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USE OF APPROXIMATE BAYESIAN METHODS FOR THE BLOCK AND BASU BIVARIATE EXPONENTIAL DISTRIBUTION

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Summary

In this paper, we present a Bayesian analysis of the bivariate exponential distribution of Block and Basu (1974) assuming different prior densities for the parameters of the model and considering Laplace's method to obtain approximate marginal posterior and posterior moments of interest. We also find approximate Bayes estimators for the reliability of two-component systems at a specified time t_0 considering series and parallel systems. We illustrate the proposed methodology with a generated data set.

Keywords: Bivariate exponential distribution, Bayesian analysis, Laplace's method.

1. Introduction

In many application of life testing, we usually have two life time X and Y associated to each unit. Among the different bivariate life models to be used in these applications, one family of models has been extensively explored in the last 30 years: the bivariate exponential distribution. There are many different versions of bivariate exponential distributions: the exponential models of Gumbel (1960), Freund (1961), Marshall and Olkin (1967), Downton (1970), Hawkes (1972), Block and Basu (1974), Sarkar (1987), and others. Among all these versions of bivariate exponential distribution, one model has been very well explored in applications: the Block and Basu exponential distribution.

In this paper, we present a Bayesian analysis of the Block and Basu bivariate exponential distribution, working with Laplace's method for approximation of integrals (see for example, Tierney and Kadane, 1986; or Kass, Tier-

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ney and Kadane, 1990), since the marginal posterior densities or posterior moments of interest are not analytically tractable.

2. The Model

Assume that we have two failure times X and Y associated to each observational unit with a bivariate exponential distribution (BVED) of Block and Basu with parameters λ_1, λ_2 and λ_3, and joint density function given by

$$f(x,y) = \begin{cases} f_1(x,y)= \dfrac{\lambda_1\, \lambda_{123}\, \lambda_{23}}{\lambda_{12}} \; exp\{-\lambda_1\, x-\lambda_{23}\, y\} \text{ if } x < y \\ f_2(x,y)= \dfrac{\lambda_2\, \lambda_{123}\, \lambda_{13}}{\lambda_{12}} \; exp\{-\lambda_{13}\, x-\lambda_2\, y\} \text{ if } x \geqslant y \end{cases} \tag{1}$$

where $\lambda_{12} = \lambda_1 + \lambda_2, \lambda_{13} = \lambda_1 + \lambda_3, \lambda_{23} = \lambda_2 + \lambda_3$ and $\lambda_{123} = \lambda_1 + \lambda_2 + \lambda_3$.

The joint generating function for the BVED is given by

$$m(s,t)=E(e^{sX+tY}) = \frac{\lambda_{123}}{\lambda_{12}(\lambda_{123}-t-s)} \left[\frac{\lambda_1\lambda_{23}}{\lambda_{23}-t} + \frac{\lambda_2\lambda_{13}}{\lambda_{13}-s} \right]. \tag{2}$$

From (2), we get the moments of interest for X and Y. The correlation coefficient of X and Y is given by

$$\varrho(X,Y)= \frac{\lambda_3[(\lambda_1^2 + \lambda_2^2)\, \lambda_{123} + \lambda_1\lambda_2\lambda_3]}{[\lambda_{12}^2\lambda_{13}^2+\lambda_2(\lambda_2+2\lambda_1)\lambda_{123}^2]^{1/2}[\lambda_{12}^2\lambda_{23}^2+\lambda_1(\lambda_1+2\lambda_2)\lambda_{123}^2]^{1/2}} \,. \tag{3}$$

We have $0 \leqslant \varrho(X,Y) \leqslant 1$ and $\varrho(X,Y) = 0$ only for the trivial cases $\lambda_3 = 0$ or $\lambda_1 = \lambda_2 = 0$.

Considering a random sample of size n, $(X_1,Y_1),\ldots,(X_n,Y_n)$ of the BVED, the likelihood function for λ_1, λ_2 and λ_3 is given by

$$L\,(\lambda_1,\lambda_2,\lambda_3) = \prod_{i=1}^{n} f_1^{\delta_i}\,(x_i,y_i)\, f_2^{1-\delta_i}\,(x_i,y_i) \tag{4}$$

where $\delta_i = 1$ if $X_i < Y_i$ and $\delta_i = 0$ if $X_i \geqslant Y_i$.

That is,

$$L(\lambda_1,\lambda_2,\lambda_3) = \frac{\lambda_1^r\, \lambda_{123}^n\, \lambda_2^{n-r}\, \lambda_{23}^r\, \lambda_{13}^{n-r}}{\lambda_{12}^n} \times \\ \times\, exp\{-\lambda_1 n\bar{x}-\lambda_2 n\bar{y}-\lambda_3 R\} \tag{5}$$

where $\lambda_{12}, \lambda_{13}, \lambda_{23}$ and λ_{123} are given in (1), $n\bar{x} = \sum_{i=1}^{n} x_i, n\bar{y} = \sum_{i=1}^{n} y_i, r = \sum_{i=1}^{n} \delta_i$ and $R = \sum_{i=1}^{n} [\delta_i y_i + (1 - \delta_i)x_i]$.

