ARKIV FÖR MATEMATIK Band 8 nr 27

1.70 11 2 Communicated 9 September 1970 by LARS GÅRDING and LARS HÖRMANDER

Propagation of analyticity of solutions of partial differential equations with constant coefficients

By KARL GUSTAV ANDERSSON

1. Introduction

In [6] L. Hörmander discusses the following problem: Given an open set Ω in $\mathbf{R}^n$, two relatively closed subset X_1, X_2 of Ω and a partial differential operator $P(D)$ with constant coefficients, which is then the smallest set $X \subset \Omega$ such that

$$\text{s.s.}\,U \subset X_1,\ \text{s.s.}\,P(D)\,U \subset X_2 \Rightarrow \text{s.s.}\,U \subset X,\ U \in \mathcal{D}'(\Omega). \tag{1.1}$$

Here s.s.U denotes the singular support of U, i.e. the smallest, relatively closed subset of Ω such that U is an infinitely differentiable function in $\Omega \setminus \text{s.s.}\,U$.

In some cases the problem has been solved completely. Let $P_m(D)$ be the principal part of $P(D)$. $P(D)$ is said to be of principal type if grad $P_m(\xi) \neq 0$ when $\xi \in \dot{\mathbf{R}}^n = \mathbf{R}^n \setminus \{0\}$ and $P(D)$ is called real if it has real coefficients. If $P(D)$ is of principal type and $P_m(D)$ is real then a line with direction grad $P_m(\xi)$, for some $\xi \in \dot{\mathbf{R}}^n$ satisfying $P_m(\xi) = 0$, is called bicharacteristic. The following results are known [4, 6, 11]:

Theorem 1.1. *Suppose that $P(D)$ is of principal type and that $P_m(D)$ is real. If l is a bicharacteristic line and I is any closed interval (finite or infinite) contained in l, then there is a distribution F such that* s.s.$F = I$ *and $P(D)F$ is infinitely differentiable except at the (finite) endpoints of I.*

Theorem 1.2. *Let $P(D)$ be as in Theorem 1.1 and let H be a closed cone in $\mathbf{R}^n$ containing one half ray of every bicharacteristic line for $P(D)$ through the origin. Then there is a fundamental solution E of $P(D)$ with* s.s.$E \subset H$.

These two thereoms together easily give:

Theorem 1.3. *If $P(D)$ is of principal type and $P_m(D)$ is real then* (1.1) *is valid if and only if*

$$X \supset X_1 \cap X_2 \textit{ and, for every bicharacteristice line } l \textit{, the set } X \textit{ contains any component } I \textit{ of } l \cap (\Omega \setminus X_1 \cap X_2) \textit{ such that } I \subset X_1. \tag{1.2}$$

The main purpose of this paper is to prove that Theorem 1.3 is valid also when infinite differentiability is replaced by real analyticity. To do so, we will extend Theorems 1.1 and 1.2 to the anlytic case. Theorem 1.1 extends word for word, but in the theorem corresponding to Theorem 1.2 it seems necessary to strengthen the assumption as follows. If S is the set of all bicharacteristic half rays through the origin, we will suppose that, for any $b \in S$, the set H contains a neighborhood in S of either b or $-b$. The weaker result that $P(D)$ has a fundamental solution, which is analytic outside all of S, has been proved by Trèves and Zerner [10] (see also [1]).

Our version of Theorems 1.1 and 1.2 will be valid for a much more general class of differential operators than the one consisting of operators of principal type with real coefficients in their principal part. In particular, all hyperbolic operators are included.

The construction of the fundamental solution E will be carried out roughly as follows. We cover $\dot{\mathbf{R}}^n$ by open cones Δ^l, $l=1, \dots, p$, such that in each cone Δ^l there is defined a bounded C^∞ vector field $\xi \to v^l(\xi)$ satisfying $P(\xi + iv^l(\xi)) \neq 0$. Following Hörmander [6], we then construct a partition of unity $\{\phi_l\}_{l=1}^p$ in $\mathbf{C}^n$, subordinate to the covering $\{\Delta^l + i\mathbf{R}^n\}_{l=1}^p$, such that $\bar\partial\phi_l(\zeta)$ tends very rapidly to zero as ζ tends to infinity in some open cone in $\mathbf{C}^n$ containing Δ^l and put

$$E(y) = \sum_{l=1}^{p} E^l(y),$$

where
$$E^l(y) = (2\pi)^{-n} \int_{\zeta = \xi + iv^l(\xi)} e^{i\langle y, \zeta\rangle} P(\zeta)^{-1} \phi_l(\zeta)\, d\zeta. \tag{1.3}$$

Of course, this has to be interpreted in the distribution sense.

In order to show that $E^l(y)$ is analytic at x, we shall construct a vector field $\xi \to w_x^l(\xi)$, such that for some $\varepsilon > 0$

(i) $P(\xi + isw_x^l(\xi)) \neq 0$, when $\xi \in \Delta^l$ and $1 \leqslant s \leqslant \varepsilon|\xi|$

(ii) $P(\xi + i(tw_x^l(\xi) + (1-t)v^l(\xi))) \neq 0$, when $\xi \in \Delta^l$ and $0 \leqslant t \leqslant 1$

(iii) $\langle x, w_x^l(\xi)\rangle \geqslant \varepsilon$, when $\xi \in \Delta^l$.

When y is close to x, we then obtain by means of Stokes' formula

$$E^l(y) = (2\pi)^{-n} \int_{\zeta = \xi + i\varepsilon|\xi| w_x^l(\xi)} e^{i\langle y, \zeta\rangle} P(\zeta)^{-1} \phi_l(\zeta)\, d\zeta + (2\pi)^{-n} \int_\gamma e^{i\langle y, \zeta\rangle} P(\zeta)^{-1} \phi_l(\zeta)\, d\zeta$$
$$+ (2\pi)^{-n} \int_B e^{i\langle y, \zeta\rangle} P(\zeta)^{-1} \bar\partial\phi_l(\zeta)\, d\zeta,$$

where γ is compact and B is an $(n+1)$-dimensional chain in $\mathbf{C}^n$. The first two terms obviously define functions holomorphic in a neighborhood of x. Furthermore, mainly because $\bar\partial\phi_1(\zeta)$ tends very rapidly to zero as $|\zeta|$ tends to infinity, this is also the case for the third term. However, we have to choose our partition of unity $\{\phi_l\}$ much more carefully than in [6]. The construction we give in section 2 was suggested by Lars Hörmander. In section 3 we analyze the possible choices of the vector fields w_x^l. Here we follow Atiyah, Bott, Gårding [2] very closely. The extensions of Theorem 1.1 and 1.2 are proved in section 4. The distribution F occurring in Theorem 1.2 will be constructed in much the same way as E^l. The main idea will be to insert a factor $e^{-g(\zeta)}$ in the integrand in (1.3). These theorems will then be applied in section 5 to prove Theorem 1.3 with infinite differentiability replaced by real analyticity.

We should mention that at the Nice Congress Sato [12] announced that, for operators of principal type with analytic coefficients and real principal part, results similar to those of this paper had been obtained for hyperfunctions by Kawai, Kashiwabara and himself.

2. A partition of unity

Let K be a compact subset of the open set $\Omega \subset \mathbf{R}^n$ and let ψ be a function in $C_0^\infty(\Omega)$ equal to 1 in a neighborhood of K. Suppose further that χ is a non-negative function in $C^\infty(\mathbf{R}^n)$ with support in $\{\xi;\ |\xi| \leqslant \varepsilon\}$ such that $\int \chi(\xi)\,d\xi = 1$. With $\chi_{(j)}(\xi) = j^n \chi(j\xi)$ we form the following convolution

$$\psi_k = \psi * \prod_{1 \leqslant 4^\nu \leqslant k} * \chi^{*[k \cdot 4^{-\nu}]}_{(k \cdot 2^{1-\nu})} \tag{2.1}$$

where $[a]$ denotes the integral part of the number a, $f^{*[a]} = f * \ldots * f$, $[a]$ times, and $\prod_{1 \leqslant i \leqslant q} * f_i = f_1 * \ldots * f_q$. Then $\psi_k \in C_0^\infty(\Omega)$ and equals 1 on K if ε is small enough, independently of k. By letting the derivatives fall on different $\chi_{(j)}$:s one easily verifies that

$$|D^\alpha \psi_k(\xi)| \leqslant M(c2^{-\nu}k)^{|\alpha|},\ |\alpha| \leqslant k \cdot 4^{-\nu}, \tag{2.2}$$

for some constants M and c.

Constructions similar to the one above have been used for a long time; see e.g. Mandelbrojt [7] or Ehrenpreis [3]. The one we have chosen is tailored to ensure that there are functions ϕ belonging to the class $L(\Delta)$ which we are now going to define. The functions in this class have the important property that their Fourier transforms are analytic in $\dot{\mathbf{R}}^n$.

Definition 2.1. Let $\Delta \subset \dot{\mathbf{R}}^n$ be an open cone. By $L(\Delta)$ we denote the set of functions $\phi \in C^\infty(\dot{\mathbf{R}}^n)$ such that $\operatorname{supp} \phi \subset \Delta$ and, for some constants M and c,

$$|D^\alpha \phi(\xi)| \leqslant M|\xi|(c2^{-\nu})^{|\alpha|},\ \text{if } 4^\nu|\alpha| \leqslant |\xi|.$$

Lemma 2.1. *Let $\{\Delta^l\}_{p=1}^l$ be a covering of $\dot{\mathbf{R}}^n$ with open cones. Then there are $\phi_l \in L(\Delta^l)$ such that $\sum_1^p \phi_l(\xi) = 1$ when $|\xi| \geqslant 1$.*

Proof. Let $\{g^l\}_{p=1}^l$ be a partition of unity in $\{\xi;\ 1/3 \leqslant |\xi| \leqslant 3\}$ with $g^l \in C_0^\infty(\Delta^l)$, i.e. $\sum_1^p g^l(\xi) = 1$ when $1/3 \leqslant |\xi| \leqslant 3$. Apply the procedure described above in order to get a sequence $\{g_k^l\}$, $k = 1, 2, \ldots$, of partitions of unity in $\{\xi;\ 1/2 \leqslant |\xi| \leqslant 2\}$ with $g_k^l \in C_0^\infty(\Delta^l)$, such that

$$|D^\alpha g_k^l(\xi)| \leqslant M(c2^{-\nu}k)^{|\alpha|},\ \text{when } 4^\nu|\alpha| \leqslant k.$$

Starting with a function $b \in C_0^\infty(\{\xi;\ |\xi| < 1)$ which equals 1 in a neighborhood of $\{\xi;\ |\xi| \leqslant 1/2\}$, we obtain in the same way functions $b_k \in C_0^\infty(\{\xi;\ |\xi| < 1\})$ equal to 1 in a neighborhood of $\{\xi;\ |\xi| \leqslant 1/2\}$ such that

$$|D^\alpha b_k(\xi)| \leqslant M(c2^{-\nu}k)^{|\alpha|},\ \text{when } 4^\nu|\alpha| \leqslant k.$$

Put $\gamma_k^l(\xi) = g_k^l(\xi/k)$ and $\chi_k(\xi) = b_{k+1}(\xi/k+1) - b_k(\xi/k)$, $k \geqslant 1$. Then $k/2 < |\xi| < k+1$ for any ξ with $\chi_k(\xi) \neq 0$ and $\sum_l \gamma_k^l(\xi) = 1$ in the support of χ_k. We now define

$$\phi_l(\xi) = \sum_{k \geqslant 1} \chi_k(\xi)\,\gamma_k^l(\xi).$$

By first summing over l and then over k it follows that $\Sigma\, \phi_l(\xi) = 1$ when $|\xi| \geqslant 1$. It remains to estimate the derivatives of ϕ_l. If $k \leqslant |\xi| < k+1$, then $4^\nu|\alpha| \leqslant |\xi|$

if and only if $4^\nu|\alpha|\leqslant k$. Furthermore,

$$\phi_l(\xi)=\sum_{k\leqslant j\leqslant 2k+1}\chi_j(\xi)\gamma_j^l(\xi).$$

According to Leibniz' formula we have

$$D^\alpha(\chi_j(\xi)\gamma_j^l(\xi))=\sum_{\beta\leqslant\alpha}D^\beta\chi_j(\xi)D^{\alpha-\beta}\gamma_j^l(\xi)\,\alpha!/(\alpha-\beta)!\,\beta!,$$

where, as usual, $\beta\leqslant\alpha$ means $\beta_i\leqslant\alpha_i$, $\forall_i$, and $\alpha!$ means $\prod_i\alpha_i!$. Now (2.2) together with the binomial theorem give that

$$|D^\alpha(\chi_j(\xi)\gamma_j^l(\xi))|\leqslant 2^{|\alpha|}M(c2^{-\nu})^{|\alpha|}.$$

Thus

$$|D^\alpha\phi_l(\xi)|\leqslant M_1|\xi|(c_1 2^{-\nu})^{|\alpha|}$$

with some new constants M_1 and c_1.

