1.65 06 1 Communicated 14 April 1965 by OTTO FROSTMAN

Some orthogonal matrices and related orthogonal functions systems

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In an earlier paper [3], one of the present authors showed that given any n mutually orthogonal unit vectors in l^2, there exists a uniquely determined infinite orthogonal matrix (i.e. unitary transformation of l^2 onto itself) $\|a_{ij}\|$ having the given vectors as its first n rows, and satisfying $a_{ij}=0$ for $j>i>n$. This matrix was used to show that every subspace of $L^2(\Gamma)$ (where Γ is the circle group) having finite deficiency has a basis consisting of trigonometric polynomials. In [3] also several questions were raised concerning the behavior of the Fourier expansion of a smooth function, when the expansion is with respect to a complete orthonormal system of smooth functions and it was pointed out that certain other types of orthogonal matrices, if they could be constructed, might be relevant to these questions. In the present note these questions are answered, the essence of the results being that no amount of smoothness (not even the requirement that all functions involved be uniformly bounded trigonometric polynomials) can guarantee either smallness of the Fourier coefficients beyond what is implied by Bessel's inequality, nor convergence of the series at every point. We also show that the functions of a uniformly bounded complete orthonormal system of smooth functions may have a common zero; in such a system the Fourier series of "most" functions converges to the wrong value at a point, and *a fortiori* cannot converge uniformly. Questions of almost everywhere convergence in the context of the present investigation we have not, however, been able to settle.

Our main tool is the construction of an orthogonal matrix with n prescribed (orthonormal) *columns*, and with $a_{ij}=0$ for $j>i+n$. Actually we only need these matrices for $n=1$, but as the matrix theorem has perhaps independent interest, we prove it for arbitrary n, and complex entries.

1. A class of unitary matrices

Theorem 1. *Suppose* $\|a_{ij}\|$ $(1\leqslant i<\infty,\ 1\leqslant j\leqslant n)$ *is a complex matrix whose columns are mutually orthogonal unit vectors. Then there exists a unique unitary matrix* $A=\|a_{ij}\|$ $(1\leqslant i<\infty,\ 1\leqslant j<\infty)$ *having the given matrix as its first n columns, and satisfying the additional conditions*

(i) $a_{ij}=0$ *for* $j>j(i)=i+\text{rank } J_i$, $J_k=\sum_{i=k+1}^{\infty} a_i^* a_i$.

(ii) *If* $\text{rank } J_{i-1}=\text{rank } J_i$, *then* $a_{i,j(i)}$ *is (strictly) negative.*

Here a_i *denotes the n-dimensional i-th row*

$$a_i=(a_{i1}, \ldots, a_{in}).$$

Proof. The heuristic background for the following construction of the matrix A is found in [2]. Let $j>n$ and denote by $i(j)$ the smallest solution i of the equation $j=i+\text{rank } J_i$. The number $i(j)$ then satisfies

$$\text{rank } J_{i(j)-1}=\text{rank } J_{i(j)}.$$

Put $i(j)=k$. The above condition then means that if d is an n-dimensional column vector,

$$J_k d=0 \quad \text{implies} \quad (J_k+a_k^* a_k)\, d=0. \tag{1}$$

Condition (1), however, implies by the non-negativity of the matrices involved, that

$$a_k d=0 \quad \text{if} \quad J_k d=0.$$

But this is just the condition for solvability of the systems of equations

$$J_k d_k=a_k^*.$$

Fix a set of solutions d_k of these systems and put

$$\lambda_k d_k=b_k, \quad \text{i.e.,} \quad J_k b_k=\lambda_k a_k^*,$$

where the positive numbers λ_k are determined by

$$\lambda_k^2=(1+a_k d_k)^{-1}=(1+d_k^* J_k d_k)^{-1}, \quad k=i(j), \quad j>n.$$

Evidently, $0<\lambda_k<1$ for all $k=i(j)$.

We now claim that the following matrix (which obviously satisfies conditions (i) and (ii) of Theorem 1 is unitary.

$$A=\begin{bmatrix} a_1 & & & 0 & \\ a_2 & & & 0 & \\ \vdots & & & 0 & \\ a_r & a_r b_{i(n+1)} \ldots & a_r b_{i(j-1)} & -\lambda_r & 0 \ldots \end{bmatrix}, \quad r=i(j). \tag{2}$$

Let us first verify that the jth column is orthogonal to the first n columns; we have

$$-\lambda_r a_r^* + \sum_{i=r+1}^{\infty} a_i^* a_i b_r = -\lambda_r a_r^* + J_r b_r = 0.$$

