4.22}$$

If $\varphi\in\mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)$, we have $\mu_k*\varphi\in\mathcal{D}_{\boldsymbol{\omega}}(\Omega)$ by Theorem 1.7.3, and hence we may apply (3.4.22) to $\psi=\mu_k*\varphi$, which gives

$$u(P(-D)\,\mu_k*\varphi)=\sum_{j=1}^\infty\check{u}_j(\tau_{-x_j}(\mu_k*\varphi)). \tag{3.4.23}$$

Since $\operatorname{supp}(\mu_k*\varphi)\subset\operatorname{supp}\mu_k+\Omega_k$, it follows from (3.4.18) that all terms in (3.4.23) with $j>k$ must vanish, and we get as in [H]

$$\check{u}_k(\tau_{-x_k}(\mu_k*\varphi))=u(P(-D)\,\mu_k*\varphi)-\sum_{j=1}^{k-1}\check{u}_j(\tau_{-x_j}(\mu_k*\varphi)).$$

Since $\check{v}(\tau_{-x}\psi)=(v*\psi)(x)$, this may be written

$$(u_k*\mu_k*\varphi)(x_k)=u(P(-D)\,\mu_k*\varphi)-\sum_{j=1}^{k-1}(u_j*\mu_k*\varphi)(x_j)\quad(\forall\varphi\in\mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)). \tag{3.4.24}$$

We will now estimate the various terms of (3.4.24). We first note that using (3.4.19) and (3.4.21) we have

$$|u_j * \mu_k * \varphi| = |\mu_k * (u_j * \varphi)| \leqslant (2\pi)^{-n} |||\mu_k|||_{-s_k} ||u_j * \varphi||_{s_k}$$
$$\leqslant C |||\mu_k|||_{-s_k} ||\varphi||_{l_j + s_k} \quad (\forall \varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)).$$

Since $\{l_j\}$ is increasing, this gives by the choice of λ_k

$$|u_j * \mu_k * \varphi| \leqslant C_k ||\varphi||_{\lambda_k} \quad (j < k,\ \forall \varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)). \tag{3.4.25}$$

Let $\chi \in \mathcal{D}_{\boldsymbol{\omega}^c}(\Omega)$ be a local unit for K such that $\tilde{K} = \overline{\Omega}_1 + \operatorname{supp} \chi \subset \Omega$. Then if b is the constant of condition (γ), applied to $\boldsymbol{\omega}$, and m is the order of P, we have

$$|||\nu_k'|||_{-s_k - m/b} < \infty \quad \text{and} \quad \nu_k'' \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega),$$

where we have written

$$\nu_k' = \chi P(-D)\, \mu_k \quad \text{and} \quad \nu_k'' = (1-\chi)\, P(-D)\, \mu_k.$$

Since $\operatorname{supp} \nu_k'' * \varphi$ is contained in the compact set $\operatorname{supp} \mu_k + \overline{\Omega}_k \subset \Omega$, there are constants C and λ such that

$$|u(\nu_k'' * \varphi)| \leqslant C ||\nu_k'' * \varphi||_\lambda \quad (\forall \varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)),$$

and thus
$$|u(\nu_k'' * \varphi)| \leqslant C \int |\hat{\nu}_k''(\xi)\, \hat{\varphi}(\xi)|\, e^{\lambda \omega(\xi)}\, d\xi \leqslant C_{ks} ||\varphi||_s \tag{3.4.26}$$

for all s and all $\varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)$ (with $C_{ks} = C |||\nu_k''|||_{\lambda - s}$). To estimate $u(\nu_k' * \varphi)$ finally, we note that $\operatorname{supp}(\nu_k' * \varphi) \subset \tilde{K}$ when $\varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)$ and that we have for all σ

$$||\nu_k' * \varphi||_\sigma = \int e^{\sigma \omega(\xi)}\, |\hat{\nu}_k'(\xi)\, \hat{\varphi}(\xi)|\, d\xi \leqslant |||\nu_k'|||_{-s_k - m/b}\, ||\varphi||_{\sigma + s_k + m/b}.$$

If σ is so chosen that it can be used as λ in (1.6.1) with K replaced by $\tilde{K}$, we therefore obtain

$$|u(\nu_k' * \varphi)| \leqslant C ||\varphi||_{\sigma + s_k + m/b} \quad (\forall \varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)). \tag{3.4.27}$$

Summing up (3.4.24)–(3.4.27), we have proved

$$|(u_k * \mu_k * \varphi)(x_k)| \leqslant C ||\varphi||_{\lambda_k} \quad (\forall \varphi \in \mathcal{D}_{\boldsymbol{\omega}}(\Omega_k)), \tag{3.4.28}$$

if k is so large that $\sigma + m/b \leqslant l_{k-1} + 1$.