3. The maximum likelihood estimators for λ_1, λ_2 and λ_3.

The maximum likelihood estimators (MLE) $\hat{\lambda}_1$, $\hat{\lambda}_2$ and $\hat{\lambda}_3$ are solutions of the likelihood equations,

$$\frac{r}{\lambda_1} + \frac{n}{\lambda_{123}} + \frac{(n-r)}{\lambda_{13}} - \frac{n}{\lambda_{12}} = n\bar{x},$$

$$\frac{n}{\lambda_{123}} + \frac{(n-r)}{\lambda_2} + \frac{r}{\lambda_{23}} - \frac{n}{\lambda_{12}} = n\bar{y}, \tag{6}$$

$$\text{and } \frac{n}{\lambda_{123}} + \frac{r}{\lambda_{23}} + \frac{(n-r)}{\lambda_{13}} = R.$$

To find the MLE for λ_1, λ_2 and λ_3, we should use an iterative procedure; for example, the Newton-Raphson procedure. To construct hypotheses tests or confidence intervals for the parameters λ_1, λ_2 and λ_3, we usually use the asymptotic normality of the MLE,

$$(\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3) \overset{a}{\sim} N\,\{(\lambda_1, \lambda_2, \lambda_3);\, I^{-1}\}, \tag{7}$$

where I is the Fisher information matrix for λ_1, λ_2 and λ_3.

For small or moderate sample sizes, usually the asymptotic normality of the MLE $\hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}_3$ can be very poor. In this case, we could explore an appropriate reparametrization (see for example, Anscombe, 1964; Sprott, 1973, 1980; or Kass and Slate, 1992), or use the asymptotic distribution of the likelihood ratio statistic to get better inferences on λ_1, λ_2 and λ_3 (see for example, Lawless, 1982).

4. A Bayesian analysis of the BVED model

For a Bayesian analysis of the BVED with density (1), we observe that the marginal posterior densities or posterior moments of interest are not analytically tractable. Therefore, we should consider one among different strategies to solve the integrals for a Bayesian analysis of the model: the use of numerical methods (see for example, Naylor and Smith, 1982), the use of approximation methods (see for example, Tierney and Kadane, 1986; or Lindley, 1980), the use of Monte Carlo methods (see for example, Kloek and Van Dijk, 1978), or the use of Gibbs sampling (see for example Gelfand and Smith, 1990). In this paper, we use Laplace's method for approximation of integrals (see for example, Kass, Tierney and Kadane, 1990) to get approximate marginal posterior densities or posterior moments of interest, considering different prior densities for the parameters.

4.1. *The Laplace's method for approximation of integrals*

Assuming h is a smooth function of an m-dimensional parameter ϑ with $-h$ having a maximum at $\hat{\vartheta}$, the Laplace's method approximates an integral of the form,

$$I = \int f(\vartheta) exp[-nh(\vartheta)] d\vartheta \tag{8}$$

by expanding h and f in a Taylor series about $\hat{\vartheta}$ (see for example, Kass, Tierney and Kadane, 1990).

Considering first the case in which ϑ is one-dimensional, the Laplace's method gives the approximation,

$$\hat{I} \cong \left(\frac{2\pi}{n} \right)^{1/2} \sigma f(\hat{\vartheta}) \; exp\{-nh(\hat{\vartheta})\} \tag{9}$$

where $\sigma = \{h''(\hat{\vartheta})\}^{-1/2}$.

In the multiparameter case, with $\vartheta \in R^m$, we have

$$\hat{I} \cong (2\pi)^{m/2} \{det\,(nD^2h(\hat{\vartheta}))\}^{-1/2} f(\hat{\vartheta}) exp\{-nh(\hat{\vartheta})\} \tag{10}$$

where $\hat{\vartheta}$ maximizes $-h(\vartheta)$ and $D^2h(\hat{\vartheta})$ is the Hessian matrix of h evaluated at $\hat{\vartheta}$.

In Bayesian applications, we get inferences on the parameters based on the posterior density $\pi(\vartheta|data) \propto \pi(\vartheta)L(\vartheta)$, where $L(\vartheta)$ is the likelihood function based on n observations and $\pi(\vartheta)$ is a prior density. One important problem is to compute the posterior expectation of a real-valued function $g(\vartheta)$

$$E\,\{g(\vartheta)|data\} = \frac{\int g(\vartheta)L(\vartheta)\pi(\vartheta)d\vartheta}{\int L(\vartheta)\pi(\vartheta)d(\vartheta)}\,. \tag{11}$$

By applying Laplace's method with $L(\vartheta)\pi(\vartheta) = exp\{-nh(\vartheta)\}$ for both the numerator and denominator of (11) (with f defined in (8) respectively equal to g and 1), we get

$$\hat{E}\ \{g(\vartheta)|data\} \cong g\ (\hat{\vartheta})\ \{1 + o\ (n^{-1})\}. \tag{12}$$