For the orthogonality of the columns j and h, with $h > j > n$, $r = i(j)$, $s = i(h)$, we have

$$\begin{aligned} & -\lambda_s (a_s b_r)^* + \sum_{i=1}^{\infty} (a_{s+i} b_r)^* (a_{s+i} b_s) \\ &= -\lambda_s b_r^* a_s^* + \sum_{i=1}^{\infty} b_r^* a_{s+i}^* a_{s+i} b_s \\ &= -b_r (\lambda_s a_s^* - J_s b_s) = 0. \end{aligned}$$

In the same way, we see that the jth column is a unit vector:

$$\lambda_r^2 + \sum_{i=0}^{\infty} (a_{r+i} b_r)^* (a_{r+i} b_r) = \lambda_r^2 + a_r \lambda_r b_r = \lambda_r^2 (1 + a_r d_r) = 1.$$

Finally, we have to show that the system of column vectors in the matrix A is complete. Assume, then, that

$$e = \{e_1, \ldots, e_r, \ldots\}$$

is orthogonal to all columns in A, and that e_r is actually the first non-vanishing entry. Then

$$a_r^* = -\frac{1}{e_r} \sum_{i=1}^{\infty} e_{r+i} a_{r+i}^* \tag{3}$$

by the orthogonality with the first n columns in A. If d is an n-dimensional vector in the null space of J_r, we have by the positivity of the matrices involved

$$a_s d = 0 \quad \text{for} \quad s > r.$$

But, by (3), this shows that $a_r d = 0$ so that, in fact, $J_r d = 0$ implies $J_{r-1} d = 0$ which in turn means that

$$\operatorname{rank} J_{r-1} = \operatorname{rank} J_r \quad \text{or} \quad r = i(j) \quad \text{for some} \quad j > n.$$

Now we use the orthogonality of the vector e with the jth column in A:

$$\begin{aligned} 0 &= -\lambda_r e_r + \sum_{i=1}^{\infty} b_r^* a_{r+i}^* e_{r+i} \\ &= -e_r (\lambda_r + b_r^* a_r^*) = -e_r \lambda_r (1 + d_k^* J_k d_k) = -e_r \lambda_r^{-1}, \end{aligned}$$

This contradiction proves that A is a unitary matrix. To see that A is unique, suppose that we have constructed another matrix A' which satisfies the conditions of Theorem 1. Assume that this new matrix coincides with A (as given by equation (2)) in all columns of index less than j for some $j>n$. Then the elements in the jth column of A' with row index less than $r=i(j)$ must vanish, since the squares of the absolute values of the elements in each of these rows sum to one already over the first $j-1$ columns. The element in position (r,j) of A' must be $-\lambda_r$ since the subsequent elements in the rth row vanish by hypothesis. Finally, the remaining elements in the jth column of A' are determined by the orthogonality of the corresponding rows with the rth row, and hence coincide with their counterparts in A. Thus $A'=A$, as asserted.

Remark. In the finite-dimensional case this proves the existence and uniqueness (up to postmultiplication by a diagonal unitary matrix) of a unitary completion of a given orthonormal set of vectors with (in view of the results of [2]) the largest possible domain of zeros in the upper right corner, if one measures the domain by the number of those index pairs (i,j) such that $a_{rs}=0$ for $r\leqslant i$, $s\geqslant j$.

2. Some orthogonal function systems

In the present section we shall examine the case $n=1$ in more detail; suppose now that a_i are real numbers with $a_i\geqslant 0$, $a_i>0$ for infinitely many i and $\sum_{i=1}^{\infty} a_i^2=1$. We have in this case

$$\lambda_k^2=\frac{c_k^2}{c_{k-1}^2},\quad k\geqslant 1, \tag{4}$$

where we have introduced the notation

$$c_k^2=\sum_{i=k+1}^{\infty} a_i^2,\quad k\geqslant 0. \tag{5}$$

We define c_k to be the positive number defined by (5). Note that $\{c_k\}$ is non-increasing and tends to zero. We have from (2), $b_k=a_k/(c_{k-1}c_k)$ so that our orthogonal matrix takes the form

$$A=\begin{bmatrix} a_1 & -\frac{c_1}{c_0} & 0 & \dots & & & \\ a_2 & \frac{a_1a_2}{c_0\,c_1} & -\frac{c_2}{c_1} & \dots & & & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ a_n & \frac{a_1a_n}{c_0\,c_1} & \frac{a_2a_n}{c_1c_2} & \dots & \frac{a_{n-1}a_n}{c_{n-2}c_{n-1}} & -\frac{c_n}{c_{n-1}} & 0\ \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{6}$$