We will now prove that (3.4.28) implies

$$u_k * \mu_k \in \mathcal{B}^{\text{loc}}_{\infty, \exp(-\lambda_k \boldsymbol{\omega})}(x_k + \Omega_k). \tag{3.4.29}$$

We may assume that $x_k = 0$, and then we obtain if $\psi \in \mathcal{D}_{\omega^c}(\Omega_k)$,

$$|(\check{\psi}(u_k * \mu_k))(\check{\varphi})| = |(u_k * \mu_k)(\check{\psi}\check{\varphi})| = |(u_k * \mu_k * \psi\varphi)(0)| \leqslant C ||\psi\varphi||_{\lambda_k}$$
$$\leqslant C' ||\varphi||_{\lambda_k} \quad (\forall \varphi \in S_{\omega^c}).$$

Here we have used Theorem 2.2.7. Writing $\check{\psi}\cdot(u_k * \mu_k) = w$, we have thus proved that w is a continuous linear form on $\mathcal{B}_{1,k}$ with $k = \exp(\lambda_k \boldsymbol{\omega})$. Thus by a generalized Th. 2.2.9 of [H], $w \in \mathcal{B}_{\infty, \exp(-\lambda_k \boldsymbol{\omega})}$ for each $\psi \in \mathcal{D}_{\omega^c}(\Omega_k)$. This proves (3.4.29). But (3.4.29) contradicts (3.4.20), and the assumption that $P(D)u = f$ is thus disproved. The proof of the theorem is complete.

We note that we have in fact proved the necessity of a condition, apparently stronger than strong (P, ω)-convexity, as stated in the following theorem, which collects the main results of this section:

Theorem 3.4.12. *Let $\omega \in \mathfrak{M}$, and let Ω be an open set in R^n. Then the following three conditions on Ω and the differential operator $P(D)$ are equivalent:*

(i) $P(D)(\mathcal{D}'_\omega(\Omega)) = \mathcal{D}'_\omega(\Omega)$.

(ii) *Ω is strongly (P, ω)-convex (cf. Definition* 3.4.1).

(iii) *Ω is strongly (P, ω)-convex and* (3.4.1) *holds for all $\mu \in \mathcal{E}'_\omega(\Omega)$.*

Chapter IV. Interior regularity

4.0. *Introduction*

In this chapter we will study a concept called ω-hypoellipticity, for $\omega(\xi) = \log(1 + |\xi|)$ reducing to hypoellipticity. Friedman [9] has studied this concept with $\omega(\xi) = |\xi|^{1/\lambda}$. In particular, the proof of Lemma 4.1.3 is adapted from Chap. 11, Sect. 2, of [9]. Otherwise, the present chapter is closer related to Chap. IV of [H].

The classical condition of hypoellipticity is the property that every $u \in \mathcal{D}'$ with $P(D)u = 0$ is in fact in $\mathcal{E}$. The corresponding algebraic property of the polynomial P is that $\operatorname{Im} \zeta \to \infty$ if $\zeta \to \infty$ on the surface $P(\zeta) = 0$. As is well known, this condition is equivalent to the following one: For some $c > 0$ and each C there exists B such that $P(\zeta) = 0$ implies $|\eta| \geq C|\xi|^c - B$. Thus the (a priori) intermediate condition "For each A there exists B such that $P(\zeta) = 0$ implies that $|\eta| \geq A \log(1 + |\xi|) - B$" is also equivalent to hypoellipticity. Precisely this condition generalizes to our situation. Replacing $\log(1 + |\xi|)$ by $\omega(\xi)$ with $\omega \in \mathfrak{M}$, we get a necessary and sufficient condition that every $u \in \mathcal{D}'_\omega$ with $P(D)u = 0$ is in fact in $\mathcal{E}$ and that every $u \in \mathcal{D}'$ with $P(D)u = 0$ is in fact in $\mathcal{E}_\omega$ (Theorem 4.1.1). Thus a hypoelliptic equation may have "wild" solutions, provided they are sufficiently "wild".

At the end of the chapter we discuss relations between ellipticity and ω-hypoellipticity.

4.1. *ω-hypoelliptic operators*

We collect our main results in the following theorem.

Theorem 4.1.1. *Let ω_1 and $\omega_2 \in \mathfrak{M}$ and let $\omega = \omega_1 + \omega_2$. Let $P(D)$ be a differential operator with constant coefficients. Then the following four conditions are equivalent:*

(i) *For each $A > 0$ there exists B such that*

$$P(\zeta) = 0 \textit{ implies } |\eta| \geq A\omega(\xi) - B.$$

(ii) *P has a fundamental solution $E \in \mathcal{D}'(R^n)$ such that $E \in \mathcal{E}_\omega(R^n \cap \complement\{0\})$.*

(iii) *For any open set Ω in R^n, from $u \in \mathcal{D}'_{\omega_2}(\Omega)$ and $P(D)u \in \mathcal{E}_{\omega_1}(\Omega)$ it follows that* $u \in \mathcal{E}_{\omega_1}(\Omega)$.

(iv) *For some non-empty open set Ω in R^n, each solution $u \in \mathcal{D}'_{\omega_2}(\Omega)$ of $P(D)u=0$ is in fact in $\mathcal{E}_{\omega_1}(\Omega)$.*

Definition 4.1.2. *If $P(D)$ satisfies these conditions, we say that $P(D)$ is ω-hypoelliptic.*

Proof of Theorem 4.1.1. It is trivial that (iii) implies (iv). We will now prove first that (iv) implies (i), then that (ii) implies (iii) and finally that (i) implies (ii).