We are now going to extend the elements in $L(\Delta)$ appropriately into the complex domain. The construction is similar to the one in [6]. Let $\lambda\in C^\infty(\mathbf{R})$ have its support in $\{t;\,t>0\}$ and be equal to 1 when $t\geqslant 1$. Given $\phi\in C^\infty(\mathbf{R}^n)$ we define

$$\phi(\xi+i\eta)=\sum_\alpha\phi^{(\alpha)}(\xi)\,(i\eta)^\alpha\lambda(|\xi|-(|\alpha|+1))/\alpha!\,. \tag{2.3}$$

Lemma 2.2. *If $\phi\in L(\Delta)$ is extended into $\mathbf{C}^n$ by (2.3), then there are positive constants M and c such that*

$$\xi\notin\Delta\Rightarrow\phi(\xi+i\eta)=0 \tag{2.4}$$

$$M(|\eta|+1)\leqslant|\xi|\Rightarrow|\bar\partial\phi(\xi+i\eta)|\leqslant M|\xi|e^{-|\xi|} \tag{2.5}$$

$$M2^\nu(|\eta|+2^\nu)\leqslant|\xi|\Rightarrow|\phi(\xi+i\eta)|\leqslant M|\xi|^2e^{c|\eta|/2^\nu}. \tag{2.6}$$

Proof. (2.4) is obvious. Put $A_1(\xi)=\{\alpha;\ |\alpha|+1\leqslant|\xi|-1\}$, $A_2(\xi)=\{\alpha;\ |\xi|-1<|\alpha|+1<|\xi|\}$ and $A_3(\xi)=\{\alpha;\ |\alpha|+1=[|\xi|-1]\}$. Then

$$\begin{aligned}&(\partial/\partial\xi_j+i\partial/\partial\eta_j)\,\phi(\xi+i\eta)\\&=(\partial/\partial\xi_j+i\partial/\partial\eta_j)\Big\{\sum_{\alpha\in A_1(\xi)}\phi^{(\alpha)}(\xi)\,(i\eta)^\alpha/\alpha!\\&\quad+\sum_{\alpha\in A_2(\xi)}\phi^{(\alpha)}(\xi)\,(i\eta)^\alpha\lambda(|\xi|-(|\alpha|+1))/\alpha!\Big\}\\&=\sum_{\alpha\in A_3(\xi)}\phi^{(\alpha+1_j)}(\xi)\,(i\eta)^\alpha/\alpha!\\&\quad+(\partial/\partial\xi_j+i\partial/\partial\eta_j)\Big(\sum_{\alpha\in A_2(\xi)}\phi^{(\alpha)}(\xi)\,(i\eta)^\alpha\lambda(|\xi|-(|\alpha|+1))/\alpha!\Big).\end{aligned}$$

According to Stirling's formula $p^p/e^pp!$ is a bounded function of p. Furthermore, $\sum_{|\alpha|=p}p!/\alpha!=n^p$. If we put $p=[|\xi|-1]$, Definition 2.1 thus gives

$$\left|\sum_{\alpha\in A_3(\xi)} \phi^{(\alpha+1_j)}(\xi)\,(i\eta)^\alpha/\alpha!\right| \leqslant M|\xi|\,(c|\eta|)^p \sum_{|\alpha|=p} 1/\alpha!$$
$$\leqslant M|\xi|\,(c|\eta|/p)^p\,(p^p/e^p p!)\,\Big(\sum_{|\alpha|=p} p!/\alpha!\Big)\,e^p \leqslant M_1|\xi|\,(c_1|\eta|/p)^p$$
$$\leqslant M_1|\xi|\,(c_2|\eta|/|\xi|)^p.$$

If in addition $e\,c_2|\eta| \leqslant |\xi|$, we get

$$\left|\sum_{\alpha\in A_3(\xi)} \phi^{(\alpha+1_j)}(\xi)\,(i\eta)^\alpha/\alpha!\right| \leqslant M_1|\xi|\,e^{-p} \leqslant M_2|\xi|\,e^{-|\xi|}.$$

The second term admits a similar estimate. This proves (2.5).

To prove (2.6) we define functions ϕ_ν by

$$\phi_\nu(\xi+i\eta) = \sum_\alpha \phi^{(\alpha)}(\xi)\,(i\eta)^\alpha \lambda(|\xi| - 4^\nu(|\alpha|+1))/\alpha!.$$

It follows directly from Definition 2.1 that ϕ_ν satisfies the estimate in (2.6) in all of $\mathbf{C}^n$. In fact

$$|\phi_\nu(\xi+i\eta)| \leqslant M|\xi| \sum_\alpha (c2^{-\nu}|\eta|)^{|\alpha|}/\alpha! = M|\xi|\,e^{nc|\eta|/2^\nu}.$$

We will finish the proof by showing that $|\phi(\xi+i\eta) - \phi_\nu(\xi+i\eta)| \leqslant \text{constant}\cdot|\xi|^2$, when $M2^\nu(|\eta| + 2^\nu) \leqslant |\xi|$.

$$\phi(\xi+i\eta) - \phi_\nu(\xi+i\eta) = \sum_{\alpha\in A_4(\xi)} \phi^{(\alpha)}(\xi)\,(i\eta)^\alpha(\lambda(|\xi| - (|\alpha|+1)) - \lambda(|\xi| - 4^\nu(|\alpha|+1)))/\alpha!,$$

where $A_4(\xi) = \{\alpha;\ |\alpha|+1 \leqslant |\xi| \leqslant 4^\nu(|\alpha|+1)+1\}$. If we denote by $\theta = \theta(|\alpha|)$ the largest integer such that $4^\theta|\alpha| \leqslant |\xi|$ then Definition 2.1 gives that

$$|\phi(\xi+i\eta) - \phi_\nu(\xi+i\eta)| \leqslant M|\xi| \sum_{\alpha\in A_4(\xi)} (c2^{-\theta}|\eta|)^{|\alpha|}/\alpha!$$

As in the proof of (2.5) we get, with $|A_4|(\xi) = \{|\alpha|;\ \alpha\in A_4(\xi)\}$,

$$|\phi(\xi+i\eta) - \phi_\nu(\xi+i\eta)| \leqslant M_1|\xi| \sum_{p\in|A_4|(\xi)} (c_1 2^{-\theta}|\eta|/p)^p$$
$$\leqslant M_1|\xi| \sum_{p\in|A_4|(\xi)} (c_1 2^{\theta+2}|\eta|/|\xi|)^p.$$

The last inequality follows from the fact that $4^{\theta+1}p > |\xi|$. Now $4^\theta p \leqslant |\xi| \leqslant 4^\nu(p+1)+1$ when $p\in|A_4|(\xi)$. This together with the assumptions of (2.6) give that

$$|\phi(\xi+i\eta) - \phi_\nu(\xi+i\eta)| \leqslant M|\xi|^2\,e^{-|\xi|/4^{\nu+1}}.$$

3. Localizations and vector fields

We are now going to define the class of differential operators $P(D)$ which will be studied in the rest of the paper. In most of this section we will restrict ourselves to the homogeneous case, but at the end we will show that the main results are valid also for inhomogenous polynomials P weaker than their principal part P_m, provided that P_m belongs to the class in question.

Most of the results will be quite straightforward extensions of the corresponding ones for hyperbolic polynomials in [2]. However, for the convenience of the reader, we shall repeat the proofs of all facts that we need.

To begin with, we denote by V the topological space of all vector fields $\xi \to v(\xi)$ from $\dot{\mathbf{R}}^n$ to $\mathbf{R}^n$, homogeneous of degree zero, with the topology of uniform convergence.

Remark. Observe that we don't require any continuity of the vector fields in V.

Definition 3.1. If P is a homogeneous polynomial and ε a positive number, we define $V(P, \varepsilon)$ as the subspace of V consisting of all vector fields v in V with the property that to every $\xi_0 \in \dot{\mathbf{R}}^n$ there is a neighborhood O of $\xi_0 + iv(\xi_0)$ in $\mathbf{C}^n$ such that

$$(\xi + i\theta) \in O, \ |s\theta| < \varepsilon|\xi_0| \text{ and } \operatorname{Im} s \neq 0 \Rightarrow P(\xi + s\theta) \neq 0 \tag{3.1}$$

The union $\bigcup_{\varepsilon>0} V(P, \varepsilon)$ is denoted by $V(P)$.

Remark. (3.1) says that all sufficiently small zeros of the polynomials $s \to P(\xi + s\theta)$, $\xi + i\theta \in O$, are real. When $P(\xi_0) \neq 0$ there are no small zeros so that (3.1) is non-empty only if $P(\xi_0) = 0$. Note that it is sufficient to assume (3.1) when $\operatorname{Re} s = 0$.

Definition 3.2. A homogeneous polynomial P is called locally hyperbolic with respect to the vector field $v \in V$ if

$$P(v(\xi)) \neq 0 \text{ for all } \xi \in \dot{\mathbf{R}}^n \tag{3.2}$$

$$v \in V(P). \tag{3.3}$$

The class of polynomials locally hyperbolic with respect to v will be denoted by $Hyp_{\text{loc}}(v)$.

Examples

1. The homogeneous polynomials hyperbolic with respect to the vector N, i.e. $Hyp(N)$. This class is obviously contained in $Hyp_{\text{loc}}(N)$. Here the vector field $\xi \to N$ is constant and the number ε in Definition 3.1 may be taken as $+\infty$.

2. The homogeneous polynomials P, with real coefficients, such that grad $P(\xi) \neq 0$ whenever $P(\xi) = 0$ and $\xi \in \dot{\mathbf{R}}^n$. If, for $z, \zeta \in \mathbf{C}^n$, we put $\langle z, \zeta \rangle = z_1\zeta_1 + \dots + z_n\zeta_n$, then any v in V, such that $\langle \operatorname{grad} P(\xi), v(\xi) \rangle \neq 0$, for all $\xi \in \dot{\mathbf{R}}^n$ with $P(\xi) = 0$, belongs to $V(P)$. In fact

$$P(\xi + i\sigma\theta) = P(\xi) + i\sigma \langle \operatorname{grad} P_m(\xi), \theta \rangle + 0(\sigma^2)$$

Here the first term is real, whilhe second is imaginary and non-vanishing when $\xi + i\theta$ is close to $\xi_0 + iv(\xi_0)$ and $P(\xi_0) = 0$. This proves that $v \in V(P)$. Furthermore the set of $w \in V$ such that $P(w(\xi)) \neq 0$, for all $\xi \in \dot{\mathbf{R}}^n$, is dense in V. Thus there are vector fields v in V, such that $P \in Hyp_{\text{loc}}(v)$, arbitrarily close to e.g. $\xi \to \operatorname{grad} P(\xi)/|\operatorname{grad} P(\xi)|$.

3. Homogeneous polynomials P with the following property: every $\xi_0 \in \dot{\mathbf{R}}^n$ with $P(\xi_0) = 0$ has a complex neighborhood $O(\xi_0)$ where $P(\zeta)$ is a product $F_1(\zeta) \dots F_k(\zeta)$ of holomorphic factors F_j such that $F_j(\xi_0) = 0$, grad $F_j(\xi) \neq 0$ and $F_j(\xi)$ is real for real

ξ in $O(\xi_0)$. For such P:s we have, as in example 2, that $v \in V(P)$ if $\langle F_j(\xi), v(\xi)\rangle \neq 0$, $j=1, \ldots, k$, when $\xi \in O(\xi_0)$ is real and $F_j(\xi)=0$. As above we may perturb such a v arbitrarily little to get $w \in V$ such that $P \in Hyp_{\mathrm{loc}}(w)$.

Finally we give two examples of polynomials which are not locally hyperbolic with respect to any $v \in V$ as will follow directly from Lemma 3.1 below.

4. Homogeneous polynomials P such that grad Re $P(\xi_0)$ and grad Im $P(\xi_0)$ are linearly independent at some $\xi_0 \in \dot{\mathbf{R}}^n$ satisfying $P(\xi_0)=0$.

When grad Re $P(\xi)$ and grad Im $P(\xi)$ are linearly independent at all points $\xi \in \dot{\mathbf{R}}^n$ with $P(\xi)=0$ results, similar to those in this paper, have been obtained by somewhat different methods in [1].

5. $P(\zeta)=\zeta_n Q(\zeta_1, \ldots, \zeta_{n-1})+R(\zeta_1, \ldots, \zeta_{n-1})$, where Q and R are homogeneous polynomials of degree $m-1$ and m respectively, and Q is not hyperbolic.

Our next step is to study the localizations of locally hyperbolic polynomials. Let P be a homogeneous polynomial of degree m. We develop $P(\xi+\tau\zeta)$ around ξ

$$P(\xi+\tau\zeta)=\tau^p P_\xi(\zeta)+\text{terms of higher order in } \tau.$$

If $P_\xi(\zeta)\neq 0$ for some ζ, then $P_\xi(\zeta)$ is called the localization of P at ξ and $p=m_\xi(P)$ the multiplicity of P at ξ. Under these circumstances it is obvious that

$$\tau^{m-p}P(\tau^{-1}\xi+\zeta)=P_\xi(\zeta)+\text{terms of positive order in } \tau$$

tends to $P_\xi(\zeta)$, locally unifirmly in ζ, when τ tends to zero.

The localizations above have been utilized by Atiyah–Bott–Gårding in [2]. Hörmander [6] uses a more sophisticated process of localization of non-homogeneous polynomials.

Examples

1. When $P(\xi)\neq 0$ then $P_\xi(\zeta)=P(\xi)$ is a non-zero constant.
2. If $P(\xi)=0$ but grad $P(\xi)\neq 0$, then $P_\xi(\zeta)=\langle \text{grad } P(\xi), \zeta\rangle$ is a first order polynomial.
3. If $P(\zeta)=\zeta_n Q(\zeta_1, \ldots, \zeta_{n-1})+R(\zeta_1, \ldots, \zeta_{n-1})$ is a polynomial satisfying the conditions in example 5 above and $e_n=(0, \ldots, 0, 1)$, then $P_{e_n}(\zeta)=Q(\zeta_1, \ldots, \zeta_{n-1})$ and $m_{e_n}(P)=m-1$.

Lemma 3.1. *Suppose that P is a homogeneous polynomial, $v \in V(P)$ and $\xi_0 \in \dot{\mathbf{R}}^n$. Then $P_\xi \in Hyp\ (v(\xi_0))$ when ξ is close to ξ_0.*

Proof. Suppose that $\eta \in \mathbf{R}^n$ and $s \in \mathbf{C}\setminus\mathbf{R}$ are given. Since $v \in V(P)$ it is obvious that the distance from $(\xi+\tau(\eta+sv(\xi_0)))$ to $\{\zeta; P(\zeta)=0\}$ is bounded from below by some positive constant times τ when ξ is close to ξ_0 and τ is small. If we denote $m_\xi(P)$ by p then, according to Lemma 4.1.1 in [5],

$$0<c\leqslant \tau^{-p}\, P(\xi+\tau(\eta+sv(\xi_0)))=P_\xi(\eta+sv(\xi))+0(\tau).$$

Since η and s were arbitrary, this proves that $P(\eta+sv(\xi_0))\neq 0$ when $\eta \in \mathbf{R}^n$ and Im $s\neq 0$. In particular it follows from the homogeniety of P that $P_\xi(v(\xi_0))\neq 0$.