This approximation usually is called modal approximation because $\hat{\vartheta}$ is the mode of the posterior density. Considering f equal to 1 for both integrals in (11), that is,

$$E\ \{g(\vartheta)|data\} = \frac{\int e^{-nh^*(\vartheta)}d\vartheta}{\int e^{-nh(\vartheta)}d\vartheta} \tag{13}$$

where $g(\vartheta)$ is a positive function, $-nh^*(\vartheta) = lng(\vartheta) + ln\pi(\vartheta) + lnL(\vartheta)$ and $-nh(\vartheta) = ln\pi(\vartheta) + lnL(\vartheta)$, we get more accurate Laplace approximations for the posterior expectation of $g(\vartheta)$, given by

$$\hat{E}\ \{g(\vartheta)|data\} \cong (\sigma^* / \sigma)\ exp\{-[h^*(\hat{\vartheta}^*) - h(\hat{\vartheta})]\} \tag{14}$$

where $\hat{\vartheta}$ maximizes $-h(\vartheta)$, $\hat{\vartheta}^*$ maximizes $-h^*(\vartheta)$, $\sigma = \{det\ (nD^2h(\hat{\vartheta}))\}^{-1/2}$ and $\sigma^* = \{det\ (nD^2h^*(\hat{\vartheta}^*))\}^{-1/2}$.

The approximation (14) satisfies,

$$E\ \{g(\vartheta)|data\} = \hat{E}\ \{g(\vartheta)|data\}\ (1 + o\ (n^{-2})). \tag{15}$$

(see for example, Tierney and Kadane, 1986; or Tierney, Kass and Kadane, 1989).

4.2. *Prior density for λ_1, λ_2 and λ_3*

When engineers or users of the BVED have opinions about λ_1, λ_2 and λ_3 (see appendix, for some considerations about the derivation of BVED), it is appropriate to consider informative prior densities for the parameters. The joint density for $\lambda_1/\lambda_{123}, \lambda_2/\lambda_{123}$ and λ_{123} can be written in the form,

$$\pi\,(\lambda_1/\lambda_{123},\lambda_2/\lambda_{123},\lambda_{123}) = \\ = \pi\,(\lambda_1/\lambda_{123},\lambda_2/\lambda_{123}\mid\lambda_{123})\,\pi_0\,(\lambda_{123}). \tag{16}$$

Assuming a Gamma prior density for λ_{123},

$$\pi_0\,(\lambda_{123}) \propto \lambda_{123}^{\alpha-1}\,exp\,\{-\beta\lambda_{123}\} \tag{17}$$

and given λ_{123}, a Dirichlet joint prior density for λ_1/λ_{123} and λ_2/λ_{123},

$$\pi\,(\lambda_1/\lambda_{123},\lambda_2/\lambda_{123}|\lambda_{123}) \propto \\ \propto \left(\frac{\lambda_1}{\lambda_{123}}\right)^{a_1-1}\left(\frac{\lambda_2}{\lambda_{123}}\right)^{a_2-1}\left(1-\frac{\lambda_1}{\lambda_{123}}-\frac{\lambda_2}{\lambda_{123}}\right)^{a_3-1} \tag{18}$$

we have (from (16)), a Gamma-Dirichlet joint prior density for $\lambda_1/\lambda_{123},\lambda_2/\lambda_{123}$ and λ_{123}, given by,

$$\pi\,(\lambda_1/\lambda_{123},\lambda_2/\lambda_{123},\,\lambda_{123}) \propto \\ \propto \lambda_{123}^{\alpha-1}exp\{-\beta\lambda_{123}\}\left(\frac{\lambda_1}{\lambda_{123}}\right)^{a_1-1}\left(\frac{\lambda_2}{\lambda_{123}}\right)^{a_2-1}\left(1-\frac{\lambda_1}{\lambda_{123}}-\frac{\lambda_2}{\lambda_{123}}\right)^{a_3-1} \tag{19}$$

where $\lambda_{123} \geqslant 0$, $\lambda_1/\lambda_{123} \geqslant 0$, $\lambda_2/\lambda_{123} \geqslant 0$ and $0 \leqslant \lambda_1/\lambda_{123} + \lambda_2/\lambda_{123} \leqslant 1$. Considering a transformation of variables, we get a joint prior density for λ_1,λ_2 and λ_3 given by,

$$\pi\,(\lambda_1,\lambda_2,\lambda_3) \propto \lambda_{123}^{\alpha-a}\pi_{i=1}^{3}\lambda_i^{a_i-1}e^{-\beta\lambda_i} \tag{20}$$

where $\lambda_1,\lambda_2,\lambda_3 \geqslant 0$ and $a = a_1 + a_2 + a_3$.

The prior density (20) was introduced by Peña and Gupta (1990) for a Bayesian analysis of another bivariate exponential distribution: the distribution of Marshall and Olkin (1967). Observe that if $\alpha = a$, we have prior independence, where λ_i is Gamma distributed with parameters a_i and β.