Let Ω be as in (iv). Let S be an open ball with $S \subset\subset \Omega$ and let H be the support function of $\bar{S}$. Let $\lambda > 0$ be fixed and consider the two Fréchet spaces $\mathcal{D}_{\omega_1}(\bar{S})$ and $\mathcal{J} = \{u \in \mathcal{B}^{\text{loc}}_{1,\exp(-\lambda\omega_2)}(\Omega);\ P(D)u = 0 \text{ in } \Omega\}$, where in $\mathcal{J}$ we take the topology induced by that of $\mathcal{B}^{\text{loc}}_{1,\exp(-\lambda\omega_2)}$. Choose ψ in $\mathcal{D}_{\omega^c}(S)$ such that $\hat{\psi}(0) \neq 0$. We claim that the mapping $u \to \psi u$ maps $\mathcal{J}$ into $\mathcal{D}_{\omega_1}(\bar{S})$ and that this mapping is closed.

First, by condition (iv), if $u \in \mathcal{J}$, then $u \in \mathcal{E}_{\omega_1}(\Omega)$ and hence $\psi u \in \mathcal{D}_{\omega_1}(S)$. Second, suppose that $u_\nu \to 0$ in $\mathcal{J}$ and $\psi u_\nu \to v$ in $\mathcal{D}_{\omega_1}(\bar{S})$. Considering in $\mathcal{J}$ the semi-norm given by $u \to \|\psi u\|_{1,\exp(-\lambda\omega_2)}$, we see that $\int_{B_1} |(\psi u_\nu)\hat{}\,(\xi)| \exp(-\lambda\omega_2(\xi)) \to 0$, which implies $\int_{B_1} |\hat{v}(\xi)|\, d\xi = 0$. Thus $v = 0$, since $\hat{v}$ is entire.

We now apply the closed graph theorem and Corollary 1.4.3. We conclude that there exist a constant C and a function $\varrho \in \mathcal{D}_\omega(\Omega)$ such that

$$\sup_{\zeta \in C^n} |(\psi u)\hat{}\,(\zeta)|\, e^{\lambda\omega_1(\xi) - H(\eta) - |\eta|} \leqslant C \int |(\varrho u)\hat{}\,(\xi)|\, e^{-\lambda\omega_2(\xi)}\, d\xi \quad (\forall u \in \mathcal{J}). \tag{4.1.1}$$

Let $\zeta_0 \in C^n$ be such that $P(\zeta_0) = 0$ and define u by $u(x) = e^{i\langle x, \zeta_0\rangle}$. Clearly, $u \in \mathcal{J}$, and we have $(\varrho u)\hat{}\,(\xi) = \hat{\varrho}(\xi - \zeta_0)$ and $(\psi u)\hat{}\,(\zeta) = \hat{\psi}(\zeta - \zeta_0)$. We now apply (4.1.1) to the present u and estimate the sup in the left-hand side by the value for $\zeta = \zeta_0$. We get

$$\begin{aligned} |\hat{\psi}(0)|\, e^{\lambda\omega_1(\xi_0) - H(\eta_0) - |\eta_0|} &\leqslant C \int |\hat{\varrho}(\xi - \zeta_0)|\, e^{-\lambda\omega_2(\xi)}\, d\xi \\ &\leqslant C e^{-\lambda\omega_2(\xi_0)} \int |\hat{\varrho}(\xi - \xi_0 - i\eta_0)|\, e^{\lambda\omega_2(\xi_0 - \xi)}\, d\xi. \end{aligned} \tag{4.1.2}$$

Since $\varrho \in \mathcal{D}_{\omega_2}(\Omega)$, the last integral can by Theorem 1.4.1 (with $\varepsilon = 1$) be estimated by $C_\lambda \exp(H_\varrho(\eta_0) + |\eta_0|)$, where H_ϱ is the support function of the convex hull of $\operatorname{supp} \varrho$. Thus, if $H(\eta) + H_\varrho(\eta) \leqslant K|\eta|$, we get from (4.1.2):

$$(K+2)|\eta_0| \geqslant \lambda(\omega_1(\xi_0) + \omega_2(\xi_0)) - \log (CC_\lambda / |\hat{\psi}(0)|).$$

Since λ is any positive number and ζ_0 is any element of C^n with $P(\zeta_0) = 0$ and since the last term does not depend on ζ_0, we have proved (i).

Next we prove the implication (ii) $\Rightarrow$ (iii). Let Ω and u be as in the hypothesis of (iii) and let $E \in \mathcal{D}' \cap \mathcal{E}_\omega(R^n \cap \complement\{0\})$ be the fundamental solution whose existence is guaranteed by (ii). Let U be an arbitrary (bounded) open set such that $U \subset\subset \Omega$. It is then enough to prove that $u \in \mathcal{E}_{\omega_1}(U)$. Let $\delta > 0$ be so small that $U + B_\delta \subset\subset \Omega$, and let $\varrho \in \mathcal{D}_\omega(\Omega)$ be a local unit for $\bar{U} + \bar{B}_\delta$ and $\alpha \in \mathcal{D}_\omega(B_\delta)$ a local unit for $\bar{B}_{\frac{1}{2}\delta}$. Since $\varrho u \in \mathcal{E}'_\omega$ we have $\varrho u = E * P(D)(\varrho u)$, and we may thus write $\varrho u = u_1 + u_2$ with

$$u_1 = P(D)(1-\alpha)E * (\varrho u)$$

and
$$u_2 = \alpha E * P(D)(\varrho u).$$

Since $(1-\alpha)E \in \mathcal{E}_\omega(R^n)$, we have $P(D)(1-\alpha)E \in \mathcal{E}_\omega(R^n)$ by Theorem 1.3.27, and thus (by Theorem 2.3.10) we have $u_1 \in \mathcal{E}_\omega(R^n)$. On the other hand, we have $\varrho u = u$ in $U + B_\delta$, and hence by hypothesis, $P(D)(\varrho u) \in \mathcal{E}_{\omega_1}(U+B_\delta)$. Since $\alpha E \in \mathcal{E}'_{\omega_1}(B_\delta)$, we get by Theorem 2.3.10 that $u_2 \in \mathcal{E}_{\omega_1}(U)$. Since $u_1 + u_2 = u$ in U, the result follows.

