Acta Math., 195 (2005), 253–264

Decay of correlations for Hénon maps

by

TIEN-CUONG DINH

Université Paris VI
Paris, France

1. Introduction

Exponential mixing is an important statistical property in dynamics. It is often difficult to prove this non-linear property for a non-uniformly hyperbolic system. See Benedicks–Young [4], [5] and the references therein for the case of real Hénon maps. Here we will study a large class of polynomial automorphisms in $\mathbf{C}^k$. We note that exponential decay of correlations has been proved for polynomial-like maps and meromorphic maps in the case of large topological degree, which is the opposite of the invertible case (see [14], [8] and [9]).

Given a polynomial automorphism f of $\mathbf{C}^k$, we will extend it to a birational map of $\mathbf{P}^k$. We say that f is a *regular automorphism* in the sense of Sibony if the indeterminacy sets $I_\pm$ of $f^{\pm 1}$ (i.e. the sets of points at infinity where the birational maps $f^{\pm 1}$ are not defined) satisfy $I_+\cap I_-=\varnothing$. We recall here some properties of regular automorphisms (see [2], [1] and [13] for dimension 2 and [20] for $k\geqslant 2$). Note that when $k=2$, the regular automorphisms are finite compositions of generalized Hénon maps (see Friedland and Milnor [15]). As was shown in [15], these are the dynamically interesting polynomial automorphisms of $\mathbf{C}^2$.

The indeterminacy sets $I_\pm$ are contained in the hyperplane at infinity L_∞. When f is regular, there exists an integer s such that $\dim I_+=k-1-s$ and $\dim I_-=s-1$. We have $f(L_\infty\setminus I_+)=I_-$ and $f^{-1}(L_\infty\setminus I_-)=I_+$. Moreover, I_- is attractive for f, and I_+ is attractive for f^{-1}. Let $\mathcal{K}_+$ (resp. $\mathcal{K}_-$) denote the *filled Julia set* of f (resp. of f^{-1}), i.e. the set of points $z\in\mathbf{C}^k$ such that the orbit $(f^n(z))_{n\in\mathbf{N}}$ (resp. $(f^{-n}(z))_{n\in\mathbf{N}}$) is bounded in $\mathbf{C}^k$. Then $\mathcal{K}_\pm$ are closed in $\mathbf{C}^k$ and satisfy $\overline{\mathcal{K}}_\pm\cap L_\infty=I_\pm$. The open set $\mathbf{P}^k\setminus\overline{\mathcal{K}}_+$ (resp. $\mathbf{P}^k\setminus\overline{\mathcal{K}}_-$) is the immediate basin of I_- for f (resp. I_+ for f^{-1}). If d_+ and d_- are the algebraic degrees of f and f^{-1}, respectively, then $d_+^s=d_-^{k-s}>1$. In particular, we have $d_+=d_-$ when $k=2s$.

By $T_\pm$, we denote the *Green currents of bidegree* $(1,1)$ associated to $f^{\pm 1}$ (see also §2). These are currents with total mass 1 which have continuous local potentials in $\mathbf{P}^k \setminus I_\pm$. They satisfy the transformation formulas $f^*(T_+)=d_+T_+$ and $f_*(T_-)=d_-T_-$. In $\mathbf{C}^k$ we have $T_\pm=\mathrm{dd^c}G^\pm$ with $G^\pm(z):=\lim_{n\to\infty} d_\pm^{-n}\max\{\log\|f^n(z)\|,0\}$. Recall that $\mathrm{d^c}:=(i/2\pi)(\bar\partial-\partial)$ and $\mathrm{dd^c}=(i/\pi)\partial\bar\partial$. The Green functions $G^\pm$ are continuous plurisubharmonic and they give the rate of escape to infinity.

Sibony constructed an invariant probability measure as the exterior product of positive closed $(1,1)$-currents:

$$\mu = T_+^s \wedge T_-^{k-s}.$$

The current T_+^s (resp. T_-^{k-s}) is supported in the boundary of $\bar{\mathcal{K}}_+$ (resp. $\bar{\mathcal{K}}_-$); it is the *Green current of bidegree* (s,s) (resp. $(k-s,k-s)$) associated to f (resp. to f^{-1}). The measure μ is supported in the boundary of the compact set $\mathcal{K}:=\mathcal{K}_+\cap\mathcal{K}_-$.