Corollary 3.1. $P \in \mathrm{Hyp}_{\mathrm{loc}}(v)$, $\xi_0 \in \dot{\mathbf{R}}^n \Rightarrow P_\xi \in \mathrm{Hyp}\ (v(\xi_0))$ *when ξ is close to ξ_0.*

Remark. This corollary and the examples 2 and 3 preceding the lemma now immediately show that the polynomials in the examples 4 and 5 above are not locally hyperbolic with respect to any $v \in V$.

Semicontinuity of the local cones. Because P_ξ is hyperbolic with respect to $v(\xi_0)$ when ξ is close to ξ_0, we may define the local cone $\Gamma(P_\xi, v(\xi_0))$ as the component of $v(\xi_0)$ in $\{\zeta \in \mathbf{R}^n;\ P_\xi(\zeta) \neq 0\}$. P_ξ is hyperbolic with respect to any $\eta \in \Gamma(P_\xi, v(\xi_0))$ (see Theorem 5.5.5 in [5]). The dual cone of the cone $\Gamma(P_\xi, v(\xi_0))$ is defined by

$$K(P_\xi, v(\xi_0)) = \{x \in \mathbf{R}^n;\ \langle x, \eta\rangle \geqslant 0,\ \forall \eta \in \Gamma(P_\xi, v(\xi_0))\}.$$

For polynomials $P \in \mathrm{Hyp}_{\mathrm{loc}}(v)$ we shall now study the set $V(P)$ more closely. We are going to show that to any $\eta \in \Gamma(P_{\xi_0}, v(\xi_0))$ there is a neighborhood U of ξ_0, such that the constant vector field $U \ni \xi \to \eta$ may be extended to an element of $V(P)$. First we need a technical lemma about the small zeros of $s \to P(\xi + sv(\xi_0))$ when ξ is close to ξ_0.

Lemma 3.2. *Let $v \in V(P, \varepsilon)$ and $P \in Hyp_{\mathrm{loc}}(v)$. Then there is a neighborhood U of ξ_0 in $\dot{\mathbf{R}}^n$ with the following property. Given $\xi \in U$ and $\eta \in \mathbf{R}^n$ then the zeros of $s \to P(\xi + t\eta + sv(\xi_0))$ satisfying $|sv(\xi_0)| < \varepsilon |\xi_0|$ can be labelled so that they are differentiable functions $s_k(\xi, \eta, t)$ of t, $k = 1, ..., p$, when t is small.*

Proof. Let O be the neighborhood of $\xi_0 + iv(\xi_0)$ occurring in Definition 3.1 and choose U and δ such that $U + t\eta + iv(\xi_0) \subset O$ when t is real and $|t| < \delta$. Now (3.1) implies that $s_k(\xi, \eta, t)$, $k = 1, ..., p$, is real when t is real and $|t| < \delta$. Furthermore, because $P(v(\xi_0)) \neq 0$, the functions $t \to s_k(\xi, \eta, t)$ may be developed into convergent Puiseux series around a real t_0

$$s_k(\xi, \eta, t) = s_k(\xi, \eta, t_0) + c_k(t - t_0)^{r_k}(1 + o(1)), \tag{3.4}$$

with $c_k \neq 0$ and $r_k > 0$ rational. If now $|t_0| < \delta$, then r_k has to be an integer, because $s_k(\xi, \eta, t)$ is real when t is real and close to t_0. This gives the differentiability of $t \to s_k(\xi, \eta, t)$.

We are now ready to prove the main result of this section (compare Lemma 5.1 in [2]).

Lemma 3.3. *Suppose that $P \in Hyp_{\mathrm{loc}}(v)$, $\xi_0 \in \dot{\mathbf{R}}^n$ and that M is a compact subset of $\Gamma(P_{\xi_0}, v(\xi_0))$. Then there is a neighborhood U of ξ_0 in $\dot{\mathbf{R}}^n$ such that*

$$\eta \in M,\ \xi \in U \text{ and } \operatorname{Im} t \neq 0 \Rightarrow P(\xi + t\eta) \neq 0, \tag{3.5}$$

provided that t is sufficiently small.

Proof. Because of (3.2), we may suppose that $P(v(\xi_0)) = 1$. If m is the degree of P and $p = m_{\xi_0}(P)$, we have

$$P(\xi_0 + sv(\xi_0)) = s^p P_{\xi_0}(v(\xi_0)) + \dots + s^m \tag{3.6}$$

Now, because of the continuity of the zeros and the fact that $P_{\xi_0}(v(\xi_0)) \neq 0$, p zeros $\mu_1(\xi_0 + \zeta), ..., \mu_p(\xi_0 + \zeta)$ of $P(\xi_0 + \zeta + sv(\xi_0))$ will be small, while the remaining ones are bounded away from the origin, provided that $\zeta \in \mathbf{R}^n$ is small. (3.1) shows then that the numbers $\mu_1(\xi_0 + \zeta), ..., \mu_p(\xi_0 + \zeta)$ are real. We shall relate these zeros to the zeros

$\mu_1^0(\zeta), \ldots, \mu_p^0(\zeta)$ of the localized equation $P_{\xi_0}(\zeta+sv(\xi_0))=0$. Since

$$P(\xi_0+\zeta+sv(\xi_0)) = \prod_1^m (s-\mu_k(\xi_0+\zeta)) \tag{3.7}$$

we get

$$q_\tau(s) = \tau^{-p} P(\xi_0+\tau\zeta+\tau sv(\xi_0))/\prod_{p+1}^m (\tau s-\mu_k(\xi_0+\tau\zeta)) = \prod_1^p (s-\tau^{-1}\mu_k(\xi_0+\tau\zeta)).$$

If s and ζ are bounded, then $q_\tau(\zeta)$ converges to $P_{\xi_0}(\zeta+sv(\xi_0))/\Pi_{p+1}^m(-\mu_k(\xi_0))$, uniformly in s and ζ. By choosing $p+1$ numbers $s_0, \ldots, s_p$, such that $(1, s_0, \ldots, s_0^p), \ldots, (1, s_p, \ldots, s_p^p)$ are linearly independent, it follows that the coefficients, and thus the zeros, of $q_\tau(s)$ converge to those of $P_{\xi_0}(\zeta+sv(\xi_0))/\Pi_{p+1}^m(-\mu_k(\xi_0))$, uniformly in ζ. With a suitable labelling we then have

$$\tau^{-1}\mu_k(\xi_0+\tau\zeta)\to\mu_k^0(\zeta),\ k=1, \ldots, p,\ \text{uniformly in } \zeta, \text{ when } \tau\to 0.$$

Using that all $\mu_k(\xi_0+\tau\zeta)$ and $\mu_k^0(\zeta)$ are real and that, furthermore, $\mu_k^0(\zeta)$ is homogeneous of degree one, we get

$$\mu_k(\xi_0+\zeta) = \mu_k^0(\zeta)+\varrho_1(\zeta)\,|\zeta|, \tag{3.8}$$

where $\varrho_1(\zeta)$ is real, for small $\zeta\in\mathbf{R}^n$, and tends to zero when $\zeta\to 0$.

If ζ_1, ζ_2 belong to a compact subset of $\mathbf{R}^n$ we define $\lambda_{\zeta_1,k}^0(\zeta_2)$ as the $\mu_j^0(\zeta_2)$ such that $|\mu_k^0(\zeta_1)-\mu_j^0(\zeta_2)|$ attains its minimum. Because a continuous function is uniformly continuous on compact sets, the function $\mu_k^0(\zeta_1)-\lambda_{\zeta_1,k}^0(\zeta_2)$ tends to zero together with $|\zeta_1-\zeta_2|$. Not exhibiting the dependence of $\lambda_{\zeta_1,k}^0(\zeta_2)$ on ζ_1 and again using its homogeneity of degree 1 we get, with $\zeta_1=\zeta+t\eta$, $\zeta_2=t\eta$ and $\eta\in M$,

$$\mu_k^0(\zeta+t\eta) = t\lambda_k^0(\eta)+\varrho_2(\zeta, t\eta), \tag{3.9}$$

where the real function $\varrho_2(\zeta, t\eta)$ tends to zero, uniformly with respect to $\eta\in M$ and t small, when $\zeta\to 0$. Replacing ζ by $\zeta+t\eta$ in (3.8), this together with (3.9) finally gives

$$\mu_k(\xi_0+\zeta+t\eta) = t(\lambda_k^0(\eta)+\varrho_3(\zeta, t\eta))+\varrho_4(\zeta, t\eta), \tag{3.10}$$

where ϱ_i, $i=3, 4$, are realvalued and $\varrho_3(\zeta, t\eta)\to 0$ as $(\zeta, t)\to 0$, uniformly when $\eta\in M$, while $\varrho_4(\zeta, t\eta)\to 0$ as $\zeta\to 0$, uniformly when $\eta\in M$ and t is small.

We now put $s=0$ in (3.7)

$$P(\xi_0+\zeta+t\eta) = (-1)^m \prod_1^p \mu_k(\xi_0+\zeta+t\eta) \prod_{p+1}^m \mu_k(\xi_0+\zeta+t\eta).$$

According to Lemma 3.2, we may suppose that the functions $t\to\mu_k(\xi_0+\zeta+t\eta)$, $k=1, \ldots, p$, are differentiable for every fixed ζ, small enough. Because we know that $t\to P(\xi_0+\zeta+t\eta)$ has exactly p small zeros, it now only remains to show that, given $\varepsilon>0$, there is a $\delta>0$, such that to every $\zeta\in\mathbf{R}^n$ with $|\zeta|<\delta$, every $\eta\in M$ and every k with $1\leqslant k\leqslant p$ there is a real t with $\mu_k(\xi_0+\zeta+t\eta)=0$. This, however, follows directly from (3.10) since, because $P_{\xi_0}(\eta)\neq 0$ when $\eta\in M$, the real numbers $\lambda_k^0(\eta)$ are bounded away from zero on M and since $\varrho_4(\zeta, t\eta)\to 0$, uniformly in $t\eta$ when $\zeta\to 0$. This finishes the proof.

As in [2] it now easily follows that the local cones $\Gamma(P_\xi, v(\xi_0))$ contain any preassigned subset M of $\Gamma(P_{\xi_0}, v(\xi_0))$, if only ξ is close enough to ξ_0.

Lemma 3.4. *Suppose that $P \in Hyp_{\mathrm{loc}}(v)$, $\xi_0 \in \dot{\mathbf{R}}^n$ and that M is a compact subset of $\Gamma(P_{\xi_0}, v(\xi_0))$. Then $M \subset \Gamma(P_\xi, v(\xi_0))$ for all $\xi \in \mathbf{R}^n$ sufficiently close to ξ_0.*

Proof. We suppose that $P(v(\xi_0)) = 1$. Because of Corollay 3.1. $P_\xi \in \mathrm{Hyp}\,(v(\xi_0))$ if ξ is close to ξ_0. Thus $P_\xi(v(\xi_0)) \neq 0$ and

$$P(\xi + sv(\xi_0)) = s^q(P_\xi(v(\xi_0)) + 0(s)),$$

where $q = q(\xi) = m_\xi(P)$.

We may suppose that M is convex and contains $v(\xi_0)$. Then it suffices to show that $P(\xi + t\eta)$ vanishes precisely of order q at $t = 0$, when ξ close to ξ_0 and $\eta \in M$. In fact, then $P_\xi(\eta) \neq 0$ when $\eta \in M$ so that $M \subset \Gamma(P_\xi, v(\xi_0))$. By (3.7), we have

$$P(\xi + t\eta + sv(\xi_0)) = \prod_1^p (s - \mu_k(\xi + t\eta)) \prod_{p+1}^m (s - \mu_k(\xi + t\eta)). \tag{3.11}$$

By Lemma 3.3 and the convexity of $\Gamma(P_\xi, v(\xi_0))$

$$\mathrm{Im}\ s \geq 0,\ \mathrm{Im}\ t \geq 0 \text{ and } \mathrm{Im}\ (s+t) > 0 \Rightarrow P(\xi + t\eta + sv(\xi_0)) \neq 0,$$

provided that ξ is close to ξ_0 and s, t small. Thus

$$\mathrm{Im}\ t > 0 \Rightarrow \mathrm{Im}\ \mu_k(\xi + t\eta) < 0,\ 1 \leq k \leq p.$$

According to (3.4) this is only possible if $r_k = 1$ and $c_k < 0$. This gives

$$\mu_k(\xi + t\eta) = \mu_k(\xi) + c_k t(1 + o(1)),\ c_k < 0,\ 1 \leq k \leq p.$$

Now the result follows from (3.11) with $s = 0$ and $t = 0$ respectively.

Dual cones and vector fields. Before proceeding further, we shall summarize some general facts about conical algebraic hypersurfaces in $\mathbf{C}^n$.

Let P be a homogeneous polynomial. The lineality $\Lambda_{\mathbf{C}}(P)$ of P is defined by

$$\Lambda_{\mathbf{C}}(P) = \{\eta \in \mathbf{C}^n;\ P(\zeta + t\eta) = P(\zeta),\ \forall t, \zeta\}$$

The polynomial P_ξ, obtained by localizing P at $\xi \in \dot{\mathbf{C}}^n$, always has a non-empty lineality. In fact, due to the homogeneity of P,

$$\begin{aligned} &s^p(1 + st)^{m-p} P_\xi(\zeta) + 0(s^{p+1}) = \\ &= P(\xi + st\xi + s\zeta) = s^p P_\xi(t\xi + \zeta) + 0(s^{p+1}) \end{aligned}$$

where $p = m_\xi(P)$. Identification of the coefficients of s^p gives that $\xi \in \Lambda_{\mathbf{C}}(P_\xi)$.