The final implication (i) $\Rightarrow$ (ii) we formulate as a separate lemma:

Lemma 4.1.3. *Let $\omega \in \mathfrak{M}$. Suppose that for each $A>0$ there exists B such that $P(\zeta)=0$ implies $|\eta| \geq A\omega(\xi) - B$. Then P has a fundamental solution*

$$E \in \mathcal{D}'(R^n) \cap \mathcal{E}_\omega(R^n \cap \complement\{0\}).$$

Proof. If necessary, we first make an orthogonal coordinate transformation to arrange that all pure powers in the principal part of P have non-zero coefficients. That is, for $i = 1, 2, \ldots, n$, the form of P is $a_i \zeta_i^m +$ lower order terms in ζ_i (with $a_i \neq 0$). The hypothesis of the lemma is not affected, since $|\eta|$ is invariant under any orthogonal transformation. Let $k = \min_i |a_i|$.

We now define E in the following classical way. If $\psi \in \mathcal{D}(R^n)$, we take

$$E(\psi) = (2\pi)^{-n} \int_T \frac{\hat{\psi}(-\zeta)}{P(\zeta)} d\zeta,$$

where the integration is over a "Hörmander ladder" T (see e.g. [9], p. 285). On T we have $|P(\zeta)| > k$. Outside some cube Q, the integration can by our hypothesis be chosen to be over R^n. From the classical construction it follows that E is a fundamental solution in $\mathcal{D}'(R^n)$. Thus it only remains to prove that $E \in \mathcal{E}_\omega(R^n \cap \complement\{0\})$. Let $\varphi \in \mathcal{D}_\omega(R^n \cap \complement\{0\})$. We have to prove that $\varphi E \in \mathcal{D}_\omega$, that is, that for any $\lambda > 0$ we have

$$\sup_{\tau \in R^n} |(\varphi E)\hat{\ }(\tau)|\, e^{\lambda\omega(\tau)} < \infty. \tag{4.1.3}$$

Since $\varphi E \in \mathcal{E}'$, we may use the Fourier–Laplace transform:

$$(\varphi E)\hat{\ }(\tau) = (\varphi E)_x (e^{-i\langle x, \tau\rangle}) = E_x(\varphi(x)\, e^{-i\langle x,\tau\rangle}).$$

Taking $\psi(x) = \varphi(x)\, e^{-i\langle x,\tau\rangle}$, we then get from the definition of E:

$$(2\pi)^n (\varphi E)\hat{\ }(\tau) = f(\tau) + g(\tau)$$

with
$$g(\tau) = \int_{\zeta \in T,\, \xi \in Q} \frac{\hat{\varphi}(-\zeta+\tau)}{P(\zeta)} d\zeta$$

and
$$f(\tau) = \int_{R^n \cap \complement Q} \frac{\hat{\varphi}(-\xi+\tau)}{P(\xi)} d\xi.$$

Here $Q = \{\xi; \max_i |\xi_i| \leq M\}$ is the above-mentioned cube. We will prove (4.1.3) by proving that for any choice of M, we have $\sup |g(\tau)|\, e^{\lambda\omega(\tau)} < \infty$ $(\forall \lambda)$, and that given λ it is possible to choose M so large that

$$\sup_{\tau \in R^n} |f(\tau)| e^{\lambda\omega(\tau)} < \infty. \tag{4.1.4}$$

To prove the first of these results it suffices to notice that the integration is over a compact set where $|P(\zeta)| > k$ and where we may obtain from Theorem 1.4.1 an estimate of the form

$$|\hat{\varphi}(-\zeta+\tau)| \leqslant C e^{-\lambda\omega(\tau)+\lambda\omega(\xi)}. \tag{4.1.5}$$

It remains to prove the second result.

Let us fix $\lambda > 0$. Choose A in such a way that (with δ to be determined below)

$$\int_{R^n} e^{(\lambda-\delta A)\,\omega(\xi)}\, d\xi < \infty \tag{4.1.6}$$

(cf. Proposition 1.3.26). Let B be the number whose existence is guaranteed by hypothesis. We now choose M so large that $A\omega(\xi) - B - 2 > 0$ outside Q. We claim that with this choice of M, (4.1.4) holds. If so, the lemma is proved.