It was recently proved in [11] and [18] that μ is mixing. This generalizes results of Bedford–Smillie [2] and Sibony [20]. The proofs follow the same approach and use the property that T_+^s and T_-^{k-s} are extremal currents. In this paper, we use another method to show that μ is mixing and that the speed of mixing is exponential when $k=2s$. Our strategy is to consider some natural regular automorphisms in $\mathbf{C}^{2k}$ or $\mathbf{C}^{4k}$ in order to reduce the problem to a linear one. We will obtain the desired estimates using the solution of the $\partial\bar\partial$-equation given by a kernel due to Bost, Gillet and Soulé [16], [6].

Let φ and ψ be real-valued continuous functions on $\mathbf{C}^k$. Define *the correlation of order n between φ and ψ* by

$$I_n(\varphi,\psi) := \int (\varphi\circ f^n)\psi\,\mathrm{d}\mu - \biggl(\int\varphi\,\mathrm{d}\mu\biggr)\biggl(\int\psi\,\mathrm{d}\mu\biggr).$$

Recall that μ is *mixing* if $I_n(\varphi,\psi)$ tends to 0 as n tends to infinity for all φ and ψ.

MAIN THEOREM. *Let f and μ be as above. Assume that $k=2s$. Then, μ is exponentially mixing. More precisely, for all α and β, $0<\alpha,\beta\leqslant 2$, there exists a constant $c>0$ depending on f, α and β such that*

$$|I_n(\varphi,\psi)| \leqslant c d_+^{-n\alpha\beta/8}\|\varphi\|_{\mathcal{C}^\alpha}\|\psi\|_{\mathcal{C}^\beta}$$

for all $n\geqslant 0$ and all real-valued functions φ of class $\mathcal{C}^\alpha$ and ψ of class $\mathcal{C}^\beta$ in $\mathbf{C}^k$.

Of course, this result holds for polynomial automorphisms of positive entropy in $\mathbf{C}^2$, in particular, for Hénon maps. We can apply it for real Hénon maps of degree d which admit an invariant probability measure of entropy $\log d$, in which case that measure coincides with μ (see [3]).

In [1], Bedford, Lyubich and Smillie proved for complex Hénon maps that the equilibrium measure is Bernoulli. This is the strongest mixing in the sense of measures.

However, it does not imply the decay of correlations in our sense. Observe also that we cannot have $|I_n(\varphi,\psi)|\lesssim(1+\varepsilon)^{-n}\|\varphi\|_{\mathcal{C}^\alpha}\|\psi\|_\infty$ since, when φ is not constant μ-almost everywhere, $\varphi\circ f^n$ does not converge in $L^1(\mu)$ to $\int\varphi\,d\mu$. This last fact is a consequence of the fact that the μ-measure of the sets $\{z:\varphi\circ f^n(z)<c\}$ and $\{z:\varphi\circ f^n(z)>c'\}$ are independent of n, because μ is invariant.

The precise outline of the paper is as follows. In §§2 and 3, we give some properties of the Green currents and the equilibrium measure. The method of dd^c-resolution developed in [8], [9], [10] and [11] will be applied to establish the necessary estimates (Propositions 2.1 and 3.1). We then deduce in §4 the mixing and the speed of mixing. We first consider the case where $\alpha=\beta=2$, and then we obtain the general case using the theory of interpolation between the Banach spaces $\mathcal{C}^0$ and $\mathcal{C}^2$.

2. Convergence toward the Green current

Let us recall some properties of currents on $\mathbf{P}^k$ that will be used later on. A *current of bidegree* (p,q) is a differential (p,q)-form but the coefficients are distributions. A smooth form Φ of bidegree (q,q) is *weakly positive* if its restriction to every projective subspace of dimension q is a positive volume form. A current S of bidegree (p,p) is *positive* if $\langle S,\Phi\rangle\geqslant 0$ for every weakly positive test $(k-p,k-p)$-form Φ. In particular, it is of order zero.