Assuming prior independence, we also consider λ_i with Gamma distribution with shape parameter a_i and different scale parameters $b_i, i = 1,2,3$, that is,

$$\pi\,(\lambda_1,\lambda_2,\lambda_3) \propto \prod_{i=1}^{3}\,\lambda_i^{a_i-1}e^{-b_i\lambda_i} \tag{21}$$

where $\lambda_1,\lambda_2,\lambda_3 \geqslant 0$.

When we do not have prior opinion, we could consider a noninformative reference prior for the parameters. Using Jeffreys multiparameter rule (see for example, Box and Tiao, 1973), we have the prior density,

$$\pi\,(\lambda_1,\lambda_2,\lambda_3) \propto \{det I(\lambda_1,\lambda_2,\lambda_3)\}^{1/2} \tag{22}$$

where $I(\lambda_1,\lambda_2,\lambda_3)$ is the Fisher information matrix.

4.3. *Approximate marginal posterior densities for λ_1,λ_2 and λ_3*

To obtain approximations for the marginal posterior densities of λ_1,λ_2 and λ_3, we could consider different choices for $f(\vartheta)$ and $h(\vartheta)$, where $\vartheta = (\lambda_1,\lambda_2,\lambda_3)$, in the Laplace's approximation (10). As a special case, we could consider $f_{\lambda_3}(\lambda_1,\lambda_2) = \pi\,(\lambda_1,\lambda_2,\lambda_3)$ and $-h_{\lambda_3}\,(\lambda_1,\lambda_2) = l\,(\lambda_1,\lambda_2,\lambda_3)$, where $l\,(\lambda_1,\lambda_2,\lambda_3)$ is the logarithm of the likelihood function to get a simple form for an approximation of the marginal posterior density for λ_3. Another possibility, is to consider the fully exponential form $f_{\lambda_3}\,(\lambda_1,\lambda_2) = 1$ and $-h_{\lambda_3}\,(\lambda_1,\lambda_2) = ln\pi\,(\lambda_1,\lambda_2,\lambda_3) + l\,(\lambda_1,\lambda_2,\lambda_3)$, in the Laplace's approximation (10), to get an approximate marginal posterior density for λ_3. In the same way, we get approximate marginal posterior densities for λ_1 and λ_2.

4.4. *Bayes estimators for the mean life times*

With the BVED density (1), the mean life times of X and Y are given from (2) by,

$$\mu_1 = E(X) = \frac{(\lambda_{123}\,\lambda_{12} + \lambda_2\,\lambda_3)}{\lambda_{12}\,\lambda_{13}\,\lambda_{123}}$$
$$\text{and } \mu_2 = E(Y) = \frac{(\lambda_{123}\,\lambda_{12} + \lambda_1\,\lambda_3)}{\lambda_{12}\,\lambda_{23}\,\lambda_{123}} \tag{23}$$

The Bayes estimators for the mean life times of components μ_1 and μ_2 with respect to unidimensional squared error are given by

$$\tilde{\mu}_i = E\,\{\mu_i|data\} = \frac{\int \mu_i\pi(\vartheta)L(\vartheta)d\vartheta}{\int \pi(\vartheta)L(\vartheta)d(\vartheta)} \tag{24}$$

where $\vartheta = (\lambda_1,\lambda_2,\lambda_3)$, $i = 1,2$.

From (12), we get Laplace's approximations,

$$\tilde{\mu}_1 \cong \frac{(\tilde{\lambda}_{123}\,\tilde{\lambda}_{12} + \tilde{\lambda}_2\,\tilde{\lambda}_3)}{\tilde{\lambda}_{12}\,\tilde{\lambda}_{13}\,\tilde{\lambda}_{123}}$$

$$\text{and} \quad \tilde{\mu}_2 \cong \frac{(\tilde{\lambda}_{123}\,\tilde{\lambda}_{12} + \tilde{\lambda}_1\,\tilde{\lambda}_3)}{\tilde{\lambda}_{12}\,\tilde{\lambda}_{23}\,\tilde{\lambda}_{123}} \tag{25}$$

where $\tilde{\lambda}_1, \tilde{\lambda}_2$ and $\tilde{\lambda}_3$ is the mode of the joint posterior density for λ_1, λ_2 and λ_3; $\tilde{\lambda}_{12} = \tilde{\lambda}_1 + \tilde{\lambda}_2$; $\tilde{\lambda}_{13} = \tilde{\lambda}_1 + \tilde{\lambda}_3$; $\tilde{\lambda}_{23} = \tilde{\lambda}_2 + \tilde{\lambda}_3$ and $\tilde{\lambda}_{123} = \tilde{\lambda}_1 + \tilde{\lambda}_2 + \tilde{\lambda}_3$.

If we use (14), we usually get more accurate approximations for the posterior expectations of μ_i, $i = 1,2$, especially for small sample sizes.