Now, if $Z(P) = \{\xi \in \dot{\mathbf{C}}^n;\ P(\xi) = 0\}$, we define the normal $\Lambda'_{\mathbf{C}}(P_\xi)$ of $Z(P)$ at ξ by

$$\Lambda'_{\mathbf{C}}(P_\xi) = \{x \in \mathbf{C}^n;\ \langle x, \eta \rangle = 0,\ \forall\, \eta \in \Lambda_{\mathbf{C}}\,(P_\xi)\}.$$

Finally the dual $Z'(P)$, of $Z(P)$, is defined as the union of all $\Lambda_{\mathbf{C}}(P_\xi)$, when $\xi \in Z(P)$.

The following result is proved in [2] (p. 153).

Proposition 3.1. *$Z'(P)$ is contained in a proper conical subvariety of $\mathbf{C}^n$.*

Suppose now that P is locally hyperbolic and let Σ be a closed cone in $\mathbf{R}^n$. By $C^\infty V(P, \Sigma)$ we mean the set of $v \in V(P)$ such that $v \in C^\infty(\Delta)$ for some open cone containing $\dot{\Sigma}=\Sigma\setminus\{0\}$. Because of Lemma 3.1, the cone $K(P_\xi, v(\xi))$ is well defined for every $\xi\in\mathbf{R}^n$. Given $v\in C^\infty(P, \Sigma)$ we may thus define

$$H(P,\Sigma,v) = -\bigcup_{\xi\in\dot{\Sigma}} K(P_\xi, v(\xi)).$$

Examples

1. If P is an arbitrary homogeneous locally hyperbolic polynomial, $v\in V(P)$ and $\xi_0\in\dot{\mathbf{R}}^n$, then, because of (3.1), there is an open cone Δ containing ξ_0, such that the constant vector field $\Delta\ni\xi\to v(\xi_0)$ may be extended to an element in $C^\infty V(P, \bar{\Delta})$.

2. Suppose that P is as in example 2 in the beginning of this section and that ξ_0 is a point in $\dot{\mathbf{R}}^n$ satisfying $P(\xi_0)=0$. According to example 2 preceeding Lemma 3.1, $\Lambda_{\mathbf{C}}(P_{\xi_0})$ consists of those $n\in\mathbf{C}^n$ such that $\langle \text{grad } P(\xi_0), \eta\rangle=0$. Thus $\Lambda_{\mathbf{C}}(P_{\xi_0})$ is the complex line generated by grad $P(\xi_0)$. Let v be the vector field $\xi\to\text{grad } P(\xi)/|\text{grad } P(\xi)|$. Then $\Gamma(P_{\xi_0}, v(\xi_0))=\{\eta\in\mathbf{R}^n;\ \langle\text{grad } P(\xi_0), \eta\rangle>0\}$ and $K(P_{\xi_0}, v(\xi_0))=\{t \text{ grad } P(\xi_0);\ t\geqslant 0\}$. We observe that $K(P_{\xi_0}, v(\xi_0))$ is half the real line $\text{Re }\Lambda_{\mathbf{C}}'(P_{\xi_0})=\Lambda_{\mathbf{C}}'(P_{\xi_0})\cap\mathbf{R}$ Finally $H(P, \mathbf{R}^n, v)=\cup\{-t \text{ grad } P(\xi);\ t\geqslant 0\}$, where the union is taken over those $\xi\in\dot{\mathbf{R}}^n$ which satisfies $P(\xi)=0$.

We now collect all the information about homogeneous locally hyperbolic polynomails, that will be needed later on, in a theorem. As usual, we denote the sum $\sum_{|\alpha|\geqslant 0}|P^{(\alpha)}(\zeta)|^2$ by $\tilde{P}(\zeta)^2$.

Theorem 3.1. *Let P be a homogeneous locally hyperbolic polynomial, $\Sigma\subset\mathbf{R}^n$ a closed cone and suppose that $v\in C^\infty V(P,\Sigma)$. Then*

$$H(P, \Sigma, v) \textit{ is contained in a proper conical subvariety of } \mathbf{R}^n. \tag{3.12}$$

Furthermore, there is, to every $x\notin H(P, \Sigma, v)$, an element $v_x\in C^\infty V(P,\Sigma)$ and a number $\delta_x>0$, such that

$$\langle x, v_x(\xi)\rangle\geqslant\delta_x, \textit{ when } \xi\in\dot{\Sigma} \tag{3.13}$$

$$\xi\in\Sigma,\ 1\leqslant s\leqslant\delta_x|\xi| \Rightarrow |P(\xi+isv_x(\xi))|\geqslant\delta_x\tilde{P}(\xi+isv_x(\xi)) \tag{3.14}$$

$$\xi\in\Sigma,\ \ |\xi|\geqslant\delta_x^{-1},\ 0\leqslant t\leqslant 1\Rightarrow|P(\xi+i(tv_x(\xi)+(1-t)v(\xi)))|\geqslant\delta_x\tilde{P}(\xi+i(tv_x(\xi)+(1-t)v(\xi))). \tag{3.15}$$

Proof. Suppose $\zeta\in\Gamma(P_\xi, v(\xi))$. Then $\zeta+\eta\in\Gamma(P_\xi, v(\xi))$ for all $\eta\in\text{Re }\Lambda_{\mathbf{C}}(P_\xi)$. In fact $P_\xi(\zeta+t\eta)=P_\xi(\zeta)\neq 0$ for all t, so ζ and $\zeta+\eta$ must belong to the same component of $\{\theta\in\mathbf{R}^n;\ P_\xi(\theta)\neq 0\}$. Thus $\langle x, \zeta+\eta\rangle\geqslant 0$, when $x\in K(P_\xi, v(\xi))$. By letting ζ tend to zero, we get that $\langle x, \eta\rangle\geqslant 0$ and consequently, since $\Lambda_{\mathbf{C}}(P_\xi)$ is a linear space, that $\langle x, \eta\rangle=0$, for all $\eta\in\text{Re }\Lambda_{\mathbf{C}}(P_\xi)$. This gives that $K(P_\xi, v(\xi))\subset(\text{Re }\Lambda_{\mathbf{C}}(P_\xi))'$. Since P_ξ is a polynomial with real coefficients, modulo multiplication with a complex constant, $P_\xi(\zeta)=P_\xi(\zeta+t\eta)$, for real ζ and t, if and only if $P_\xi(\zeta)=P_\xi(\zeta+t\bar{\eta})$. From this it follows that $(\text{Re }\Lambda_{\mathbf{C}}(P_\xi))'=\text{Re }\Lambda_{\mathbf{C}}'(P_\xi)$. Thus $K(P_\xi, v(\xi))\subset\Lambda_{\mathbf{C}}'(P_\xi)$ and (3.12) follows from the proposition above.

To prove the remaining statements of the theorem, we observe that, if $\xi_0\in\dot{\Sigma}$, then $\Gamma(P_\xi, v(\xi))=\Gamma(P_\xi, v(\xi_0))$ when ξ belongs to a conical neighborhood of ξ_0. In fact, because of the continuity of v, $v(\xi)$ varies in a compact subset of $\Gamma(P_{\xi_0}, v(\xi_0))$ when ξ is close to ξ_0. Therefore, because of Lemma 3.4, $v(\xi)\in\Gamma(P_\xi, v(\xi_0))$ when ξ is close to ξ_0. From the definition of $H(P, \Sigma, v)$ it follows that to every $x\notin H(P, \Sigma, v)$ and to every

$\xi_0 \in \dot{\Sigma}$ there is an $\eta \in \Gamma(P_{\xi_0}, v(\xi_0))$ with $\langle x, \eta \rangle > 0$. Again according to Lemma 3.4, η belongs to $\Gamma(P_\xi, v(\xi_0))$, and thus to $\Gamma(P_\xi, v(\xi))$, when ξ belongs to some conical neighborhood of ξ_0. Cover now $\dot{\Sigma}$ with a finite number of such neighborhoods and glue the corresponding η_i:s together with a partition of unity, subordinate to this covering, homogeneous of degree zero. Because the cones $\Gamma(P_\xi, v(\xi))$ are convex we get in this way a vector field $v_x(\xi)$ such that $v_x(\xi) \in \Gamma(P_\xi, v(\xi))$, $\forall \xi \in \dot{\Sigma}$. Lemma 3.3 gives that, if the η_i:s are chosen small enough, then the distance from $\{\xi + i\sigma v_x(\xi);\ \xi \in \Sigma \cap S^{n-1}\}$ to $Z(P)$ is bounded from below by $C \cdot \sigma$. (3.14) now follows from Lemma 4.1.1 in [5] and the homogeneity of P. It is obvious, from the construction of v_x, that (3.13) is satisfied. (3.15) follows, exactly as (3.14), from Lemma 4.1.1 in [5], since, due to the convexity of $\Gamma(P_\xi, v(\xi))$, the distance from $\xi + i(tv_x(\xi) + (1-t)v(\xi_0))$ to $Z(P)$ is bounded from below if $\xi \in \sum$ and $|\xi|$ is large enough.

Remarks about the non-homogeneous case. If P is a non-homogeneous polynomial of degree m, we define the localization of P at ξ as the first not identically vanishing coefficient $P_\xi(\zeta)$ in the expansion

$$\tau^m P(\tau^{-1}\xi + \zeta) = \tau^p P_\xi(\zeta) + 0(\tau^{p+1})$$

$p = m_\xi(P)$ is called the multiplicity of P at ξ.

In the homogeneous case, this definition obviously coincides with the one we have given before.

As usual, we call the polynomial Q weaker than P and write $Q \prec P$ if $\tilde{Q}(\xi) \leqslant C \cdot \tilde{P}(\xi)$ for all $\xi \in \mathbf{R}^n$.

Lemma 3.5. *If the polynomial P is weaker than its principal part P_m, then the principal part of the localization P_ξ of P at $\xi \in \dot{\mathbf{R}}^n$ is the localization $(P_m)_\xi$ of P_m at ξ and $P_\xi \prec (P_m)_\xi$.*

Proof. Write $P = P_m + P_{m-1} + \ldots$, where P_k is homogeneous of degree k, and put $p_k = m_\xi(P_k)$. Then

$$\tau^m P(\tau^{-1}\xi + \eta) = \sum_k \tau^{m-k+p_k}((P_k)_\xi(\eta) + 0(\tau))$$

Since

$$\tau^m \tilde{P}_m(\tau^{-1}\xi + \eta) = \tau^{p_m}((\tilde{P}_m)_\xi(\eta) + 0(\tau))$$

and $P \prec P_m$ it follows that $p_m \leqslant m - k + p_k$ for all k. In fact

$$|P(\tau^{-1}\xi + \eta)| \leqslant \text{constant} \cdot \tilde{P}(\tau^{-1}\xi + \eta), \quad \forall\, \tau, \eta.$$

This proves that $(P_m)_\xi$ is the principal part of P_ξ. The rest of the lemma follows by letting τ tend to zero in

$$\tau^{m-p_m}\tilde{P}(\tau^{-1}\xi + \eta) \leqslant C\tau^{m-p_m}\tilde{P}_m(\tau^{-1}\xi + \eta)$$

Corollary 3.2. *If $P \prec P_m$, $P_m \in Hyp_{loc}(v)$ and $\xi_0 \in \dot{\mathbf{R}}^n$, then P_ξ is hyperbolic with respect to $v(\xi_0)$, when ξ is close to ξ_0.*

Proof. If Q is a polynomial of degree k such that $Q_k \in \mathrm{Hyp}\,(N)$ and $Q \prec Q_k$, then, according to Theorem 5.5.7 in [5], Q is hyperbolic with respect to N. Now the corollary follows directly from the lemma and Corollary 3.1.

Remark. For polynomials satisfying the hypothesis of this corollary one may thus define the local cones $\Gamma(P_\xi, v(\xi))$ and their duals $K(P_\xi, v(\xi))$.

We now formulate a theorem corresponding to Theorem 3.1.

Theorem 3.1′. *Let $\Sigma \subset \mathbf{R}^n$ be a closed cone and suppose that P is a polynomial such that $P \prec P_m$ and P_m is locally hyperbolic. If $v \in C^\infty V(P_m, \Sigma)$, then there is, to every $x \notin H(P_m, \Sigma, v)$, an element $w_x \in C^\infty V(P_m, \Sigma)$ and numbers r, $\delta_x > 0$ such that*

$$\langle x, w_x(\xi)\rangle \geq \delta_x |w_x(\xi)| \text{ when } \xi \in \dot{\Sigma} \tag{3.13′}$$

$$\xi \in \Sigma,\ 1 \leq s \leq \delta_x |\xi| \Rightarrow |P(\xi + isw_x(\xi))| \geq \delta_x \tag{3.14′}$$

$$\xi \in \Sigma,\ |\xi| \geq \delta_x^{-1},\ 0 \leq t \leq 1 \Rightarrow |P(\xi + i(tw_x(\xi) + (1-t)rv(\xi)))| \geq \delta_x. \tag{3.15′}$$

Proof. From Taylor's formula, it follows that, for any polynomial Q of degree k,

$$\tilde{Q}(\xi+\eta) \leq (1 + C|\eta|)^k \tilde{Q}(\xi),\ \xi,\ \eta \in \mathbf{C}^n. \tag{3.16}$$

Let $v_x(\xi)$ be the vector field corresponding to P_m in Theorem 3.1. According to (3.14), with $s=1$, we have

$$\delta_x \tilde{P}_m(\xi + iv_x(\xi)) \leq |P_m(\xi + iv_x(\xi))|,\ \xi \in \Sigma \text{ and } 1 \leq \delta_x |\xi|. \tag{3.17}$$

Because of Lemma 5.5.2 in [5] $P_k \prec P_m$, $0 \leq k < m$. Thus (3.16) and (3.17) imply that

$$|P_k(\xi + iv_x(\xi))| \leq C_x |P_m(\xi + iv_x(\xi))|,\ \xi \in \Sigma \text{ and } 1 \leq \delta_x |\xi|.$$

Finally, if we replace ξ by ξ/s and utilize the homogeneity of P_k, P_m and v_x, we get

$$|P_k(\xi + isv_x(\xi))| \leq C_x s^{k-m} |P_m(\xi + isv_x(\xi))|,\ \xi \in \Sigma \text{ and } s \leq \delta_x |\xi|.$$

Thus

$$|P_k(\xi + isrv_x(\xi)| \leq C_x r^{k-m} |P_m(\xi + isrv_x(\xi)|, \text{ when } \xi \in \Sigma \text{ and } 1 \leq s \leq \delta_x r^{-1} |\xi|.$$

If we choose r large enough, denote $\delta_x r^{-1}$ by δ_x and put $w_x(\xi) = rv_x(\xi)$, (3,14′) is proved. (3.13′) is obvious and (3.15′) follows from (3.15) in exactly the same way as (3.14′) follows from (3.14).