We start by using a partition of unity to write φ as a sum of functions φ_j with supports in half-spaces, not containing the origin. In fact, φ is zero in a neighborhood of the origin, say when $\max_i |x_i| < 3\delta$. We can thus choose functions $\chi_1, \ldots, \chi_{2n} \in \mathcal{D}_\omega$ such that $\sum \chi_j = 1$ in $\operatorname{supp}\varphi$ and such that $\operatorname{supp}\chi_{2\nu} \subset \{x;\, x_\nu > 2\delta\}$ and $\operatorname{supp}\chi_{2\nu-1} \subset \{x;\, x_\nu < -2\delta\}$ ($\nu = 1, 2, \ldots, n$). Let $\varphi_j = \chi_j\varphi$. We consider

$$f_{2\nu}(\tau) = \int_{R^n \cap \complement Q} \frac{\hat{\varphi}_{2\nu}(-\xi+\tau)}{P(\xi)}\, d\xi$$

and claim that

$$\sup_{\tau \in R^n} |f_{2\nu}(\tau)| e^{\lambda\omega(\tau)} < \infty. \tag{4.1.7}$$

To simplify notation we consider only the case $\nu = 1$. Let us thus write $\zeta = (\zeta_1, \zeta')$ where $\zeta' = (\zeta_2, \ldots, \zeta_n)$ and similarly for ξ and η. Let us define $Q' = \{\xi' \in R^{n-1};\ \max_{j\geqslant 2} |\xi_j| \leqslant M\}$ and $F(\zeta) = \hat{\varphi}_2(-\zeta+\tau)/P(\zeta)$. Then we have $f_2(\tau) = f_{Q'}(\tau) + f_{\complement Q'}(\tau)$, where we have written

$$f_{Q'}(\tau) = \int_{Q'} d\xi' \int_{|\xi_1| \geqslant M} F(\xi)\, d\xi_1$$

and

$$f_{\complement Q'}(\tau) = \int_{\complement Q'} d\xi' \int_{-\infty}^{+\infty} F(\xi)\, d\xi_1.$$

We will now deform the integration contours of the inner integrals and use the fact that $\hat{\varphi}_2$ and P are analytic. If necessary we first apply Lemma 1.2.4 and Theorem 1.3.18 to arrange that ω is sufficiently smooth. By Lemma 1.2.4 we have for some $v < \frac{1}{2}\pi$,

$$A\left|\frac{\partial\omega}{\partial\xi_1}\right| \leqslant \operatorname{tg} v \quad \text{(a.e.)}. \tag{4.1.8}$$

We will always keep within the set where ζ' is real and $|\eta| = |\eta_1| \leqslant A\omega(\xi) - B - 1$. Factoring P as a polynomial in ζ_1 we can then prove that $|P(\zeta)| \geqslant k\,(\cos v)^m$, and thus F is analytic. We claim that for each fixed $\xi' \in \complement Q'$ we have

$$\int_{-\infty}^{+\infty} F(\xi)\,d\xi_1 = \int_\gamma F(\zeta)\,d\zeta_1, \tag{4.1.9}$$

where the curve γ in the ζ_1-plane is defined by

$$\eta_1 = A\omega(\xi) - B - 1. \tag{4.1.10}$$

Since $\varphi_2 \in \mathcal{D}_\omega(\{x; x_1 > 2\delta\})$, we get from Theorem 1.4.1 (with $\varepsilon = \delta$) that if ζ' is real and $\eta_1 > 0$, then

$$|\hat{\varphi}_2(-\zeta+\tau)| \leqslant C\,e^{-\lambda\omega(-\xi+\tau)-\delta\eta_1}. \tag{4.1.11}$$

Thus for real ζ' and positive η_1 we have

$$|F(\zeta)| \leqslant \frac{C\exp(\lambda\omega(-\tau))}{k(\cos v)^m}\,e^{-\lambda\omega(-\xi)-\delta\eta_1}.$$

This implies that $\int_0^{+\infty} |F(\zeta)|\,d\eta_1 \to 0$ when $|\xi_1| \to \infty$, and hence (4.1.9) follows. Thus we have

$$f_{\complement Q'}(\tau) = \int_{\complement Q'} d\xi' \int_\gamma F(\zeta)\,d\zeta_1. \tag{4.1.12}$$

From (4.1.8), (4.1.10) and (4.1.11) we get

$$\left|\int_\gamma F(\zeta)\,d\zeta_1\right| \leqslant \frac{C}{k(\cos v)^{m+1}}\,e^{-\lambda\omega(\tau)+\delta(B+1)} \int_{-\infty}^{+\infty} e^{(\lambda-\delta A)\,\omega(\xi)}\,d\xi_1.$$

Thus by (4.1.12),

$$|f_{\complement Q'}(\tau)| \leqslant C'\,e^{-\lambda\omega(\tau)} \int_{R^n} e^{(\lambda-\delta A)\,\omega(\xi)}\,d\xi,$$

and from (4.1.6) it follows that (4.1.7) holds with $f_{\complement Q'}$ instead of f_2.

We will now consider $f_{Q'}$. We lift the integration path in the same way, this time only to the part of γ on which $|\xi_1| \geqslant M$ and use the same estimates as above. We must also estimate the integrals (where $\eta_\gamma = A\omega(\pm M, \xi') - B - 1$)

$$\int_{Q'} d\xi' \int_0^{\eta_\gamma} F(\pm M + i\eta_1, \xi')\,d\eta_1.$$

Since the integration is over a subset of $Q' \times I$, where I is a fixed interval on the η_1-axis, it suffices to use an estimate of the form (4.1.5) for $\hat{\varphi}_2$. We have thus proved (4.1.7). To prove the corresponding inequality for $f_{2\nu-1}$, we only have to choose as γ the curve $\eta_\nu = -A\omega(\xi) + B + 1$, use the fact that $\varphi_{2\nu-1} \in \mathcal{D}_\omega(\{x; x_\nu < -2\delta\})$ and proceed as above. We have thus proved (4.1.4). This completes the proof of the lemma and of Theorem 4.1.1.