Let ω denote the Fubini–Study form on $\mathbf{P}^k$, normalized so that $\int\omega^k=1$. The mass of a positive closed (p,p)-current S is given by $\|S\|=\int S\wedge\omega^{k-p}$. Since $\mathbf{P}^k$ is homogeneous, every positive closed current S on $\mathbf{P}^k$ can be regularized on every neighbourhood U of $\operatorname{supp}(S)$. This allows us to construct smooth positive closed currents supported in U and strictly positive, i.e. $\geqslant\varepsilon\omega^p$, on $\operatorname{supp}(S)$. If T is a positive closed $(1,1)$-current with local continuous potentials in a neighbourhood of $\overline{U}$, then the positive closed current $T^m\wedge S$ is well defined and depends continuously on S, see, e.g., [7] and [20]. More precisely, because of the cohomology of $\mathbf{P}^k$, if $\|T\|=c$, we can write $T=c\omega+dd^c u$, where u is a function continuous on $\overline{U}$, and we have $T\wedge S:=c\omega\wedge S+dd^c(uS)$.

Now, consider a regular automorphism f on $\mathbf{C}^k$ as in §1. We do not assume for the moment that $k=2s$. Fix neighbourhoods U_i of $\overline{\mathcal{K}}_+$ and V_i of $\overline{\mathcal{K}}_-$ such that $f^{-1}(U_i)\Subset U_i$, $U_1\Subset U_2$, $f(V_i)\Subset V_i$, $V_1\Subset V_2$ and $U_2\cap V_2\Subset\mathbf{C}^k$. This is possible since I_- has as basin of attraction $\mathbf{P}^k\setminus\overline{\mathcal{K}}_+$, and similarly for f^{-1} and I_+. Observe that $\mathcal{K}_+\cap\mathcal{K}_-\subset U_1\cap V_1$.

Let Ω be a real $(k-s+1,k-s+1)$-current with support in $\overline{V}_1$. Assume that there exists a positive closed $(k-s+1,k-s+1)$-current Ω' supported in $\overline{V}_1$ such that $-\Omega'\leqslant\Omega\leqslant\Omega'$. Define the norm $\|\Omega\|_*$ of Ω as

$$\|\Omega\|_*:=\min\{\|\Omega'\|:\Omega' \text{ as above}\}.$$

where $\|\Omega'\|:=\langle\Omega',\omega^{s-1}\rangle$ is the mass of Ω'.

Keeping the above notation, the main result of this section is the following proposition:

PROPOSITION 2.1. *Let R be a positive closed (s,s)-current of mass 1 supported in U_1 and smooth on $\mathbf{C}^k$. Let Φ be a real-valued $(k-s,k-s)$-form of class $\mathcal{C}^2$ with compact support in $V_1\cap\mathbf{C}^k$. Assume that $\mathrm{dd^c}\Phi\geqslant 0$ in U_2. Then, there exist constants $c>0$ independent of R and Φ, and $c_R>0$ independent of Φ such that*

$$\langle d_+^{-sn}(f^n)^*(R)-T_+^s,\Phi\rangle\leqslant cd_+^{-n}\|\mathrm{dd^c}\Phi\|_*$$

and

$$|\langle d_+^{-sn}(f^n)^*(R)-T_+^s,\Phi\rangle|\leqslant c_R d_+^{-n}\|\mathrm{dd^c}\Phi\|_*$$

for every $n\geqslant 0$. In particular, $d_+^{-sn}(f^n)^*(R)\to T_+^s$ as $n\to\infty$.

The current $(f^n)^*(R)$ is well defined since f^{-n} is holomorphic in U_1. We have, because of the functional equation satisfied by T_+,

$$\langle d_+^{-sn}(f^n)^*(R)-T_+^s,\Phi\rangle=d_+^{-sn}\langle(f^n)^*(R-T_+^s),\Phi\rangle=d_+^{-sn}\langle R-T_+^s,(f^n)_*\Phi\rangle. \tag{1}$$

Since the currents R and T_+^s have the same mass 1, they are cohomologous. On $\mathbf{P}^k$, $R-T_+^s$ is $\mathrm{dd^c}$-exact. Hence, the last term in (1) does not change if we subtract a $\mathrm{dd^c}$-closed form from $(f^n)_*\Phi$. We will apply the following lemma to $\mathrm{dd^c}(f^n)_*\Phi$.