Examples

1. If P is a polynomial hyperbolic with respect N, then $P_m \in \mathrm{Hyp}\,(N)$ and $P \prec P_m$. This was proved by Leif Svensson in [7].

2. If P is an polynomial such that $P_m(\xi)$ is real, when $\xi \in \mathbf{R}^n$, and grad $P_m(\xi) \neq 0$, when $P_m(\xi) = 0$ and $\xi \in \dot{\mathbf{R}}^n$, then we know, from example 2 in the beginning of this section, that $P_m \in \mathrm{Hyp}_{\mathrm{loc}}(v)$ for vectorfields $v \in V$, arbitrary close to the vectorfield $\xi \to \operatorname{grad} P_m(\xi)/|\operatorname{grad} P_m(\xi)|$. Furthermore, it is obvious that $P \prec P_m$.

4. Fundamental solutions and solutions with singularities concentrated on a local cone $\pm\mathbf{K}(\mathbf{P}_\xi, \mathbf{v}(\xi))$

Let P be a polynomial such that $P \prec P_m$ and P_m is locally hyperbolic. Suppose we are given a covering $\mathcal{D} = \{\Delta^l\}_{l=1}^p$ of $\dot{\mathbf{R}}^n$ with open cones and suppose that we also have a set of vector fields $\mathcal{V} = \{v^l\}_{l=1}^p$, such that $v^l \in C^\infty V(P_m, \Delta^l)$. We then put $\mathcal{H}(\mathcal{D}, \mathcal{V}) = \bigcup_{1 \leq l \leq p} H(P_m, \bar{\Delta}^l, v^l)$. In this section we are going to construct a fundamental solution E of $P(D)$ which is analytc outside $H(\mathcal{D}, \mathcal{V})$ by, roughly speaking, putting

$$E(y) = \sum_{l=1}^{p} (2\pi)^{-n} \int_{\zeta = \xi + irv^l(\xi)} e^{i\langle y, \zeta\rangle} P(\zeta)^{-1} \phi_l(\zeta)\, d\zeta, |\xi| \geq \delta^{-1},$$

where $\{\phi_l\}_{l=1}^p$ is a partition of unity with $\phi_l \in L(\Delta^l)$, subordinate to the covering $\mathcal{D}$ and extended to $\mathbf{C}^n$ by (2.3). $r = r^l$ is the number occurring in Theorem 3.1'. In fact, we will show that the distributions

$$E^l(y) = (2\pi)^{-n} \int_{\zeta = \xi + irv^l(\xi)} e^{i\langle y, \zeta\rangle} P(\zeta)^{-1} \phi_l(\zeta)\, d\zeta, |\xi| \geq \delta^{-1}$$

are analytic outside $H(P_m, \bar{\Delta}^l, v^l)$ by modifying the chain of integration. The presence of the non-holomorphic function ϕ_l will not cause any problem, because of the properties of ϕ_l listed in Lemma 2.2 In particular (2.5) shows that $\bar{\partial}\phi_l(\zeta)$ tends very rapidly to zero, when ζ tends to infinity in a complex cone around Δ^l.

We are also going to construct a distribution F with singularities on a single local cone $K(P_\xi, v(\xi))$, such that $P(D)F$ is analytic outside the origin. As mentioned in the introduction, this will be done by simple inserting a suitable factor $e^{-g(\zeta)}$ in the integrand defining $E^l(y)$.

To sum up, we will study integrals, interpreted in the distribution sense, of the form

$$\int_{\zeta = \xi + irv(\xi)} e^{i\langle y, \zeta\rangle} P(\zeta)^{-1} e^{-g(\zeta)} \phi(\zeta)\, d\zeta, |\xi| \geq \delta^{-1}$$

where $v \in C^\infty(P_m, \bar{\Delta})$, for some open cone $\Delta \subset \dot{\mathbf{R}}^n$, $\phi \in L(\Delta)$ and g belongs to any of the classes $M(\varrho, \Delta)$, which we are now going to define.

Defintion 4.1. Let $\Delta \subset \dot{\mathbf{R}}^n$ be an open cone and ϱ a non-negative function, positively homogeneous of degree one and continuous on a neighborhood of $\bar{\Delta} \setminus \{0\}$. Then $M(\Delta, \varrho)$ is defined as the set of functions g, such that, for some constant c,

When $|\zeta| \geq c$, then $\zeta \to g(\zeta)$ is holomorphic in some open cone Δ' in $\mathbf{C}^n$ containing Δ

and
$$|D^\alpha g(\zeta)| \leq c_\alpha (1 + |\zeta|)^{1-|\alpha|}, \forall \alpha. \tag{4.1}$$

$$\operatorname{Re} g(\xi) \geq \varrho(\xi)^k |\xi|^{1-k} - c, \text{ for some } k > 0, \text{ when } \xi \in \Delta' \cap \mathbf{R}^n. \tag{4.2}$$

$$\operatorname{Re} g(\xi + i\eta) \geq -c|\eta|(|\eta| + \varrho(\xi))|\xi|^{-1} - c, \text{ when } \xi + i\eta \in \Delta', \tag{4.3}$$

Remark. If $g(\xi)$ is real when $\xi \in \Delta$, then g automatically satisfies (4.3) if it satisfies the rest of the definition. In fact, Taylor expansion gives

$$g(\xi+i\eta)=g(\xi)+i\langle \operatorname{grad} g(\xi),\eta\rangle+|\eta|^2r(\xi,\eta),$$

where, because of (4.1), $|r(\xi,\eta)|\leqslant c|\xi|^{-1}$, if $|\eta|$ is less than some constant times $|\xi|$.

Examples

1. $0\in M(\Delta,0)$, for any open cone $\Delta\subset\dot{\mathbf{R}}^n$.

2. Let Δ be an open cone such that $\bar\Delta\setminus\{0\}$ is contained in the open half-space defined by $\xi_1>0$. Put $\tilde\zeta=(0,\zeta_2,\ldots,\zeta_n)$ and $\varrho(\xi)=|\tilde\xi|$. Then $g(\zeta)=\langle\tilde\zeta,\tilde\zeta\rangle\cdot\zeta^{-1}$ is holomorphic in an open cone Δ' in $\mathbf{C}^n$ containing $\bar\Delta\setminus\{0\}$. (4.2) is satisfied with $k=2$ so, according to the remark above, $g\in M(\Delta,\varrho)$.

3. Let g, ϱ, Δ and Δ' be as in Example 2 and suppose that ψ is a positively homogeneous function of degree one, holomorphic in Δ'. If, in addition, ψ is realvalued in Δ and grad $\psi(\xi)=0$, when $\tilde\xi=0$, then $(g+i\psi)\in M(\Delta,\varrho)$. Obviously, we only have to verify that $i\psi$ satisfies (4.3). This follows from the Taylor expansion of ψ around a point $\xi\in\Delta$:

$$i\psi(\xi+i\eta)=i\psi(\xi)-\langle \operatorname{grad}\psi(\xi),\eta\rangle+|\eta|^2r(\xi,\eta).$$

Since ψ is positively homogeneous of degree one and grad $\psi(\xi)=0$, when $\tilde\xi=0$, it follows that $|\operatorname{grad}\psi(\xi)|\leqslant c|\tilde\xi|\,|\xi|^{-1}$ in Δ and that $|r(\xi,\eta)|\leqslant c|\xi|^{-1}$, when $\xi+i\eta\in\Delta$. Because $\psi(\xi)$ is real in Δ, this proves (4.3).

Suppose we are given a polynomial P, an open cone $\Delta\subset\dot{\mathbf{R}}^n$ and two functions ϕ, g, such that $\phi\in L(\Delta)$ and $g\in M(\Delta,\varrho)$, for some ϱ. If $v\colon\Delta\to\mathbf{R}^n$ is a C^∞ vector field, homogeneous of degree zero, such tht for some $\delta>0$

$$\xi\in\Delta,\ |\xi|\geqslant\delta^{-1}\Rightarrow|P(\xi+iv(\xi))|\geqslant\delta \tag{4.4}$$

then we may define the following distribution

$$E(P,\Delta,v,\phi,g)(u)=\int_{\zeta=\xi+iv(\xi)}\tilde u(\zeta)\,P(\zeta)^{-1}e^{-g(\zeta)}\phi(\zeta)\,d\zeta,\ |\xi|\geqslant\delta^{-1}. \tag{4.5}$$

Here $u\in C_0^\infty(\mathbf{R}^n)$, $\tilde u(\zeta)=(2\pi)^{-n}\int_{\mathbf{R}^n}e^{i\langle z,\zeta\rangle}u(z)\,dz$ and $d\zeta=d\zeta_1\wedge\ldots\wedge d\zeta_n$.

Lemma 4.1. *Let $v\colon[0,1]\times\Delta\ni(t,\xi)\to v(t,\xi)\in\mathbf{R}^n$ be a C^∞ vector field, homogeneous of degree zero with respect to ξ, such that for some $d>0$*

$$\xi\in\Delta,\ |\xi|\geqslant d^{-1},\ 0\leqslant t\leqslant 1\Rightarrow P(\xi+iv(t,\xi))\geqslant d$$

and put $E_k=E(P,\Delta,v(k,\cdot),\phi,g)$, $k=0,1$. Then E_0-E_1 is an analytic function in all of $\mathbf{R}^n$.

Proof. Define, for large R, the following chains

$\alpha_k(R)\colon\zeta=\xi+iv(k,\xi),\ d^{-1}\leqslant|\xi|\leqslant R,\ \xi\in\Delta,\ k=0,1.$

$B(R)\colon\zeta=\xi+iv(t,\xi),\ 0\leqslant t\leqslant 1,\ d^{-1}\leqslant|\xi|\leqslant R,\ \xi\in\Delta.$

$\beta(R)\colon\zeta=\xi+iv(t,\xi),\ 0\leqslant t\leqslant 1,\ |\xi|=R,\ \xi\in\Delta.$

We put $\omega(\zeta)=\tilde u(\zeta)P(\zeta)^{-1}e^{-g(\zeta)}\phi(\zeta)\,d\zeta$.

Then, if the chains are suitably oriented, we have, according to (3.15′) and Stokes' formula,

$$\int_{\alpha_0(R)} \omega(\zeta) - \int_{\alpha_1(R)} \omega(\zeta) = \int_\gamma \omega(\zeta) + \int_{B(R)} \bar\partial\omega(\zeta) + \int_{\beta(R)} \omega(\zeta), \tag{4.6}$$

where γ is a compact chain, in $\mathbf{C}^n$, independent of R. Now $\tilde u(\zeta)$ tends to zero faster than any power of $|\zeta|^{-1}$, when $|\zeta|$ tends to infinity and $|\operatorname{Im}\zeta|$ remains bounded. Furthermore, it follows from (4.1) and (4.2), by means of Taylor's formula, that $\operatorname{Re} g(\zeta)$ is bounded from below when $|\operatorname{Im}\zeta|$ is bounded. These remarks, together with (2.5), give that we may let R tend to infinity in (4.6). We get

$$E_0(u) - E_1(u) = \int_\gamma \tilde u(\zeta)\, P(\zeta)^{-1} e^{-g(\zeta)} \phi(\zeta)\, d\zeta + \int_{B(\infty)} \tilde u(\zeta)\, P(\zeta)^{-1} e^{-g(\zeta)} \bar\partial\phi(\zeta) \wedge d\zeta.$$

Because of (2.5) and the compactness of γ, we may change the order of integration in the two integrals on the right-hand side, thus getting

$$E_0(u) - E_1(u) = \int_{\mathbf{R}^n} f(z)\, u(z)\, dz,$$

where

$$f(z) = (2\pi)^{-n} \int_\gamma e^{i\langle z,\zeta\rangle} P(\zeta)^{-1} e^{-g(\zeta)} \phi(\zeta)\, d\zeta + (2\pi)^{-n} \int_{B(\infty)} e^{i\langle z,\zeta\rangle} P(\zeta)^{-1} e^{g(\zeta)} \bar\partial\phi(\zeta) \wedge d\zeta.$$

It now follows, again from (2.5) and the compactness of γ, that $f(z)$ is holomorphic when $|\operatorname{Im} z| < 1$.

We are now going to state the main theorem of this paper. Because it is fairly technical, we will deduce some corollaries from it before we give the proof and also give some examples, hoping that this will facilate the reading. First of all we must, unfortunatey, introduce some more notation.

Let P be a polynomial, such that $P \prec P_m$ and P_m is locally hyperbolic, and let $\Delta \subset \dot{\mathbf{R}}^n$ be an open cone. Suppose further that we are given $v \in C^\infty V(P_m, \bar\Delta)$, $\phi \in L(\Delta)$ and $g \in M(\Delta, \varrho)$, for some ϱ. Because of (3.15′), the distribution $E(P, \Delta, rv, \phi, g)$ is defined if r is large enough. We put

$$H(P_m, \bar\Delta, v, \varrho) = -\bigcup_{\xi\in\Delta(\varrho)} K(P_\xi, v(\xi)), \quad \text{where} \quad \Delta(\varrho) = \{\xi \in \bar\Delta \setminus \{0\}; \varrho(\xi) = 0\}.$$

Obviously $H(P_m, \bar\Delta, v, 0) = H(P_m, \bar\Delta, v)$. If T is a distribution, s.T, a.s.T and s.s.T denote the smallest closed set outside which T vanishes, is analytic or is an infinitely differentiable function, respectively. Finally, $E(P_\xi, v(\xi))$ is the unique fundamental solution of $P_\xi(D)$ with support in the half-space defined by $\langle x, v(\xi)\rangle \leqslant 0$ (see Theorem 5.6.1 in [5]). Note that s.E$(P_\xi, v(\xi)) \subset -K(P_\xi, v(\xi))$.