5. Reliability function estimators for two-component systems

Assuming the BVED with density (1), the reliability function at time t_0 for a two-component system is given by $R_S(t_0) = exp\{-\lambda_{123}t_0\}$ for a series system and $R_P(t_0) = \{\lambda_{123}(e^{\lambda_1 t_0} + e^{\lambda_2 t_0} - 1) - \lambda_3\} / \{\lambda_{12}e^{\lambda_{123}t_0}\}$ for a parallel system. The Bayes estimators for $R_S(t_0)$ and $R_P(t_0)$ with respect to unidimensional squared error loss, are given by

$$\hat{R}_S(t_0) = E\{R_S(t_0) \mid data\} = \frac{\int R_S(t_0)\,\pi(\vartheta)L(\vartheta)d\vartheta}{\int \pi(\vartheta)L(\vartheta)d\vartheta} \tag{26}$$

and

$$\hat{R}_P(t_0) = E\{R_P(t_0) \mid data\} = \frac{\int R_P(t_0)\,\pi(\vartheta)L(\vartheta)d\vartheta}{\int \pi(\vartheta)L(\vartheta)d\vartheta} \tag{27}$$

where $\vartheta = (\lambda_1, \lambda_2, \lambda_3)$, $\pi(\vartheta)$ is a prior density and $L(\vartheta)$ is the likelihood function (5).

From (12), we get the modal Laplace approximations,

$$\hat{R}_S(t_0) \cong exp\{-\tilde{\lambda}_{123}t_0\} \tag{28}$$

and

$$\hat{R}_P(t_0) \cong \{\tilde{\lambda}_{123}(e^{\tilde{\lambda}_1 t_0} + e^{\tilde{\lambda}_2 t_0} - 1) - \tilde{\lambda}_3\} / \{\tilde{\lambda}_{12}e^{\tilde{\lambda}_{123}t_0}\}$$

where $\tilde{\lambda}_1, \tilde{\lambda}_2$ and $\tilde{\lambda}_3$ is the mode of the joint posterior density for λ_1, λ_2 and λ_3.
We also could use (14) to get more accurate approximations for $E\{R_S(t_0) \mid data\}$ and $E\{R_P(t_0) \mid data\}$.

6. An example

In table 1, we have 30 bivariate observations (X, Y) generated from a BVED with density (1) and parameters $\lambda_1 = 0.25$, $\lambda_4 = 0.16$ and $\lambda_3 = 0$. From table 1, we have $r = 16$, $n - r = 14$, $n = 30$, $\sum_{i=1}^{30} x_i = 114.51$, $\sum_{i=1}^{30} y_i = 165.67$ and $R = 207.77$ (see (5)). The MLE for λ_1, λ_2 and λ_3 obtained using Newton-Raphson method in the likelihood equations (6) are given by $\hat{\lambda}_1 = 0.2485$, $\hat{\lambda}_2 = 0.1698$ and $\hat{\lambda}_3 = 0.0164$.

Table 1

Generated Bivariate Life Time Data with a BVED with $\lambda_1 = 0.25$, $\lambda_2 = 0.16$ and $\lambda_3 = 0$

i	x	y	i	x	y
1	3.73	2.54	16	3.42	1.09
2	5.83	7.74	17	7.71	0.33
3	8.44	9.89	18	6.92	2.59
4	7.95	2.47	19	7.76	3.77
5	7.66	8.77	20	0.16	6.07
6	3.47	1.86	21	7.79	6.98
7	2.75	1.30	22	0.66	0.49
8	0.57	5.04	23	10.83	4.03
9	3.48	1.13	24	4.23	2.71
10	4.12	7.24	25	3.23	18.74
11	2.08	9.40	26	1.00	9.10
12	4.19	1.50	27	3.08	12.43
13	0.82	6.29	28	0.55	13.50
14	1.14	2.61	29	0.37	5.52
15	0.18	8.17	30	0.39	2.37

Considering the normal limiting distribution for the MLE $\hat{\lambda}_1, \hat{\lambda}_2$ and $\hat{\lambda}_3$ given in (7), we get approximate 95% confidence intervals for λ_1, λ_2 and λ_3 given by (0.0710;0.4260), (0.0090;0.3310) and (−0.1830;0.2160), respectively.

For a Bayesian analysis of the BVED with the data of table 1, we consider two different prior densities: the noninformative Jeffreys prior (22) and the informative prior (21) with $a_1 = 125$, $a_2 = 64$, $a_3 = 4$, $b_1 = 500$, $b_2 = 400$ and $b_3 = 200$, by observing that the independent Gamma prior densities for λ_i, $i = 1,2,3$ have means a_i/b_i and variances a_i/b_i^2, and we known the true values for λ_1, λ_2 and λ_3. In figures 1, 2 and 3, we have the graphs of the approximate marginal posterior densities for λ_1, λ_2 and λ_3 considering Laplace's approximation (10) with f equals to the prior density and $-nh$ equals to the log-likelihood function $l(\lambda_1, \lambda_2, \lambda_3)$.