LEMMA 2.2. *Let Ω be a real-valued continuous form of bidegree $(k-s+1,k-s+1)$ supported in $\overline{V}_1$ such that $\Omega\geqslant 0$ on U_2 and $\|\Omega\|_*\leqslant 1$. Assume that Ω is $\mathrm{dd^c}$-exact. Then there exist $c>0$ independent of Ω and a real-valued continuous $(k-s,k-s)$-form Ψ such that $\mathrm{dd^c}\Psi=\Omega$, $\|\Psi\|\leqslant c$, $\Psi\leqslant 0$ on U_1 and $\Psi\geqslant -c\omega^{k-s}$ on $\mathbf{P}^k\setminus V_2$.*

Proof. By Hodge theory [17], we have

$$H^{k,k}(\mathbf{P}^k\times\mathbf{P}^k,\mathbf{C})\simeq\sum_{p+p'=k}H^{p,p}(\mathbf{P}^k,\mathbf{C})\otimes H^{p',p'}(\mathbf{P}^k,\mathbf{C}).$$

Hence, if Δ is the diagonal of $\mathbf{P}^k\times\mathbf{P}^k$, there exists a smooth real-valued (k,k)-form $\alpha(x,y)$ on $\mathbf{P}^k\times\mathbf{P}^k$, cohomologous to $[\Delta]$, with $\mathrm{d}_x\alpha=\mathrm{d}_y\alpha=0$. Since $\mathbf{P}^k\times\mathbf{P}^k$ is homogeneous, following [6, Proposition 6.2.3] (see also [16], [10] and [11]), one can construct a negative $(k-1,k-1)$-form $K(x,y)$ on $\mathbf{P}^k\times\mathbf{P}^k$, smooth outside Δ, such that $\mathrm{dd^c}K=[\Delta]-\alpha$ and $|K(x,y)|\lesssim -\log|x-y|\,|x-y|^{2-2k}$ near Δ. Here $|x-y|$ denotes the distance between x and y.

Define

$$\Psi'(x):=\int_y K(x,y)\wedge\Omega(y).$$

From the bound of the singularities of K, one can easily check that Ψ' is continuous and $\|\Psi'\|\leqslant c'$, $\Psi'\leqslant c'\omega^{k-s}$ on U_1, $\Psi'\geqslant -c'\omega^{k-s}$ on $\mathbf{P}^k\setminus V_2$, where $c'>0$ is independent of Ω. Define $\Psi:=\Psi'-c'\omega^{k-s}$. We obtain $\|\Psi\|\leqslant 2c'$, $\Psi\leqslant 0$ on U_1 and $\Psi\geqslant -2c'\omega^{k-s}$ on $\mathbf{P}^k\setminus V_2$. We only have to verify that $dd^c\Psi'=\Omega$.

Since Ω is dd^c-exact and $d_x\alpha=d_y\alpha=0$, we have

$$dd^c\Psi'(x):=\int_y (dd^c)_x K(x,y)\wedge\Omega(y)=\int_y dd^c K(x,y)\wedge\Omega(y)$$
$$=\int_y([\Delta]-\alpha)\wedge\Omega(y)=\Omega(x)-\int_y\alpha\wedge\Omega(y)=\Omega(x).$$

Hence, $dd^c\Psi=dd^c\Psi'=\Omega$. □

Proof of Proposition 2.1. We can assume that $\|dd^c\Phi\|_*=1$. The constants c and c_i below are independent of Φ and R. Define $\Omega:=dd^c\Phi$. Then there exists a positive closed current Ω' of mass 1 supported in $\overline{V}_1$ such that $-\Omega'\leqslant\Omega\leqslant\Omega'$. Define $\Omega_n:=dd^c(f^n)_*\Phi=(f^n)_*\Omega$ and $\Omega'_n:=(f^n)_*\Omega'$. These currents have supports in $\overline{V}_1$ since $f^n(V_1)\Subset V_1$. We also have $-\Omega'_n\leqslant\Omega_n\leqslant\Omega'_n$ and $\Omega_n\geqslant 0$ on U_2 since $f^{-n}(U_2)\Subset U_2$. A simple calculation on cohomology gives $\|\Omega'_n\|=d_+^{(s-1)n}\|\Omega'\|=d_+^{(s-1)n}$. Lemma 2.2 implies the existence of Ψ_n cohomologous to $(f^n)_*\Phi$ such that $\Psi_n\leqslant 0$ on U_1, $\Psi_n\geqslant -cd_+^{(s-1)n}\omega^{k-s}$ on $\mathbf{P}^k\setminus V_2$ and $\|\Psi_n\|\leqslant cd_+^{(s-1)n}$. In particular, $\Psi_n\leqslant 0$ on $\operatorname{supp}(R)$. Therefore, we deduce from (1) that