Theorem 4.1. a.s.$E(P, \Delta, rv, \phi, g) \subset H(P_m, \bar\Delta, v, \varrho)$ *for large* r.

Corollary 4.1. *Let* $\mathcal{D} = \{\Delta^l\}_{l=1}^p$ *be a covering of* $\dot{\mathbf{R}}^n$ *with open cones and let* $\mathcal{V} = \{v^l\}_{l=1}^p$ *be a set of vector fields, such that* $v^l \in C^\infty V(P_m, \Delta^l)$. *Then there is a fundamental solution* E *of* $P(D)$ *with*

$$\text{a.s.}E \subset \mathcal{H}(\mathcal{D}, \mathcal{V}).$$

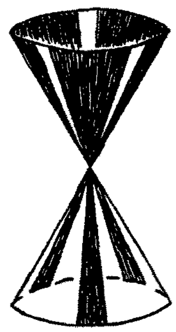

Fig. 1. The conoid generated by all the bicharacteristic lines for $P(D)=D_1^2-c^2(D_2^2+D_3^2)$. There is a fundamental solution E of $P(D)$ which is analytic outside the shaded parts of the conoid.

Example. If P is hyperbolic with respect to N, then we may take $\mathcal{D}=\{\dot{\mathbf{R}}^n\}$ and $\mathcal{V}=\{v\}$, where v is the constant vector field $\xi\to -N$. Then Corollary 4.1 says that P has a fundamental solution, E, with a.s. $E\subset W(P,N)$, where $W(P,N)$ is the wave front surface of P with respect to N (see [2] for the definition). If we cover $\dot{\mathbf{R}}^n$ by two cones $\mathcal{D}=\{\Delta^+,\Delta^-\}$ and choose v^+ to be $\xi\to -N$, in Δ^+, and v^- to be $\xi\to N$ in Δ^-, then $\mathcal{H}(\mathcal{D},\mathcal{V})$ becomes a subset of $W(P,N)\cup W(P,-N)$. If, for example, $P(D)$ is the wave operator $D_1^2-c^2(D_2^2+\ldots+D_n^2)$, $N=(1,0,\ldots,0)$ and $J_1, J_2\subset W(P,N)$ are two relatively open cones covering $W(P,N)\setminus\{0\}$, then we may in this way construct a fundamental solution E of $P(D)$, such that a.s. $E\subset J_1\cup(-J_2)$ (see Fig. 1). Constructions of this type will be used in Section 5 for general operators $P(D)$ of principal type, i.e. grad $P_m(\xi)\neq 0$ when $P_m(\xi)=0$ and $\xi\in\dot{\mathbf{R}}^n$, with real coefficients in the principal part.

Corollary 4.2. *Let $\Delta\subset\dot{\mathbf{R}}^n$ be an open cone containing ξ_0 and suppose that $v\in C^\infty V(P_m,\bar{\Delta})$. Then there is a distribution $F(v(\xi_0))$ such that*

$$\text{a.s. } P(D)F(v(\xi_0))=\{0\},\ \text{a.s. } F(v(\xi_0))\subset -K(P_{\xi_0},v(\xi_0)),\ \text{s.}E(P_{\xi_0},v(\xi_0))\subset\text{s.s. } F(v(\xi_0)).$$

Furthermore, there is a function f, analytic in all of $\mathbf{R}^n$, such that

$$P(D)\ (F(v(\xi_0))-F(-v(\xi_0))-f)=0.$$

Example. Suppose that $P(D)$ is of principal type, $P_m(D)$ has real coefficients and that $v(\xi)=\operatorname{grad}P_m(\xi)/|\operatorname{grad}P_m(\xi)|$. Then $K(P_\xi,v(\xi))=\{0\}$ if $P_m(\xi)\neq 0$ and $K(P_\xi,v(\xi))$ is the half ray $\{t\operatorname{grad}P_m(\xi);\,t\geqslant 0\}$ if $P_m(\xi)=0$, $\xi\in\dot{\mathbf{R}}^n$. Such a half ray is called a bicharacteristic half ray and the corresponding line a bicharacteristic line. If b_ξ is a bicharacteristic half ray, and we denote by x' coordinates on this line and by x'' coordinates in the hyperplane perpendicular to b_ξ, then $E(P_\xi,\ v(\xi))=ce^{ax'}\theta(x')\otimes\delta(x'')$, where $\theta(x')=0$ on b_ξ, $\theta(x')=-1$ on $-b_\xi$ and δ is the Dirac measure. Thus, according to Corollary 4.2, there is a distribution F^- such that a.s. $F^-=$s.s. $F^-=-b_\xi$ and a.s. $P(D)F^-=\{0\}$. Furthermore, for any bicharacteristic line l, there is a solution F of $P(D)F=0$ such that a.s. $F=$s.s. $F=l$. In the special case treated in this example, the following corollary gives still more information.

Corollary 4.3. *If $P(D)$ is of principal type and $P_m(D)$ has real coefficients, then, to any closed interval I (finite or infinite) contained in a bicharacteristic line l for $P(D)$, there is* *a distribution F such that a.s.F = s.s.$F = I$ and $P(D)F$ is analytic except at the (finite) endpoints of I.*

Proof of Corollary 4.1. According to Lemma 2.1, there are functions $\phi_l \in L(\Delta^l)$, such that $\sum_1^p \phi_l(\xi)=1$, when $|\xi|\geqslant 1$. Put

$$F=\sum_{l=1}^{p} E^l, \quad \text{where} \quad E^l = E(P, \Delta^l, r^l v^l, \phi_l, 0).$$

Theorem 4.1 now tells us that a.s.$F \subset \mathcal{H}(\mathcal{D}, \mathcal{V})$. We are going to show that F only differs from a fundamental solution of $P(D)$ by an analytic function. If the operator $P(D)$ is applied to E^l, we get

$$(P(D)E^l)(u) = \int_{\zeta=\xi+ir^l v^l(\xi)} \tilde{u}(\zeta)\phi_l(\zeta)\,d\zeta, \ |\xi|\geqslant \delta^{-1}$$

Because $\{\phi_l\}_{l=1}^p$ is a partition of unity, when $|\xi|\geqslant 1$, we obtain, by means of Stokes' formula,

$$(P(D)\,F)\,(u) = \int_{\mathbf{R}^n} \tilde{u}(\xi)\,d\xi - \int_{|\xi|\leqslant 1} \tilde{u}(\xi)\left(1-\sum_{l=1}^{p}\phi_l(\zeta)\right)d\xi$$
$$+\sum_{l=1}^{p}\int_{\gamma_l}\tilde{u}(\zeta)\phi_l(\zeta)\,d\zeta + \sum_{l=1}^{p}\int_{B_l}\tilde{u}(\zeta)\bar{\partial}\phi_l(\zeta)\wedge d\zeta,$$

where γ_l is a compact chain in $\mathbf{C}^n$ and B_l is the chain given by $\zeta=\xi+itr^l v^l(\xi)$, when $0\leqslant t\leqslant 1$ and $|\xi|\geqslant\delta^{-1}$.

The first term equals $u(0)$, by the Fourier inversion formula, and, exactly as in the proof of Lemma 4.1, it follows, from (2.5) and the compactness of γ_l, that the remaining terms correspond to a function $h(z)$, holomorphic when $|\operatorname{Im} z|<1$. In conclusion, we have $P(D)F=\delta+h$, where h is holomorphic when $|\operatorname{Im} z|<1$. Solve $P(D)f=h$, with f holomorphic when $|\operatorname{Im} z|<1$ (see e.g. Trèves [9] p. 477). Then $E=F-f$ is a fundamental solution of $P(D)$ with the desired properties.

Proof of Corollary 4.2. By choosing coordinates suitably, we may suppose that $\xi_0=(1, 0, ..., 0)$. We may also take Δ such that $\Delta\setminus\{0\}$ is contained in the half space $\{\xi; \langle\xi, \xi_0\rangle>0\}$. Define ϱ and g as in Example 2, following Definition 4.1, i.e. $\varrho(\xi)=|\tilde{\xi}|$ and $g(\zeta)=\langle\tilde{\zeta}, \tilde{\zeta}\rangle\cdot\zeta_1^{-1}$. Obviously $H(P_m, \tilde{\Delta}, v, \varrho) = -K(P_{\xi_0}, v(\xi_0))$. According to Lemma 2.1, there is a $\phi\in L(\Delta)$ which equals 1 in an open cone containing ξ_0. Put

$$F(v(\xi_0)) = E(P, \Delta, rv, \phi, g).$$

Theorem 4.1 gives that a.s.$F(v(\xi_0))\subset -K(P_{\xi_0}, v(\xi_0))$. Furthermore, $P(D)F(v(\xi_0)) = E(1, \Delta, rv, \phi, g)$. Since $H(1, \tilde{\Delta}, v, \varrho)=\{0\}$, it also follows from Theorem 4.1 that a.s.$P(D)F(v(\xi_0))\subset\{0\}$. Next we define the following chains

$$\alpha^k\colon \zeta=\xi+(-1)^k irv(\xi), \ |\xi|\geqslant\delta^{-1}, \ k=1,2.$$

$$B\colon \zeta=\xi+(2t-1)irv(\xi), \ 0\leqslant t\leqslant 1, \ |\xi|\geqslant\delta^{-1}.$$

If the orientations are chosen properly, Stokes' formula gives

$$\begin{aligned}(P(D)\,F(v(\xi_0)))\,(u)-(P(D)\,F(-v(\xi_0)))\,(u)\\ &=\int_{\alpha^2}\tilde{u}(\zeta)\,e^{-g(\zeta)}\phi(\xi)\,d\zeta-\int_{\alpha^1}\tilde{u}(\zeta)\,e^{-g(\zeta)}\phi(\zeta)\,d\zeta\\ &=\int_{\gamma}\tilde{u}(\zeta)\,e^{-g(\zeta)}\phi(\zeta)\,d\zeta+\int_{B}\tilde{u}(\zeta)\,e^{-g(\zeta)}\bar{\partial}\phi(\zeta)\wedge d\zeta,\end{aligned}$$

where γ is a compact chain in $\mathbf{C}^n$. Just as before in the proofs of Lemma 4.1 and Corollary 4.1, it follows, from (2.5) and the compactness of γ, that the right-hand side corresponds to a function $h(z)$, holomorphic when $|\operatorname{Im} z|<1$. As in the proof of Corollary 4.1, we solve $P(D)f=h$ in order to find a function $f(z)$, holomorphic when $|\operatorname{Im} z|<1$. It only remains to prove that $\text{s.}E(P_{\xi_0}, v(\xi_0))\subset \text{s.s.}F(v(\xi_0))$ and that $0\in \text{s.s.}P(D)\,F(v(\xi_0))$. We will do this by utilizing a technique used by Hörmander in [6].

If m is the degree of P and $p=m_{\xi_0}(P)$, we put

$$F_\tau(y)=\tau^{m-p}\,e^{-i\tau\langle y,\,\xi_0\rangle}F(v(\xi_0))\,(y).$$

After a change of variables of integration, we have

$$F_\tau(u)=\int_{\zeta=\xi+ir v(\xi+\tau\xi_0)}\tilde{u}(\zeta)\,\tau^{m-p}P(\zeta+\tau\xi_0)^{-1}\,e^{-g(\zeta+\tau\xi_0)}\phi(\zeta+\tau\xi_0)\,d\zeta,\ |\xi+\tau\xi_0|\geqslant\delta^{-1}.$$

Obviously, because $g(\zeta)=\langle\tilde{\zeta},\tilde{\zeta}\rangle\cdot\zeta_1^{-1}$,

$$F_\tau(u)\to\int_{\zeta=\xi+irv(\xi_0)}\tilde{u}(\zeta)\,P_{\xi_0}(\zeta)\,d\zeta,\quad\text{when}\quad\tau\to+\infty.$$

This, however, is just the definition of $E(P_{\xi_0}, v(\xi_0))$ (see the proof of Theorem 5.6.1 in [5]). Now it only remains to observe that, if $F(v(\xi_0))\in C^\infty(U)$ for some open set U, then $F_\tau\to 0$, in the distribution sense, in U. That $0\in\text{s.s.}P(D)\,F(v(\xi_0))$ is proved in the same way.