In table 2, we have a summary of the point and interval estimators for the parameters λ_1, λ_2 and λ_3. We observe better Bayesian intervals for λ_1, λ_2 and λ_3 considering the informative prior (21). We observe that using the asymptotical normality of the maximum likelihood estimators, we get similar results considering the noninformative prior (22), especially for λ_1 and λ_3.

We also could use any other density indicating prior opinion of the researcher, that is, our approximate Bayesian approach is very flexible to be used in applications of the BVED with bivariate life data.

In table 3, we have Laplace's approximate Bayes estimators with respect to unidimensional squared error for the mean life times μ_1 and μ_2 (see (23)).

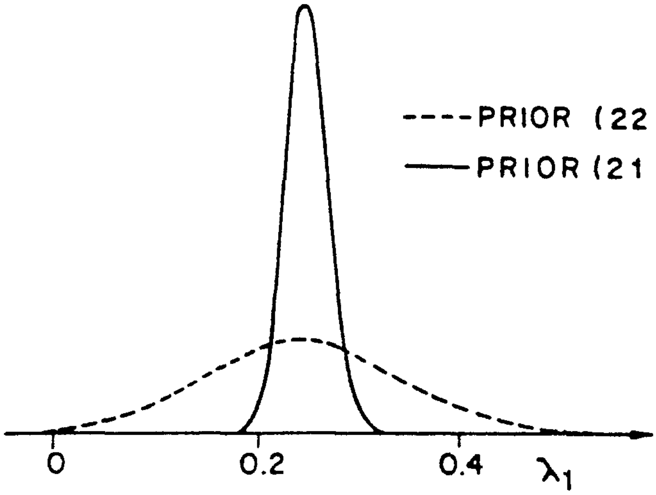

Fig. 1 – Marginal Posterior for λ_1.

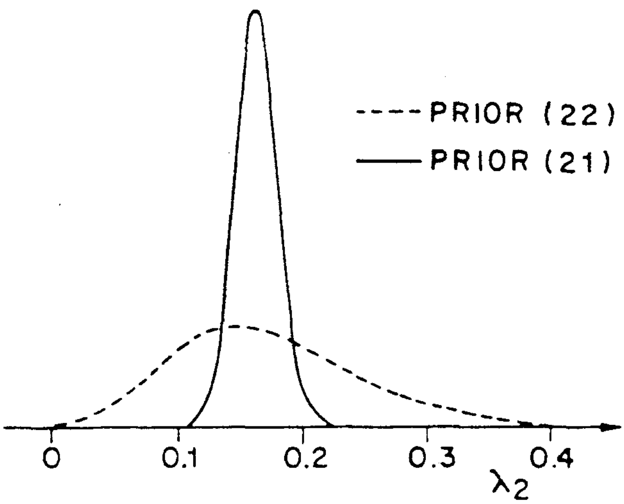

Fig. 2 – Marginal Posterior for λ_2.

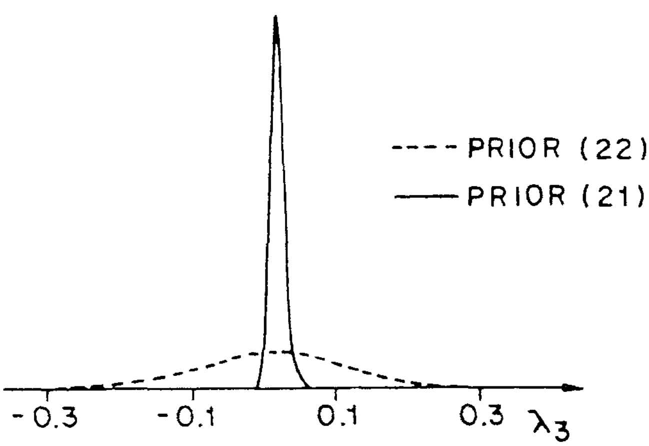

Fig. 3 – Marginal Posterior for λ_3.

If is interesting to observe that the Bayes estimators for μ_1 and μ_2 with prior (21) considering Monte Carlo procedure with $M = 200$ generated samples are given by $\bar{\mu}_1 = 3.7914$ and $\bar{\mu}_2 = 5.6213$. That is, we have close results with Laplace's approximation (14).

Table 2

Point and Interval Estimators for λ_1, λ_2 and λ_3

	MLE	Mode of Posterior Prior (22)	Mode of Posterior Prior (21)	95% Confidence Interval	95% HPD Interval Prior (22)	95% HPD Interval Prior (21)
λ_1	0.2485	0.2480	0.2480	(0.0710;0.4260)	(0.0700;0.4360)	(0.2120;0.2920)
λ_2	0.1698	0.1460	0.1600	(0.0090;0.3310)	(0.0500;0.3360)	(0.1330;0.1990)
λ_3	0.0164	0.0200	0.0150	(−0.1830;0.2160)	(−0.1820;0.2100)	(0.0060;0.0440)

Table 3

Bayes Estimators for μ_1 and μ_2

	True Values	MLE	Laplace's Approximation (12) for $E(\mu_i \mid Data)$ Prior (22)	Laplace's Approximation (12) for $E(\mu_i \mid Data)$ Prior (21)	Laplace's Appr. (14) for $E(\mu_i \mid Data)$ Prior (21)
μ_1	4	3.8328	3.7981	3.8552	3.8057
μ_2	6.25	5.4909	6.2073	5.8374	5.7001

In table 4, we have Laplace's approximate Bayes estimators with respect to unidimensional squared error for the reliability function of two-component systems at some values of t_0 considering series and parallel systems.