$$\langle d_+^{-sn}(f^n)^*(R)-T_+^s,\Phi\rangle=d_+^{-sn}\langle R-T_+^s,\Psi_n\rangle\leqslant -d_+^{-sn}\langle T_+^s,\Psi_n\rangle. \tag{2}$$

We have to bound $-\langle T_+^s,\Psi_n\rangle$. Since T_+ has local continuous potentials in $\mathbf{P}^k\setminus I_+$, we can write $T_+=\omega+dd^c u$ with $u\leqslant 0$ and u continuous on $\mathbf{P}^k\setminus I_+$. One has

$$\begin{aligned}|\langle T_+^s,\Psi_n\rangle|&=|\langle\omega\wedge T_+^{s-1}+dd^c(uT_+^{s-1}),\Psi_n\rangle|\\&\leqslant|\langle T_+^{s-1},\omega\wedge\Psi_n\rangle|+|\langle uT_+^{s-1},dd^c\Psi_n\rangle|\\&\leqslant|\langle T_+^{s-1},\omega\wedge\Psi_n\rangle|-\langle uT_+^{s-1},\Omega'_n\rangle.\end{aligned} \tag{3}$$

Since Ω'_n has support in $\overline{V}_1$ where u is bounded, the second term on the last line of (3) is dominated by $c_1\langle T_+^{s-1},\Omega'_n\rangle$. The integral $\langle T_+^{s-1},\Omega'_n\rangle$ is computed replacing each element by the associated cohomology class; it is equal to $\|\Omega'_n\|$. Hence, $-\langle uT_+^{s-1},\Omega'_n\rangle\leqslant c_1 d_+^{(s-1)n}$.

For the first term on the last line of (3), we write $T_+^{s-1}=\omega\wedge T_+^{s-2}+dd^c(uT_+^{s-2})$. Using expansions as in (3) and an induction argument, we get $|\langle T_+^{s-1},\omega\wedge\Psi_n\rangle|\leqslant c_2 d_+^{(s-1)n}$. At the last step of the induction, we use the inequality $\|\Psi_n\|\leqslant cd_+^{(s-1)n}$. Hence, the first part of Proposition 2.1 follows.

For the second part, it is sufficient to prove that $|\langle R,\Psi_n\rangle|\leqslant c'_R d_+^{(s-1)n}$ with c'_R independent of Φ. This follows directly from the smoothness of R on $\mathbf{C}^k$ and the properties that $\|\Psi_n\| \leqslant c d_+^{(s-1)n}$ and $-c d_+^{(s-1)n} \omega^s \leqslant \Psi_n \leqslant 0$ on the neighbourhood $U_1 \setminus V_2$ of the singularities of R.

Now, we show that $d_+^{-sn}(f^n)^*(R) \to T_+^s$ on $\mathbf{C}^k$. Consider a real-valued smooth test $(k-s, k-s)$-form Φ with compact support in $\mathbf{C}^k$. We want to prove that

$$\langle d^{-sn}(f^n)^*(R) - T_-^s, \Phi\rangle \longrightarrow 0.$$

In order to apply the second part of the proposition, we show that it is possible to suppose that $dd^c\Phi \geqslant 0$ in U_2.

Observe that $\mathbf{P}^k \setminus I_+$ is a union of compact algebraic sets of dimension s since $\dim I_+ = k-1-s$. Hence, we can construct a positive closed $(k-s, k-s)$-form Θ with compact support in $\mathbf{P}^k \setminus I_+$ which is strictly positive on $\operatorname{supp}(\Phi)$. Since $(f^m)^*(T_+^s) = d_+^{sm} T_+^s$ and

$$\langle d_+^{-sn}(f^n)^*(R) - T_+^s, \Phi\rangle = \langle d_+^{-s(n-m)}(f^{n-m})^*(R) - T_+^s, d_+^{-sm}(f^m)_*\Phi\rangle,$$

replacing Φ and Θ by $d_+^{-sm}(f^m)_*\Phi$ and $(f^m)_*\Theta$, m large enough, one can assume that $\operatorname{supp}(\Theta) \subset V_1$.