Proof of Corollary 4.3. Let the bicharacteristic line l be generated by grad $P_m(\xi_0)$, where $\xi_0\in\dot{\mathbf{R}}^n$ and satisfies $P_m(\xi_0)=0$. We choose our coordinates so that $\xi_0=(1, 0, \ldots, 0)$ and grad $P_m(\xi_0)$ is a positive multiple of $e_n=(0, \ldots, 0, 1)$. From the example preceding Corollary 4.3 it follows that we only need to consider the case when I is a finite interval. Obviously, we may also suppose that the endpoints are the origin and $(0, \ldots, 0, a)$, for some $a>0$. Put $\zeta'=(\zeta_1, \ldots, \zeta_{n-1}, 0)$ and choose a conic neighborhood $\Delta\subset\dot{\mathbf{R}}^n$ of ξ_0 so small that $\bar{\Delta}\setminus\{0\}$ is contained in the half space defined by $\xi_1>0$ and so that, for some open, complex cone Δ' containing Δ, the equation $P_m(\zeta'+\psi(\zeta')\,e_n)=0$ has a unique solution ψ, holomorphic when $\zeta'\in\Delta'$, with $\psi(\xi_0)=0$. This is always possible because $\langle\text{grad } P_m(\xi_0), e_n\rangle\neq 0$. The function $\psi(\zeta)$ is homogeneous of degree one and grad $\psi(\xi_0)=0$. Since $P_m(\xi)$ is real when $\xi\in\mathbf{R}^n$ it is clear that $\psi(\xi)$ is real when $\xi\in\Delta$. If, as before, we put $g(\zeta)=\langle\tilde{\zeta},\tilde{\zeta}\rangle\cdot\zeta_1^{-1}$ and $\varrho(\xi)=|\xi|$, it follows from Example 3 above that $(g+ia\psi)\in M(\Delta,\varrho)$. Since $P_m(\zeta', \psi(\zeta'))=0$ and $\langle\text{grad } P_m(\zeta'), e_n\rangle\neq 0$, when $\zeta'\in\Delta'$, we have

$$P(\zeta' + (\psi(\zeta') + s)\,e_n) = \sum_{k=0}^{m} f_k(\zeta')\,s^k, \tag{4.7}$$

where $f_0(\zeta') = 0(|\zeta'|^{m-1})$, $f_k(\zeta') = 0(|\zeta'|^{m-k})$ when $0 < k \leqslant m$, $|f_1(\zeta')| \geqslant c_1|\zeta'|^{m-1} - c_2$ and $f_m(\zeta') = P_m(e_n) \neq 0$. Now it follows from the connection between zeros and coefficients of a polynomial that there are constants c_3 and c_4 such that, when $|\zeta'| \geqslant c_3$, there is a unique zero $s = \psi_1(\zeta')$ of (4.7) with $|\psi_1(\zeta')| < c_4$. In fact, because $f_k(\zeta') = 0(|\zeta'|^{m-k})$ and $f_m(\zeta') = P_m(e_n) \neq 0$, it is obvious that the zeros are bounded from above by $c_5|\zeta'| + c_6$, for some constants c_5 and c_6. This together with the fact that $|f_1(\zeta')| \geqslant c_1|\zeta'|^{m-1} - c_2$ gives the uniqueness of a bounded zero. Finally, the existence results from the estimates $f_0(\zeta') = 0(|\zeta'|^{m-1})$ and $|f_1(\zeta'| \geqslant c_1|\zeta'|^{m-1} - c_2$. Since $\langle \operatorname{grad} P(\zeta'), e_n\rangle \neq 0$, when ζ' is large, it follows that $\psi_1(\zeta')$ is holomorphic. Furthermore, by differentiating $P(\zeta', (\psi(\zeta') + \psi_1(\zeta'))e_n)$, one verifies that $|D^\alpha(\psi + \psi_1)\ (\zeta')| \leqslant c_\alpha(1 + |\zeta'|)^{1-|\alpha|}$. Because $(g + ia\psi) \in M(\Delta, \varrho)$, we conclude that $g_1 = g + ia(\psi + \psi_1) \in M(\Delta, \varrho)$.

We choose $v(\xi) = \operatorname{grad} P_m(\xi)/|\operatorname{grad} P_m(\xi)|$ and take a $\phi \in L(\Delta)$, which equals 1 in an open cone containing ξ_0. Put $F_0 = E(P, \Delta, rv, \phi, g)$ and $F_1 = E(P, \Delta, rv, \phi, g_1)$. From Theorem 4.1 we know that a.s.F_0 and a.s.F_1 are contained in the bicharacteristic half ray $-b_{\xi_0} = \{(0, \dots, 0, t);\ t \leqslant 0\}$. Furthermore, F_0 is the distribution $F(v(\xi_0))$ occurring in Corollary 4.2 (see the proof of that corollary). Thus a.s.$F_0 = -b_{\xi_0}$. We now define F by $F(y) = F_0(y - ae_n) - F_1(y)$, i.e.

$$F(u) = \int_{\zeta = \xi + irv(\xi)} \tilde{u}(\zeta)\,f(\zeta)\,\phi(\zeta)\,d\zeta,\ |\xi| \geqslant \delta^{-1}$$

where
$$f(\zeta) = (e^{-(g(\zeta) + ia\zeta_n)} - e^{-g_1(\zeta)})/P(\zeta).$$

From the remarks above about the analyticity of F_0 and F_1, we conclude that F is analytic except possibly at points of the form $(0, \dots, 0, t)$ with $t \leqslant a$. We also know that $I = \{(0, \dots, 0, t);\ 0 \leqslant t \leqslant a\}$ is contained in a.s.F. It remains to show that F is analytic on $-b_{\xi_0} \setminus \{0\}$. To do this, we put $G_0 = E(P, \Delta, -rv, \phi, g)$, $G_1 = E(P, \Delta, -rv, \phi, g_1)$ and $G(y) = G_0(y - ae_n) - G_1(y)$. Again, it follows from Theorem 4.1 that a.s.$G \subset b_{\xi_0}$. We are now going to show that $F - G$ is an analytic function in all of $\mathbf{R}^n$. This will, of course, prove that a.s.$F = I$. Once more, we define the chains

$$\alpha^k\colon \zeta = \xi + (-1)^k irv(\xi),\ |\xi| \geqslant \delta^{-1},\ k = 1, 2.$$

$$B\colon \zeta = \xi + (2t - 1)irv(\xi),\ 0 \leqslant t \leqslant 1,\ |\xi| \geqslant \delta^{-1}.$$

If δ is small enough, then $B \subset \Delta'$ and $f(\zeta)$ is holomorphic on B. In fact, if $\zeta \in \Delta'$ and $|\zeta|$ is large enough, then $P(\zeta)$ vanishes if and only if $\zeta = \zeta' + i(\psi(\zeta') + \psi_1(\zeta'))e_n$. Thus, when $P(\zeta)$ vanishes, we have $g(\zeta) + ia\zeta_n = g(\zeta) + ia(\psi(\zeta') + \psi_1(\zeta')) = g_1(\zeta)$. Since $P(\zeta)$ only vanishes to the first order in Δ', if $|\zeta|$ is large, this proves that $f(\zeta)$ is holomorphic on B. Stokes' formula gives now

$$F(u) - G(u) = \int_{\alpha^2} \tilde{u}(\zeta)\,f(\zeta)\,\phi(\zeta)\,d\zeta - \int_{\alpha^1} \tilde{u}(\zeta)\,f(\zeta)\,\phi(\zeta)\,d\zeta$$

$$= \int_{\gamma} \tilde{u}(\zeta)\,f(\zeta)\,\phi(\zeta)\,d\zeta + \int_B \tilde{u}(\zeta)\,f(\zeta)\,\bar{\partial}\phi(\zeta) \wedge d\zeta,$$

where γ is a compact chain. Again, (2.5) together with the compactness of γ implies that this corresponds to a function $h(z)$, holomorphic when $|\operatorname{Im} z| < 1$. This finishes the proof of Corollary 4.3.

Proof of Theorem 4.1. Since $v \in C^\infty(P_m, \bar{\Delta})$, there is an open cone Δ_1 containing Δ, such that $v \in C^\infty(P_m, \bar{\Delta}_1)$. Suppose that $x \notin H(P_m, \bar{\Delta}, v, \varrho)$. Then, because of the semi-continuity of the local cones $\Gamma(P_\xi, v(\xi))$, proved in Lemma 3.4, there is an open cone Δ_2, such that $\Delta(\varrho) = \{\xi \in \bar{\Delta} \setminus \{0\};\ \varrho(\xi) = 0\} \subset \Delta_2 \subset \Delta_1$ and $x \notin H(P_m, \bar{\Delta}_2, v)$. According to Theorem 3.1′ there is an element $w_x \in C^\infty V(P_m, \bar{\Delta}_2)$ satisfying (3.13′), (3.14′) and (3.15′). Let $\delta_x < 1$ be a constant corresponding to w_x in Theorem 3.1′ and denote by c' the constant c corresponding to ϕ in Lemma 2.2. Take ν so large that

$$5c' \, 2^{-\nu} \leqslant \delta_x. \tag{4.8}$$

When ν is chosen we take another constant $\varkappa < 1$ such that

$$\varkappa < \min\,(\delta_x/5c'', 2^{-\nu}/M) \tag{4.9}$$

where c'' is the constant c corresponding to g in Definition 4.1 and M is the constant in Lemma 2.2. Let now Δ_3 and Δ_4 be open cones such that

$$\Delta(\varrho) \subset \Delta_4 \subset \bar{\Delta}_4 \setminus \{0\} \subset \Delta_3 \subset \bar{\Delta}_3 \setminus \{0\} \subset \Delta_2$$

and

$$\varrho(\xi) \leqslant \varkappa |\xi|, \text{ when } \xi \in \Delta_3. \tag{4.10}$$

Furthermore, let $\psi \in C^\infty(\mathbf{R}^n)$ be a function homogeneous of degree zero such that $0 \leqslant \psi(\xi) \leqslant 1$ for all ξ, $\psi(\xi) = 1$ when $\xi \in \Delta_4$ and $\psi(\xi)$ vanishes outside Δ_3. When $\xi \in \Delta$ and $0 \leqslant t \leqslant 1$, we define the vector field $v_x(t, \xi)$ by

$$v_x(t, \xi) = (1 - t\psi(\xi))\, r v(\xi) + t\psi(\xi)\, w_x(\xi).$$

From (3.15′) it follows that

$$|P(\xi + i v_x(t, \xi))| \geqslant \text{constant} > 0, \text{ when } \xi \in \Delta, \ |\xi| > \delta_x^{-1} \text{ and } 0 \leqslant t \leqslant 1. \tag{4.11}$$

Finally, we define the distribution F_x by

$$F_x(u) = \int_{\zeta = \xi + i v_x(1, \xi)} \tilde{u}(\zeta)\, P(\zeta)^{-1} e^{-g(\zeta)} \phi(\zeta)\, d\zeta, \ |\xi| \geqslant \delta_x^{-1}.$$

Because of (4.11), it follows from Lemma 4.1 that $E - F_x$ is analytic in all of $\mathbf{R}^n$. It only remains to prove that F_x is analytic in a neighborhood of x. Define the vector-field $w_x(s, \xi)$ by

$$w_x(s, \xi) = (1 - \psi(\xi))\, r v(\xi) + s\psi(\xi)\, w_x(\xi)$$

and form the following chains

$B(R)$: $\zeta = \xi + i w_x(s, \xi)$, $\delta_x^{-1} \leqslant |\xi| \leqslant R$, $1 \leqslant s \leqslant \varkappa |\xi| \cdot |w_x(\xi)|^{-1}$, $\xi \in \Delta_2$

$\beta(R)$: $\zeta = \xi + i w_x(s, \xi)$, $\delta_x^{-1} \leqslant |\xi| \leqslant R$, $s = \varkappa |\xi|\, |w_x(\xi)|^{-1}$, $\xi \in \Delta_2$

$\gamma(R)$: $\zeta=\xi+iw_x(s,\xi)$, $|\xi|=R$, $1\leqslant s\leqslant \varkappa R|w_x(\xi)|^{-1}$, $\xi\in\Delta_2$

$\alpha(R)$: $\zeta=\xi+iw_x(1,\xi)$, $\delta_x^{-1}\leqslant|\xi|\leqslant R, \xi\in\Delta_2$.

Since $w_x(s,\xi)=rv(\xi)$ near the boundary of Δ_2, when $|\xi|\geqslant\delta_x^{-1}$, we define the chains also when $\xi\notin\Delta_2$ by simply replacing $w_x(s,\xi)$ by $rv(\xi)$.

If we put $f(\zeta)=\tilde{u}(\zeta)P(\zeta)^{-1}e^{-g(\zeta)}$, then Stokes' formula gives

$$\int_{\alpha(R)} f(\zeta)\,\phi(\zeta)\,d\zeta=\int_\gamma f(\zeta)\,\phi(\zeta)\,d\zeta+\int_{\gamma(R)} f(\zeta)\,\phi(\zeta)\,d\zeta+\int_{B(R)} f(\zeta)\,\bar{\partial}\phi(\zeta)\wedge d\zeta$$
$$+\int_{\beta(R)} f(\zeta)\,\phi(\zeta)\,d\zeta=I_1+I_2(R)+I_3(R)+I_4(R)$$

provided that the orientations are suitably chosen. Here γ is a compact chain, independent of R. Since $F_x(u)=\int_{\alpha(\infty)}f(\zeta)\phi(\zeta)d\zeta$, we are going to study the four integrals on the right-hand side, as R tends to infinity.

Study of I_1. I_1 is independent of R, and, since γ is compact, we may change the order of integration. We get $I_1=\int_{\mathbf{R}^n}h_1(z)u(z)\,dz$, where

$$h_1(z)=(2\pi)^{-n}\int_\gamma e^{i\langle z,\zeta\rangle}P(\zeta)^{-1}e^{-g(\zeta)}\phi(\zeta)\,d\zeta.$$

Obviously, h_1 is an entire function.

Before studying the other integrals, we notice that, if $u\in C_0^\infty(\mathbf{R}^n)$, then

$$|\tilde{u}(\zeta)|\leqslant c_p(1+|\zeta|)^{-p}e^{-h(\operatorname{Im}\zeta)},\tag{4.12}$$

where p is any integer and $h(\operatorname{Im}\zeta)=\min\langle y,\operatorname{Im}\zeta\rangle$, when y belongs to the support of u. We denote by Φ the set of points $\zeta\in\mathbf{C}^n$, such that ζ lies on any of the four chains above. Since we are only interested in $F_x(u)$ when the support of u is located close to x, we may suppose that $|x-y|\leqslant\delta_x/5$ when $y\in\operatorname{supp}u$. (3.13′) and the fact that $\operatorname{Im}\zeta$ is bounded when $\zeta\in\Phi$ and $\operatorname{Re}\zeta\notin\Delta_3$ implies then that

$$h(\eta)\geqslant 4|\eta|\delta_x/5-\text{some constant, when }\xi+i\eta\in\Phi.\tag{4.13}$$

Because of (4.10), it follows from (4.3) that

$$\operatorname{Re}g(\xi+i\eta)\geqslant -c''|\eta|(|\eta|\cdot|\xi|^{-1}+\varkappa)-c'',\text{ when }\xi\in\Delta_3.$$

Since $|\eta|\leqslant\varkappa|\xi|+\text{some constant}$, when $\xi+i\eta\in\Phi$, we get

$$\operatorname{Re}\ g(\xi+i\eta)\geqslant -2c''\varkappa|\eta|-\text{constant, when }\xi+i\eta\in\Phi\text{ and }\xi\in\Delta_3.\tag{4.14}$$

Now $\varkappa<\delta_x/5c''$, so it follows from (4.14) that

$$\operatorname{Re}g(\xi+i\eta)\geqslant -2\delta_x|\eta|/5-\text{constant, when }\xi+i\eta\in\Phi\text{ and }\xi\in\Delta_3.\tag{4.15}$$

Because $\varkappa<2^{-\nu}/M$, (2.6) together with (4.8) give that

$$|\phi(\xi+i\eta)|\leqslant M|\xi|^2e^{\delta_x|\eta|/5},\quad\text{when}\quad \xi+i\eta\in\Phi.\tag{4.16}$$

From (4.12), (4.13), (4.15), (4.16) and (3.14′) it follows that

$$|f(\zeta)\,\phi(\zeta)| \leqslant c_p(1+|\zeta|)^{-p}\,e^{-\delta_x|\mathrm{Im}\,\zeta|/5}, \quad \text{when } \zeta\in\Phi \text{ and } \mathrm{Re}\,\zeta\in\Delta_3. \tag{4.17}$$

Here the c_p:*s* are new constants. We now examine the remaining three integrals.