We observe in tables 3 and 4, close results considering the two forms of Laplace's approximations (12) and (14) (see in particular the last four columns of table 4).

It is important to point out that prior density (21) should be used only when the researcher or engineer has a clearly prior opinion about λ_1, λ_2 and λ_3, since with different values for (a_j, b_j), $j = 1,2,3$ in (21) we could get very different inferences for the parameters. In figure 4, we have the graphs of Laplace's approximate marginal posterior densities for λ_1 with prior (21) and values for (a_j, b_j), $j = 1,2,3$ given in table 5.

Table 4

Bayes Estimators for $R_S\ (t_0)$ and $R_P\ (t_0)$

	True Values		Laplace's Approximation (12)				Laplace's Approx. (14)	
			Jeffreys Prior (22)		Prior (21)		Prior (21)	
t_0	$R_S(t_0)$	$R_P(t_0)$	$R_S(t_0)$	$R_P(t_0)$	$R_S(t_0)$	$R_P(t_0)$	$R_S(t_0)$	$R_P(t_0)$
1	0.6636	0.9673	0.6610	0.9657	0.6551	0.9641	0.6499	0.9624
2	0.4404	0.8922	0.4369	0.8874	0.4291	0.8826	0.4226	0.8780
3	0.2923	0.7989	0.2888	0.7907	0.2811	0.7825	0.2751	0.7748
4	0.1940	0.7012	0.1909	0.6903	0.1841	0.6792	0.1792	0.6694
5	0.1287	0.6071	0.1262	0.5943	0.1206	0.5810	0.1168	0.5699
10	0.0166	0.2674	0.0159	0.2543	0.0145	0.2393	0.0139	0.2299
15	0.0021	0.1121	0.0020	0.1038	0.0017	0.0933	0.0017	0.0884

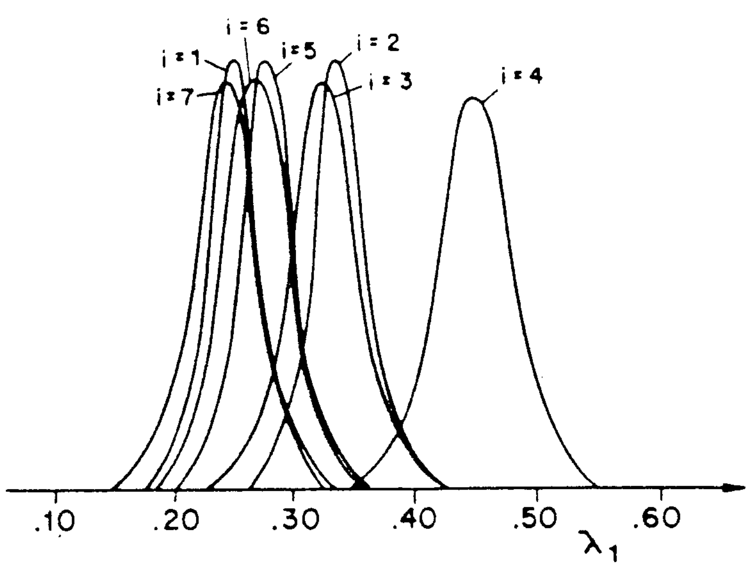

Fig. 4 – Marginal Posterior density for λ_1 considering prior (21) and different values for $(d_j^{(i)}, b_j^{(i)})$, $j = 1, 2, 3$.

Table 5

Values for $(a_j^{(i)}, b_j^{(i)})$, $j = 1,2,3$ in prior (21)

i	$a_1^{(i)}$	$a_2^{(i)}$	$a_3^{(i)}$	$b_1^{(i)}$	$b_2^{(i)}$	$b_3^{(i)}$
1	125	64	4	500	400	200
2	245	100	25	700	500	500
3	122.5	40	2.5	350	200	50
4	250	90	64	500	300	800
5	156.8	81	4	560	450	200
6	78.4	100	2.5	280	500	50
7	62.5	10	2.5	250	50	50