Consider a smooth function χ with compact support in $\mathbf{C}^k$ which is strictly plurisubharmonic on a neighbourhood of $U_2 \cap V_2$. Write $\Phi = (\Phi + A\chi\Theta) - A\chi\Theta$ with $A>0$ large enough, so that $dd^c(\Phi + A\chi\Theta)$ and $dd^c(A\chi\Theta)$ are positive on U_2. Hence, it is sufficient to consider the case where $dd^c\Phi \geqslant 0$ on U_2. The second part of the proposition implies that $\langle d^{-sn}(f^n)^*(R) - T_+^s, \Phi\rangle \to 0$. □

3. Convergence toward the Green measure

In this section, we consider the "diagonal" mapping $F(z, w) := (f(z), f^{-1}(w))$. The main result here is Proposition 3.1, which will be obtained by applying Proposition 2.1 to F.

PROPOSITION 3.1. *Let f be as above with $k=2s$. Let φ be a $\mathcal{C}^2$-function on $\mathbf{P}^k$ which is plurisubharmonic on $U_2 \cap V_2$. Let R (resp. S) be a positive closed (s,s)-current of mass 1 with support in U_1 (resp. in V_1) and smooth on $\mathbf{C}^k$. Then, there exist constants $c>0$ independent of φ, R and S, and $c_{R,S}>0$ independent of φ, such that*

$$\langle d_+^{-2sn}(f^n)^*(R) \wedge (f^n)_*(S) - \mu, \varphi\rangle \leqslant c d_+^{-n} \|\varphi\|_{\mathcal{C}^2}$$

and

$$|\langle d_+^{-2sn}(f^n)^*(R) \wedge (f^n)_*(S) - \mu, \varphi\rangle| \leqslant c_{R,S} d_+^{-n} \|\varphi\|_{\mathcal{C}^2}$$

for every $n \geqslant 0$. In particular, $d_+^{-2sn}(f^n)^*(R) \wedge (f^n)_*(S) \to \mu$ as $n \to \infty$.

We will use z, w and (z, w) for the canonical coordinates of complex spaces $\mathbf{C}^k$ and $\mathbf{C}^k \times \mathbf{C}^k$. Consider also the canonical inclusions of $\mathbf{C}^k$ and $\mathbf{C}^k \times \mathbf{C}^k$ in $\mathbf{P}^k$ and $\mathbf{P}^{2k}$. We write $[z:t]$, $[w:t]$ or $[z:w:t]$ for the homogeneous coordinates of projective spaces. The hyperplanes at infinity are defined by $t=0$. If $g\colon \mathbf{C}^k \to \mathbf{C}^k$ is a polynomial automorphism, we write g_h (resp. g_h^{-1}) for the homogeneous part of maximal degree of g (resp. of g^{-1}). They are self-maps of $\mathbf{C}^k$, not invertible in general. In the sequel, we always assume that $k=2s$.

LEMMA 3.2. *Let F be the automorphism of $\mathbf{C}^k \times \mathbf{C}^k$ defined by*

$$F(z,w) := (f(z), f^{-1}(w)).$$

Then F is regular. The indeterminacy sets $I_\pm^F$ of $F^\pm$ are defined by

$$I_\pm^F := \{[z:w:0] : f_h^{\pm 1}(z) = 0,\ f_h^{\mp 1}(w) = 0\}.$$

Let $\Delta := \{(z,w) : z=w\}$ be the diagonal in $\mathbf{C}^k \times \mathbf{C}^k$. Then the sets $I_\pm^F$ do not intersect $\bar{\Delta}$, and $F(\bar{\Delta}) \cap \{t=0\} \subset I_-^F$.