Corollary 4.1.4. *If $P(D)$ is elliptic, then $P(D)$ is ω-hypoelliptic for each $\omega \in \mathfrak{M}$. In particular, if $P(D)$ is elliptic and $u \in \mathcal{D}'_\omega(\Omega)$ for some $\omega \in \mathfrak{M}$ and $P(D)u = 0$ in Ω, then u is analytic in Ω.*

Proof. From the ellipticity it follows that there exist constants A and B such that

$$P(\zeta)=0 \quad \text{implies} \quad |\eta| \geqslant A|\xi| - B$$

(cf. [H], Cor. 4.4.1, or [9], Chap. 11, Th. 9). Then by Corollary 1.2.8, condition (i) of Theorem 4.1.1 is fulfilled for each $\omega \in \mathcal{M}$. This proves the first result. To get the last result, we apply condition (iv) of Theorem 4.1.1 to conclude that $u \in \mathcal{E}_\omega(\Omega)$ which is more than enough to prove that u is analytic, using ellipticity in the classical way.

Conversely, we will prove that elliptic operators are the only ones which are ω-hypoelliptic for each $\omega \in \mathcal{M}$. A related result is given in [7]. In our case, the result is true even in the following strong form, where we may take e.g. $\omega(\xi) \approx |\xi|/(\log|\xi|)^2$.

Theorem 4.1.5. *Let $\omega \in \mathcal{M}$ be given and suppose that for every $\gamma > 1$ we have $|\xi|^{1/\gamma} \prec \omega$. Then $P(D)$ is ω-hypoelliptic if and only if $P(D)$ is elliptic.*

Proof. We only have to prove that if $P(D)$ is ω-hypoelliptic, then $P(D)$ is elliptic. Let $P(D)u=0$ and $u \in \mathcal{D}'$. Then by hypothesis and Example 1.5.7, u is in the Gevrey class $C^{\{k^\gamma\}}$ for every $\gamma > 1$. Thus by Th. 4.4.3 of [H], for each $y \in R^n$ and each $\gamma > 1$ there exist constants $\sigma \geqslant 1$ and $C > 0$ and $c > 0$ such that

$$k^\gamma \geqslant c\, k^\sigma \quad (k=1,2,\ldots), \tag{4.1.13}$$

and

$$|\langle y, \zeta\rangle| \leqslant C(1+|\eta|)^\sigma \ \text{ if } \ P(\zeta)=0. \tag{4.1.14}$$

Then by (4.1.13), we have $1 \leqslant \sigma \leqslant \gamma$. Thus, using (4.1.14) and Def. 4.4.1 of [H], we have $\varrho(y)=1$ for all $y \in R^n$, which by Th. 4.4.6 of [H] gives the desired result.

Chapter V. Differential equations which have no solutions

5.0. *Introduction*

Let us consider the famous example of H. Lewy, namely the equation

$$-i D_1 u + D_2 u - 2(x_1 + i x_2) D_3 u = f.$$

Here f is a certain function in $C^\infty(R^3)$, such that for no open non-void Ω does there exist a solution $u \in \mathcal{D}'(\Omega)$. It is now natural to ask if for any $\omega \in \mathcal{M}$ we may choose $f \in \mathcal{E}_\omega$ in such a way that we do not even have a solution $u \in \mathcal{D}'_\omega(\Omega)$. We will prove that the answer is affirmative. In fact we will consider the necessary condition given by Hörmander ([12], [13] and [H], Chap. VI) for the local existence of a solution $u \in \mathcal{D}'$ of an equation $P(x, D)u=f$ for each $f \in \mathcal{E}$. We will prove that if $P(x, D)$ is of first order and has analytic coefficients, then the same condition is necessary for the local existence of a solution $u \in \mathcal{D}'_\omega$ for each $f \in \mathcal{E}_\omega$.

5.1. *Conditions for non-existence*

In an open set $\Omega \subset R^n$ we consider a differential operator

$$P(x,D) = \sum_{|\alpha| \leqslant m} a^\alpha(x) D^\alpha$$

of order m with coefficients in $\mathcal{E}_\omega(\Omega)$. We collect the notation of [H], Chap. VI, as follows:

Definition 5.1.1. *$C_{2m-1}(x, D)$ is the sum of the terms of order exactly $2m-1$ in the commutator $\bar{P}(x, D)P(x, D)-P(x, D)\bar{P}(x, D)$, where $\bar{P}(x, D)=\sum_{|\alpha|\leqslant m}\bar{a}^\alpha(x)D^\alpha$.*

We will now state a weak form of the main result, partially generalizing Th. 6.1.1 of [H]. We denote by $P_m(x, D)$ the principal part $\sum_{|\alpha|=m}a^\alpha(x)D^\alpha$.

Theorem 5.1.2. *Let $P(x, D)$ be a linear first order partial differential operator with analytic coefficients in Ω. Let $\omega \in \mathcal{M}$. Suppose that the equation*

$$P(x, D)u = f \tag{5.1.1}$$

has a solution $u\in\mathcal{D}'_\omega(\Omega)$ for each $f\in\mathcal{D}_\omega(\Omega)$. Then we have

$$C_1(x, \xi)=0 \quad \text{if} \quad P_1(x, \xi)=0, \quad \text{where } x\in\Omega \text{ and } \xi\in R^n.$$

Before giving the proof we will also state Theorem 5.1.4, which is a strong form of the main result, partially generalizing Th. 6.1.2 of [H]. The proof of the strong result, assuming the weak one, proceeds as in [H] with obvious changes, and we will not repeat it.