Actually it is more convenient to express the element of the matrix in terms of other parameters defined by

$$p_n = \frac{a_n}{c_{n-1} c_n}, \quad n \geq 1; \quad p_0 = 1. \tag{7}$$

Note that, since
$$p_n^2 = \frac{c_{n-1}^2 - c_n^2}{c_{n-1}^2 c_n^2} = \frac{1}{c_n^2} - \frac{1}{c_{n-1}^2},$$

we have
$$\sum_{i=0}^{n} p_i^2 = c_n^{-2} = \left(\sum_{i=n+1}^{\infty} a_i^2 \right)^{-1}, \tag{8}$$

and so
$$\sum_{i=0}^{\infty} p_i^2 = \infty. \tag{9}$$

Any sequence of non-negative numbers p_i satisfying (9) thus uniquely determines an orthogonal matrix of the form (6), where a_n and c_n are then determined by (8).

We shall now use the matrix (6) to define an orthogonal function system, as was done similarly in [3]. Let

$$f_0(x) = \sqrt{\frac{1}{\pi}}, \quad f_n(x) = \sqrt{\frac{2}{\pi}} \cos nx, \quad n = 1, 2, \ldots . \tag{10}$$

This system is a complete orthonormal system (CONS) on $[0, \pi]$, and hence so is the system $\{g_n(x)\}$, $n = 1, 2, \ldots$ where

$$g_n(x) = \sum_{j=1}^{n+1} a_{nj} f_{j-1}(x). \tag{11}$$

Here we use $\|a_{ij}\|$ to denote the matrix A. More concretely, we have

$$g_n(x) = a_n \sum_{i=0}^{n-1} p_i f_i(x) - \frac{c_n}{c_{n-1}} f_n(x), \quad n \geq 1. \tag{12}$$

Note that $g_n(x)$ is a trigonometric (cosine) polynomial of order n. The necessary and sufficient condition that the g_n be uniformly bounded is clearly the existence of a positive M with

$$a_n \sum_{i=0}^{n-1} p_i \leq M. \tag{13}$$

Let now $f(x) \in L^2[0, \pi]$ and denote by $s_n(x)$, $t_n(x)$ respectively the nth partial sum of the Fourier development of f in the f-system and in the g-system. Now, s_n is the orthogonal projection of f on S_n, the span of $f_0, f_1, \ldots f_n$ and t_n is the orthogonal projection of f on T_n, the span of $g_1, \ldots g_n$. Moreover, the function

$$h_n(x) = \frac{\sum_{i=0}^{n} p_i f_i(x)}{\left(\sum_{i=0}^{n} p_i^2 \right)^{\frac{1}{2}}} \tag{14}$$

is in S_n and orthogonal to T_n. Since T_n is a subspace of S_n of deficiency one, we conclude

$$s_n(x) = t_n(x) + \langle f, h_n \rangle h_n(x), \tag{15}$$

where $\langle f, g \rangle$ denotes the inner product $\int_0^\pi fg\,dx$.

By means of (15) it is quite simple to compare the behavior of the $t_n(x)$ with that of the ordinary Fourier partial sums $s_n(x)$.

Theorem 2. *There exists, for each of the following properties a), b), c), d), e), a complete orthonormal system $\{g_n\}$, $n = 1, 2, \ldots$ on $[0, \pi]$ such that g_n is a cosine polynomial of order n, and moreover with respect to this system:*

a) The Fourier coefficients of $f_0(x) \equiv \frac{1}{\sqrt{\pi}}$ are a prescribed unit vector in l^2.

b) The Fourier series of $f_0(x) \equiv 1$ diverges to $-\infty$ at $x = 0$.

c) The Fourier series of $f_0(x) \equiv 1$ oscillates between finite limits at $x = 0$.

d) The $g_n(x)$ all vanish at $x = 0$.

e) The Fourier series of $f_0(x) \equiv 1$ diverges on a dense set having the cardinality of the continuum.

Moreover, in cases b), c), d), e) we may take $g_n(x)$ uniformly bounded.