7. Some conclusions

The use of Laplace's method could be a suitable alternative to develop a Bayesian analysis of the BVED with density (1), since all Bayesian solutions for this model requires the solution of integrals that are not analytically tractable. The Laplace's approximations to marginal posterior densities and posterior moments of interest are very simple to be obtained with no need of computer expertise and also is very flexible in terms of choice of priors. We can justify the use of Laplace's approximation to marginal posterior densities for the parameters of BVED, by comparing Monte Carlo integrated posteriors to Laplace's approximations. In table 6, we have the values of the marginal posterior density for λ_1 considering the prior density (21) with $a_1 = 125$, $a_2 = 64$, $a_3 = 4$, $b_1 = 500$, $b_2 = 400$ and $b_3 = 200$ using Monte Carlo with $M = 400$ generated pairs (λ_2, λ_3) and Laplace's approximation (10). As we see in figure 5, the accuracy of Laplace's approximation is very good. Observe that using Laplace's method, we only need to find maximums and second derivatives. The use of Monte Carlo procedure requires some computational expertise, choice of an appropriate «importance density» (see for example, Kloek and Van Dijk, 1978) and a long time in computer. It is very important to point out that the Laplace's approximations are not invariant to different parametrizations (see for example, Achcar and Smith, 1990) and a good reparametrization could improve the approximate results, especially for small sample sizes (see for example, Sprott, 1973, 1980; or Kass and Slate, 1992).

Table 6

Some value of marginal posterior density for λ_I considering prior density (21)

λ_I	Laplace's Method	Monte Carlo Procedure
0.15	0.0000	0.0000
0.16	0.0001	0.0001
0.17	0.0017	0.0017
0.18	0.0208	0.0207
0.19	0.1609	0.1596
0.20	0.8224	0.8173
0.21	2.8917	2.8783
0.22	7.2404	7.2153
0.23	13.3009	13.2794
0.24	18.3945	18.3787
0.25	19.5988	19.6131
0.26	16.4164	16.4314
0.27	11.0040	11.0265
0.28	5.9992	6.0151
0.29	2.6983	2.7072
0.30	1.0143	1.0182
0.31	0.3224	0.3237
0.32	0.0876	0.0879
0.33	0.0205	0.0207
0.34	0.0042	0.0042
0.35	0.0007	0.0008
0.36	0.0001	0.0001
0.37	0.0000	0.0000

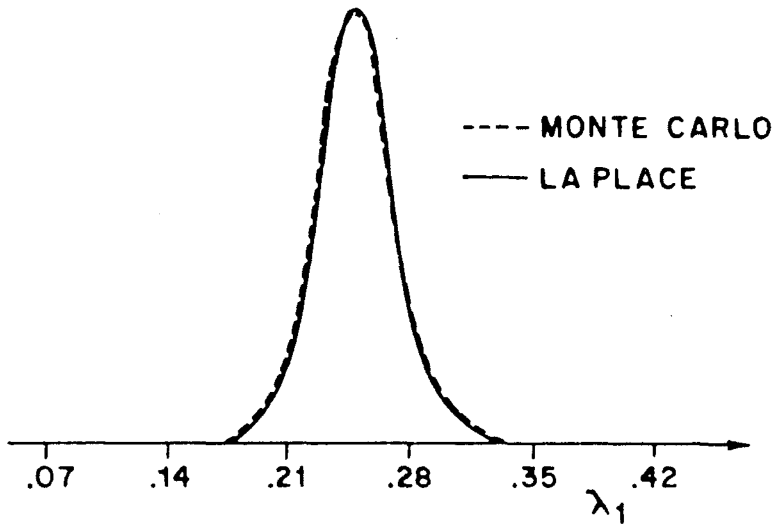

Fig. 5 – Marginal Posterior density for λ_1 using Monte Carlo and Laplace's method.

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Appendix

A derivation of the BVED

The BVED can be derived using a model suggested by Freund (1961). Let (X, Y) denote the life length of a two component system. Consider X and Y individually having exponential distributions with means $\alpha_1^{-1} (\alpha_1, \beta_1 > 0)$ when they are free from external forces. If component 1 fails before component 2, an additional strain is placed on component 2 reducing its mean life time from β_1^{-1} to $\beta_1'^{-1}$, when $\beta_1' > \beta_1$. Similarly, if component 2 fails before component 1, an additional strain is placed on component 1 reducing its mean lifetime from α_1^{-1} to $\alpha_1'^{-1}$, when $\alpha_1' > \alpha_1$. Using Freund's derivation for $\alpha_1 = \lambda_1 + \lambda_3\lambda_1/\lambda_{12}$, $\alpha_1' = \lambda_1 + \lambda_3$, $\beta_1 = \lambda_2 + \lambda_3\lambda_2/\lambda_{12}$, $\beta_1' = \lambda_2 + \lambda_3$, where $\lambda_1, \lambda_2, \lambda_3 > 0$, $\lambda_{12} = \lambda_1 + \lambda_2$, it follows that $\alpha_1 < \alpha_1'$, $\beta_1 < \beta_1'$ and (X, Y) has density (1) (see Block and Basu, 1974).

In applied work, engineers could have prior opinions about the mean lifetimes of X and Y when the units are free or not from external forces. Thus, having prior opinion on α_1, α_1', β_1 and β_1', they have an informative prior density for the parameters λ_1, λ_2 and λ_3, as it was considered in (21).

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