Definition 5.1.3. *Let $\omega\in\mathcal{M}_c$. We denote by $S_\omega(\Omega)$ the Fréchet space which is the closure of $\mathcal{D}_\omega(\Omega)$ in S_ω.*

Theorem 5.1.4. *Suppose that the coefficients of the first order operator $P(x, D)$ are analytic in Ω. Suppose that N is dense in Ω, where N is defined as the set of points x in Ω for which there exists $\xi\in R^n$ with*

$$C_1(x, \xi)\neq 0 \quad \text{but} \quad P_1(x, \xi)=0.$$

Let $\omega\in\mathcal{M}_c$. Then there exist functions $f\in S_\omega(\Omega)$ such that the equation (5.1.1) *does not have any solution $u\in\mathcal{D}'_\omega(\Omega_1)$ for any open non-void set $\Omega_1\subset\Omega$. The set of such functions f is of the second category.*

We will prepare for the proof of Theorem 5.1.2 by deducing an inequality from the hypothesis. We prove the following lemma (cf. Lem. 6.1.2 of [H]), where ${}^tP$ is defined by the identity $\int vPu\,dx=\int({}^tPv)u\,dx$ when v or u has compact support.

Lemma 5.1.5. *Let $\omega\in\mathcal{M}$. Suppose that $P(x, D)$ is a linear partial differential operator (of any order) with coefficients in $\mathcal{E}_\omega(\Omega)$. Suppose that the equation*

$$P(x, D)u=f$$

has a solution $u\in\mathcal{D}'_\omega(\Omega)$ for each $f\in\mathcal{D}_\omega(\Omega)$. Let Ω_1 be an open set $\subset\subset\Omega$. Then there exist constants C and λ such that

$$\left|\int fv\,dx\right|\leqslant C|||f|||_\lambda\,|||{}^tPv|||_\lambda \quad (\forall f, v\in\mathcal{D}_\omega(\Omega_1)). \tag{5.1.2}$$

Proof. $\int fv dx$ is a bilinear form defined for $f\in\mathcal{D}_\omega(\bar{\Omega}_1)$ which is a Fréchet space and $v\in\mathcal{N}$ which is a metric space defined as follows. $\mathcal{N}$ consists of the same elements

v as $\mathcal{D}_\omega(\Omega_1)$ but is equipped with the semi-norms $v \to \|{}^tPv\|_\lambda$ (corresponding to all positive numbers λ). The bilinear form is obviously continuous in f for a fixed v (by Parseval's formula). On the other hand, when f is fixed, we can by hypothesis choose $u \in \mathcal{D}'_\omega(\Omega)$ such that $P(x, D)u = f$. Hence

$$\int fv\,dx = (Pu)(v) = u({}^tPv),$$

which proves the continuity in v for a fixed f. Thus by [6], Chap. III, § 4, the bilinear form is continuous, which proves the lemma.

Proof of Theorem 5.1.2. To save ξ for use as the variable on the Fourier transform side, we write θ instead of ξ in the hypothesis of Theorem 5.1.2. Making the same reductions as in [H] we thus have to prove that if for some $\theta \in R^n$ we have

$$P_1(0, \theta) = 0 \quad \text{and} \quad C_1(0, \theta) < 0, \tag{5.1.3}$$

then (5.1.2) does not hold for any choice of λ and C. We will use the following lemma:

Lemma 5.1.6. *Let $P(x, D)$ be a linear partial differential operator of order* 1 *with analytic coefficients in a neighborhood Ω of the origin in R^n such that* (5.1.3) *holds. Then in some neighborhood O of the origin in C^n there is an analytic function w such that*

$$P_1(x, \operatorname{grad} w) = 0 \quad (x \in \Omega \cap O), \tag{5.1.4}$$

$$w(z) = \langle z, \theta\rangle + \tfrac{1}{2}\sum \alpha_{jk} z_j z_k + O(z^3) \quad (z \to 0), \tag{5.1.5}$$

where the matrix α_{jk} is symmetric and has a positive definite imaginary part.

The proof of Lemma 5.1.6 is given in [H], Lem. 6.1.3. (Since the coefficients of P_1 are analytic, no modifications of the coefficients are needed. Also the function W of [H] can be used as it is, but of course at the cost of having it defined only in O.)

Proof of Theorem 5.1.2, continued. By the Cauchy–Kovalevsky theorem there is in some neighborhood Ω_2 of the origin in R^n an analytic solution ψ of the equation

$${}^tP(x, D)\psi = 0,$$

and we may assume that $\psi(0) = 1$.

Let $\Omega_3 \subset\subset \Omega_1 \cap \Omega_2 \cap O$ with the Ω_1 of Lemma 5.1.5 and the O of Lemma 5.1.6. Let $0 \in \Omega_4 \subset\subset \Omega_3$ and let $\chi \in \mathcal{D}_{\omega^c}(\Omega_3)$ be a local unit for $\bar{\Omega}_4$. Define $\varphi = \chi\psi$ (defined as zero outside Ω_3). By Corollary 1.5.15, $\varphi \in \mathcal{D}_\omega(\Omega_3)$. We also have $\varphi(0) = 1$ and

$${}^tP(x, D)\varphi = 0 \text{ in } \Omega_4. \tag{5.1.6}$$

Let us now define for positive τ (which we shall let $\to +\infty$) and positive K

$$F_\tau(x) = (K\tau)^n e^{-i\tau^2\langle x, \theta\rangle} F(K\tau x) \text{ with } F(x) = e^{-\frac{1}{2}|x|^2},$$

$$f_\tau(x) = \varphi(x) \cdot F_\tau(x)$$

and

$$v_\tau(x) = \varphi(x) \cdot e^{i\tau^2 w(x)}$$

with the w of Lemma 5.1.6.