Proof. a) is just the observation that the first column in A may be any unit vector. Now, for $f_0 \equiv 1$, we have, taking $f = 1$ in (15), and noting that $s_n(x) \equiv 1$,

$$t_n(x) = 1 - \frac{\sqrt{2}\left(\frac{p_0}{\sqrt{2}} + p_1 \cos x + \ldots p_n \cos nx\right)}{p_0^2 + p_1^2 + \ldots p_n^2}. \tag{16}$$

If we choose $p_n = n^{-\frac{1}{2}}$, we have $t_n(0) \to -\infty$, proving b). On the other hand, the ratio

$$\frac{\frac{p_0}{\sqrt{2}} + p_1 + \ldots p_n}{p_0^2 + \ldots p_n^2}$$

may be made to oscillate between finite positive limits a and b, by choosing $p_n = a^{-1}$ for a long block of n, then $p_n = b^{-1}$ for a suitably long block of n, and so on alternately. This proves c). As for d) we have simply to satisfy the equations (see (12)).

$$a_n\left(\frac{p_0}{\sqrt{2}} + p_1 + \ldots p_{n-1}\right) = \frac{c_n}{c_{n-1}}, \quad n \geq 1$$

or, in terms of the p_i (see (7), (8))

$$p_n\left(\frac{p_0}{\sqrt{2}} + p_1 + \ldots p_{n-1}\right) = p_0^2 + p_1^2 + \ldots p_{n-1}^2$$

the solution of which is $p_0=1$, $p_1=p_2=\ldots=\sqrt{2}$. For this choice we have $c_n^2=1/(2n+1)$, $a_n^2=2/(4n^2-1)$, and

$$g_n(x)=\sqrt{\frac{2}{\pi}}\,(4n^2-1)^{-\frac{1}{2}}(1+2\cos x+\ldots 2\cos(n-1)x-(2n-1)\cos nx)$$

$$=\sqrt{\frac{2}{\pi}}\,(4n^2-1)^{-\frac{1}{2}}\left[\frac{\sin(n-\frac{1}{2})x-(2n-1)\cos nx\sin\frac{1}{2}x}{\sin\frac{1}{2}x}\right],\qquad n\geqslant 1.$$

Of course, that these special $\{g_n(x)\}$ are a CONS could also be verified by direct computation. Thus d) has been proved. As for e), let n_k denote an increasing sequence of positive integers, and define

$$p_n=0,\qquad n\neq \text{any } n_k$$

$$=\frac{1}{\sqrt{k}},\qquad n=n_k.$$

Choosing n_k rapidly increasing (e.g. $n_k=6^k$) it is a simple exercise to show that there is a dense set E having the cardinality of the continuum, such that for x in E, $\cos n_k x\geqslant\frac{1}{2}$ for almost all indices k. For such x, $t_n(x)\to-\infty$, from (16) (similarly, we could make $t_n(x)\to+\infty$ on another such set E'). This proves e). There remains only the question of uniform boundedness. By (13) the uniform boundedness of the $\{g_n\}$ is implied by the boundedness of the sequence $a_n\sum_{i=0}^{n-1}p_i$. Now,

$$a_n^2=\frac{p_n^2}{(p_0^2+\ldots p_{n-1}^2)(p_0^2+\ldots p_n^2)}\leqslant\frac{p_n^2}{(p_0^2+\ldots p_{n-1}^2)^2},$$

hence a sufficient condition for the uniform boundedness of the $\{g_n\}$ is:

$$p_n(p_0+p_1+\ldots p_{n-1})\leqslant M(p_0^2+\ldots p_{n-1}^2)\tag{17}$$

and the proof is now completed by remarking that (17) holds for the systems we have constructed in b), c), d), e).

Remarks 1. Note that the choice $p_n=1/\sqrt{n+1}$ (which satisfies (17)) leads to a_n which are asymptotically $n^{-\frac{1}{2}}\log n$. Thus we see that even for a uniformly bounded, smooth CONS the constant function may have Fourier coefficients a_n satisfying $\sum a_n^2\log^2 n=\infty$, i.e. violating the hypothesis of the Menšov–Rademacher theorem ([1], p. 76). This suggests the possibility that in such a system the Fourier series of 1 might diverge everywhere, but we have not been able to construct an example giving divergence even on a set of positive measure.

2. By adjoining to the $\{g_n\}$ the functions $\sqrt{2/\pi}\sin nx$ we can get CONS on the circle group which exhibit the same pathologies.

3. If we drop the requirement of uniform boundedness d) becomes quite trivial to prove: simply take functions complete in $L^2[0, \pi]$, all vanishing at 0, and orthonormalize them. In this way we can also construct a CONS consisting of C^∞ functions, all of which vanish on a prescribed closed set of measure zero.

4. That smoothness alone cannot imply rapid decrease of the Fourier coefficients may be seen readily by simply permuting the ordinary trigonometric functions to form a new CONS. On the other hand, in all of the systems so obtained a twice differentiable function has an absolutely and uniformly convergent Fourier series.

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Added in proof: Paper [4] contains information on and applications of finite dimensional matrices of the type studied in [3].

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Tryckt den 16 december 1965

Uppsala 1965. Almqvist & Wiksells Boktryckeri AB