Study of $I_2(R)$. Since $\mathrm{Im}\,\zeta$ is bounded on Φ, when $\mathrm{Re}\,\zeta\notin\Delta_3$, it follows from (4.12) and (4.17) that

$$|f(\zeta)\phi(\zeta)| \leqslant c_p(1+R)^{-p}, \text{ when } \zeta \text{ lies on } \gamma(R).$$

Here p is arbitrary, so it follows that $I_2(R)\to 0$, when R tends to infinity.

Study of $I_3(R)$. Because $g(\xi+i\eta)=g(\xi)+|\eta|\,r(\xi,\eta)$, where $r(\xi,\eta)$ is bounded, (4.2) gives that $\mathrm{Re}\,g(\xi+i\eta)$ is bounded from below, when $\xi+i\eta\in\Phi$ and $\xi\notin\Delta_3$. In fact, $|\eta|$ is bounded on this set. When $\xi+i\eta\in\Phi$ and $\xi\in\Delta_3$, it follows from (4.14), since $|\eta|<\varkappa|\xi|$ + some constant on Φ, that

$$\mathrm{Re}\,g(\xi+i\eta)\geqslant -2c''\varkappa^2|\xi| - \text{constant}.$$

Because δ_x, $\varkappa<1$, it follows from (4.9) that $\varkappa^2<\varkappa\leqslant 1/4c''$. In conclusion, we thus have

$$\mathrm{Re}\,g(\xi+i\eta)\geqslant -|\xi|/2 - \text{constant, when } \xi+i\eta\in\Phi. \tag{4.18}$$

This, together with (4.12), (4.13), (3.14′) and (2.5), gives that

$$|f(\zeta)\bar\partial\phi(\zeta)| \leqslant \text{constant}\cdot e^{-|\xi|/2}, \text{ when } \zeta\in\Phi.$$

Thus $I_3(R)$ converges, when R tends to infinity. Furthermore, we may change the order of integration and get

$$I_3(\infty)=\int_{\mathbf{R}^n} h_3(z)\,u(z)\,dz,$$

where

$$h_3(z)=(2\pi)^{-n}\int_{B(\infty)} e^{i\langle z,\zeta\rangle}P(\zeta)^{-1}\,e^{-g(\zeta)}\,\bar\partial\phi(\zeta)\,d\zeta.$$

Again, it follows from (4.13), (4.18), (3.14′) and (2.5) that $h_3(z)$ is holomorphic in a neighborhood of the support of u.

Study of $I_4(R)$. (4.17) implies that

$$|f(\xi+i\eta)\,\phi(\xi+i\eta)| \leqslant \text{constant}\cdot e^{-\delta_x\varkappa\psi(\xi)|\xi|/5}, \text{ when } \xi+i\eta \text{ lies on } \beta(R). \tag{4.19}$$

Since $g(\xi+i\eta)=g(\xi)+|\eta|\,r(\xi,\eta)$, where $r(\xi,\eta)$ is bounded and $\Delta(\varrho)\subset\Delta_4$, it follows from (4.2) that, for some positive constant c_1

$$\mathrm{Re}\,g(\xi+i\eta)\geqslant (c_1-c_2\psi(\xi))|\xi| - \text{constant, when } \xi+i\eta\in\Phi \text{ and } \xi\notin\Delta_4. \tag{4.20}$$

(4.19) and (4.20), together with (4.12), (4.13), (4.16) and (3.14′), give that, for some constant $c>0$,

$$|f(\zeta)\phi(\zeta)| \leqslant \text{constant}\cdot e^{-c|\zeta|}, \text{ when } \zeta \text{ lies on } \beta(R).$$

This implies that $I_4(R)$ converges, when R tends to infinity. In a similar way as for $I_3(R)$, we conclude that

$$I_4(\infty) = \int_{\mathbf{R}^n} h_4(z)\, u(z)\, dz,$$

where
$$h_4(z) = (2\pi)^{-n} \int_{\beta(\infty)} e^{i\langle z,\zeta\rangle} P(\zeta)^{-1} e^{-g(\zeta)} \phi(\zeta)\, d\zeta$$

$h_4(z)$ is holomorphic in a neighborhood of the support of u. This completes the proof of Theorem 4.1.

5. Application to the problem of propagation of analyticity

We are now going to use Corollary 4.1 and Corollary 4.3 to prove a counterpart of Theorem 1.3 in the analytic case. Of course, by using the more general Corollary 4.2 instead of Corollary 4.3, we could obtain similar results for any differential operator $P(D)$, such that $P \prec P_m$ and P_m is locally hyperbolic. However, since the results are only complete for operators of principal type with real principal part, we restrict ourselves to that case. Once again, we are going to use functions defined as in (2.1). More specifically, we will use functions satisfying (2.2) for $\nu=0$ and some constants M and c. It follows from Leibniz' formula that these functions form an algebra. The existence of suitable cut-off functions belonging to this algebra was proved in Section 2. For the sake of convenience, we start by formulating the following very simple lemma.

Lemma 5.1. *Let O be an open bounded set in $\mathbf{R}^n$ and suppose that the functions $f_k \in C_0(O)$ and $g_k \in C_0^\infty(O)$ satisfy*

$$|f_k(x)| \leqslant Mk^p \tag{5.1}$$

$$|D^\alpha g_k(x)| \leqslant M(ck)^{|\alpha|},\ |\alpha| \leqslant k. \tag{5.2}$$

Then, with some new constants M and c,

$$|D^\alpha(f_{|\alpha|} * g_{|\alpha|})(x)| \leqslant M(c|\alpha|)^{|\alpha|},\ |\alpha| \geqslant 1. \tag{5.3}$$

Proof. $|D^\alpha(f_{|\alpha|} * g_{|\alpha|})(x)| = |\int f_{|\alpha|}(x-y)\, D^\alpha g_{|\alpha|}(y)\, dy| \leqslant M_1(c|\alpha|)^{|\alpha|+p} \leqslant M_1(c_1|\alpha|)^{|\alpha|}$

Lemma 5.2. *Let J be a closed cone in $\mathbf{R}^n$, such that $P(D)$ has a fundamental solution E with* a.s.$E \subset -J$. *If $U \in \mathcal{D}'(\Omega)$, $P(D)U$ is analytic in Ω and $x \in$* a.s.U, *then x does not belong to a compact component of* (a.s.U)$\cap(\{x\}+J)$.

Proof. If x belongs to a compact component of (a.s.U)$\cap(\{x\}+J)$, then there is an open set ω with $x \in \omega \subset \Omega$, such that $K = \omega \cap$ (a.s.U) $\cap (\{x\}+J)$ is compact in ω. Take

$\phi_k \in C_0^\infty(\omega)$ such that $\phi_k = 1$ in an open ε-neighborhood K_ε of K and the derivatives of ϕ_k satisfy (5.2). If m is the degree of $P(D)$, we denote $P(D)\ (\phi_{k+m} U)$ by T_k. Obviously

$$\phi_{k+m} U = E * T_k. \tag{5.4}$$

Furthermore, T_k satisfies (5.2) locally in $K_\varepsilon \cup \complement_\omega(\text{a.s.}U)$.

If R and S are two distributions, one of which has compact support, then $\text{s.}(R * S) \subset \text{s.}R + \text{s.}S$. Thus (5.4) implies that, if $\psi_k \in C_0^\infty(\mathbf{R}^n)$ is 1 on a large open set and $E_k = \psi_k E$

$$U(y) = (E_k * T_k)\ (y), \text{ when } y \text{ is in a fixed neighborhood of } x.$$

For the same reason it follows from the fact that $x \notin (\omega \setminus K_\varepsilon) \cap (\text{a.s.}U) - J$ that $x \notin \text{s.}(\sigma_k E_k * \chi_k T_k)$, provided that σ_k, $\chi_k \in C^\infty(\mathbf{R}^n)$ have their supports in fixed small neighborhoods of $-J$ and $(\omega \setminus K_\varepsilon) \cap (\text{a.s.}U)$ respectively. This gives that, when y is close to x,

$$U(y) = (1 - \sigma_k) E_k * T_k(y) + \sigma_k E_k * (1 - \chi_k) T_k(y). \tag{5.5}$$

Suppose now that σ_k and χ_k are 1 in fixed small neighborhoods of $-J$ and $(\omega \setminus K_\varepsilon) \cap (\text{a.s.}U)$ respectively and that the derivatives of ψ_k, σ_k and χ_k satisfy (5.2). Then also the derivatives of $(1-\sigma_k) E_k$ and $(1-\chi_k) T_k$ satisfy (5.2). We may suppose that the functions ϕ_k and ψ_k vanishes outside a fixed compact set. Since, on a bounded set, any distribution is a finite linear combination of derivatives of a continuous function, the functions $T_k = P(D)\ (\phi_{k+m} U)$ and $\sigma_k E_k = \sigma_k \psi_k E$ may be written as finite sums of distributions of the form $D^\beta h_k \cdot D^\gamma f$, where h_k satisfies (5.2) and f is continuous. "Integrating by parts" one may, in (5.5), remove the derivatives from f and write $U(y)$ as a finite sum of terms of the form $(D^{\beta'} h_k \cdot f) * D^{\gamma'} j_k(y)$, where j_k is one of the functions $(1-\sigma_k) E_k$, $(1-\chi_k) T_k$. If we put $f_k = D^{\beta'} h_{k+|\gamma'|} \cdot f$ and $g_k = j_{k+|\gamma'|}$, then the assumptions of Lemma 5.1 are fulfilled. In particular p equals $|\beta'|$. Thus, according to (5.3),

$$|D^\alpha U(y)| \leq M(c|\alpha|)^{|\alpha|}.$$

However, this means that U is analytic in a neighborhood of x.

Theorem 5.1. *Let $P(D)$ be of principal type with real principal part and let X_1, X_2, be two relatively closed subsets of the open set Ω. Then the set X has the following property*

$$\text{a.s.}U \subset X_1,\ \text{a.s.}P(D)\, U \subset X_2 \Rightarrow \text{a.s.}U \subset X,\ U \in \mathcal{D}'(\Omega) \tag{5.6}$$

if and only if

$X \supset X_1 \cap X_2$ and, for every bicharacteristic line l, the set X contains any component I of $l \cap (\Omega \setminus X_1 \cap X_2)$ such that $I \subset X_1$. (5.7)

Proof. The necessity of (5.7) follows directly from Corollary 4.3. To prove the sufficiency, we note that if a bicharacteristic half ray connects $x \in X_1$ with a point in $\Omega \setminus X_1$, while remaining in $\Omega \setminus X_1 \cap X_2$ then this is true for all neighbouring half rays from x. Thus, if for every bicharacteristic line l through x, x does not belong to a component of $l \cap (\Omega \setminus X_1 \cap X_2)$ lying entirely in X_1, then we may cover $\mathbf{R}^n$ with two open cones $\mathcal{D} = \{\Delta^+, \Delta^-\}$, such that, if $\mathcal{V} = \{v^+, v^-\}$, where $v^\pm(\xi) = \pm \operatorname{grad} P_m(\xi) / |\operatorname{grad} P_m(\xi)|$, then x belongs to a compact component of $X_1 \cap (\{x\} - \mathcal{H}(\mathcal{D}, \mathcal{V}))$. Since a.s. $U \subset X_1$, the theorem follows from Corollary 4.1 and Lemma 5.2.

REFERENCES

1. ANDERSSON, K. G., Analyticity of fundamental solutions, Ark. Mat. *8*, 73–81 (1970).
2. ATIYAH, M. F., BOTT, R. and GÅRDING, L., Lacunas for hyperbolic differential operators with constant coefficients I, Acta Math. *124*, *109–189 (1970)*.
3. EHRENPREIS, L., Solutions of some problems of division IV. Invertible and elliptic operators, Amer. J. Math. *82*, 522–588 (1960).
4. GRUŠIN, V. V., The extension of smoothness of solutions of differential equations of principal type, Soviet Math. *4*, 248–251 (1963).
5. HÖRMANDER, L., Linear partial differential operators. Springer, 1963.
6. — On the singularities of solutions of partial differential equations, Comm. Pure Appl. Math. *23* (1970).
7. MANDELBROJT, S., Séries adhérentes. Régularisation des suites. Applications. Gauthier-Villars, 1952.
8. SVENSSON, S. L., Necessary and sufficient conditions for the hyperbolicity of polynomials with hyperbolic principal part, Ark. Math. *8*, 145–162 (1970).
9. TRÈVES, F., Linear partial differential equations with constant coefficients. Gordon & Breach, 1966.
10. TRÈVES, F. and ZERNER, M., Zones d'analyticité des solutions élémentaires, Bull. Soc. Math. Fr. *95*, 155–191 (1967).
11. ZERNER, M., Solutions singulières d'équations aux dérivées partielles, Bull. Soc. Math. Fr. *91*, 203–226 (1963).
12. SATO, M., Hyperfunctions and partial differential equations, Proc. Int. Congr. Math. Nice 1970.

Tryckt den 18 februari 1971

Uppsala 1971. Almqvist & Wiksells Boktryckeri AB