We have by Corollary 1.5.15 that f_τ and $v_\tau \in \mathcal{D}_\omega(\Omega_3)$. A change of variable gives

$$\int f_\tau(x)\, v_\tau(x)\, dx = \int F(x)\, (\varphi(x/K\tau)^2 \cdot e^{-i\tau\langle x,\theta\rangle/K + i\tau^2 w(x/K\tau)}\, dx.$$

If Ω_3 is chosen sufficiently small, $\operatorname{Re}(i\tau^2 w(x/K\tau) < 0$ when $x/K\tau \in \Omega_3$. Thus we may apply the theorem of dominated convergence to get from (5.1.5) that

$$\int f_\tau(x)\, v_\tau(x)\, dx \to (\varphi(0))^2 \int \exp\left(-\tfrac{1}{2}|x|^2 + \tfrac{1}{2} i K^{-2} \sum \alpha_{jk} x_j x_k\right) dx$$

when $\tau \to +\infty$.

We claim that the right-hand side is different from zero, at least if K is sufficiently large. In fact, by dominated convergence it tends to $\int \exp(-\frac{1}{2}|x|^2)\, dx$ when $K \to \infty$. Thus to prove that (5.1.2) is not valid, it is enough to fix a suitable K and prove that

$$|||f_\tau|||_\lambda\, |||{}^tPv_\tau|||_\lambda \to 0 \quad \text{when} \quad \tau \to +\infty. \tag{5.1.7}$$

We will first estimate $|||{}^tPv_\tau|||_\lambda$. By (5.1.4) (with $m=1$) we have

$${}^tPv_\tau = e^{i\tau^2 w} \cdot {}^tP\varphi, \tag{5.1.8}$$

and by (5.1.6) we have thus

$${}^tPv_\tau = 0 \text{ outside } \Omega_5, \tag{5.1.9}$$

where we have written $\Omega_5 = \Omega_3 \cap \complement\overline{\Omega}_4$. We now claim that for some $\delta > 0$ it is possible to choose Ω_3 and Ω_4 in such a way that $\operatorname{Re}(iw(x)) \leqslant -3\delta$ for $x \in \Omega_5$. In fact, by Lemma 5.1.6, all we have to do is to choose Ω_3 so small that the remainder term in (5.1.5) does not destroy the effect of the positivity of $\operatorname{Im} \alpha_{jk}$. Thus by continuity we can find a complex neighborhood O_1 of Ω_5 such that

$$|e^{i\tau^2 w(z)}| \leqslant e^{-2\delta\tau^2} \text{ for } z \in O_1. \tag{5.1.10}$$

Combining (5.1.8), (5.1.9) and (5.1.10) and applying Theorem 1.5.16, we obtain

$$|||{}^tPv_\tau|||_\lambda \leqslant C\, e^{-2\delta\tau^2},$$

and thus to prove (5.1.7) it is enough to prove that

$$|||f_\tau|||_\lambda \leqslant C\, e^{\delta\tau^2}. \tag{5.1.11}$$

We may assume that $\omega \in \mathfrak{M}_c$. Since φ and $F_\tau \in S_\omega$, it follows from the proof of Proposition 1.8.3 that to prove (5.1.11) it suffices to prove the $\pi_{0,\lambda}(F_\tau) \leqslant C\, e^{\delta\tau^2}$. Evidently, $\hat{F}_\tau(\xi) = \hat{F}((\xi + \tau^2\theta)/K\tau)$, and thus

$$\pi_{0,\lambda}(F_\tau) = \sup_\xi |\hat{F}(\xi)|\, e^{\lambda\omega(K\tau\xi - \tau^2\theta)} \leqslant (2\pi)^{\frac{1}{2}n}\, e^{\lambda\omega(-\tau^2\theta)} \sup_\xi e^{-\frac{1}{2}|\xi|^2 + \lambda\omega(K\tau\xi)}.$$

Since $\omega(-\tau^2\theta)/\tau^2 \to 0$ when $\tau \to \infty$, it is enough to prove that for some constant C' we have

$$\sup_\xi (-|\xi|^2 + 2\lambda\omega(K\tau\xi)) \leqslant \delta\tau^2 + C'. \tag{5.1.12}$$

For each $\varepsilon > 0$ there is a number X_ε such that $\lambda\omega(x) \leqslant \varepsilon|x|$ when $|x| \geqslant X_\varepsilon$. Thus if $K\tau|\xi| \geqslant X_\varepsilon$, we get

$$-|\xi|^2 + 2\lambda\omega(K\tau\xi) \leqslant -(|\xi| - \varepsilon K\tau)^2 + \varepsilon^2 K^2\tau^2 \leqslant \varepsilon^2 K^2 \tau^2.$$

Taking $\varepsilon=\delta^{\frac{1}{2}}/K$, we have then proved (5.1.12) with

$$C'=2\lambda \sup_{|x|<X_\varepsilon} \omega(x).$$

This completes the proof of Theorem 5.1.2.

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