Proof. Since $k=2s$, we have $d_+=d_-$ and $F_h^{\pm 1}(z,w) = (f_h^{\pm 1}(z), f_h^{\mp 1}(w))$. It follows that

$$I_\pm^F = \{[z:w:0] : F_h^{\pm 1}(z,w) = 0\} = \{[z:w:0] : f_h^{\pm 1}(z) = f_h^{\mp 1}(w) = 0\}.$$

We also have

$$I_\pm := \{[z:0] : f_h^{\pm 1}(z) = 0\}$$

and, since f is regular,

$$\{z \in \mathbf{C}^k : f_h(z) = f_h^{-1}(z) = 0\} = \{0\}.$$

This implies that $I_+^F \cap I_-^F = \varnothing$. Hence, F is regular. We also have

$$I_\pm^F \cap \bar{\Delta} = \{[z:z:0] : f_h(z) = f_h^{-1}(z) = 0\} = \varnothing. \qquad \square$$

LEMMA 3.3. *With the notation of Lemma 3.2, the Green current of bidegree $(2s,2s)$ of F is equal to $T_+^s \otimes T_-^s$.*

Proof. Let R and S be as in Proposition 3.1. Replacing R and S by $d_+^{-s} f^*(R)$ and $d_+^{-s} f_*(S)$, we get $\operatorname{supp}(R) \cap \{t=0\} \subset I_+$ and $\operatorname{supp}(S) \cap \{t=0\} \subset I_-$.

Consider the current $R \otimes S$ in $\mathbf{C}^k \times \mathbf{C}^k$ and in $\mathbf{P}^{2k}$. Lemma 3.2 implies that

$$\overline{\operatorname{supp}(R \otimes S)} \cap \{t=0\} \subset I_+^F.$$

Since $\dim I_+^F = 2s-1$, the trivial extension of $R \otimes S$ to $\mathbf{P}^{2k}$ (which we also denote by $R \otimes S$) is a positive closed current [19], [21]. One can check that the mass of $R \otimes S$ is equal to 1. Proposition 2.1 applied to F implies that $d_+^{-2sn}(F^n)^*(R \otimes S)$ converge to the Green current of bidegree $(2s,2s)$ of F. On the other hand, we have

$$d_+^{-2sn}(F^n)^*(R \otimes S) = d_+^{-2sn}(f^n)^*(R) \otimes (f^n)_*(S) \to T_+^s \otimes T_-^s$$

in $\mathbf{C}^k \times \mathbf{C}^k$. Hence, $T_+^s \otimes T_-^s$ is the Green current of bidegree $(2s,2s)$ of F. $\square$

Proof of Proposition 3.1. We can assume that φ has compact support in $\mathbf{C}^k$ and that $\|\varphi\|_{C^2}=1$. As in Lemma 3.3, we can assume that the current $R\otimes S$ in $\mathbf{P}^{2k}$ satisfies $\operatorname{supp}(R\otimes S)\cap\{t=0\}\subset I_+^F$.

Define $\widehat{\varphi}(z,w):=\varphi(z)$. Since $T_\pm$ are invariant and have continuous potentials away from $I_\pm$, we can write

$$\langle d_+^{-2sn}(f^n)^*(R)\wedge(f^n)_*(S)-\mu,\varphi\rangle=\langle d_+^{-2sn}(f^n)^*(R)\otimes(f^n)_*(S)-T_+^s\otimes T_-^s,\widehat{\varphi}[\Delta]\rangle.$$

Using a regularization of $[\Delta]$, one may find a smooth current Θ of mass 1 supported in a small neighbourhood $\mathcal{W}$ of $\bar{\Delta}$, with $\mathcal{W}\cap I_+^F=\varnothing$ (see Lemma 3.2), such that

$$\begin{aligned}|\langle d_+^{-2sn}(f^n)^*(R)\otimes(f^n)_*(S)-T_+^s\otimes T_-^s,\widehat{\varphi}[\Delta]\rangle&\\ -\langle d_+^{-2sn}(f^n)^*(R)\otimes(f^n)_*(S)-T_+^s\otimes T_-^s,\widehat{\varphi}\Theta\rangle|&\leqslant d_+^{-n}.\end{aligned}$$

The current Θ depends on n.

We have to estimate

$$\langle d_+^{-2sn}(f^n)^*(R)\otimes(f^n)_*(S)-T_+^s\otimes T_-^s,\widehat{\varphi}\Theta\rangle.$$

Fix an integer $m>0$ large enough. Write

$$\begin{aligned}&\langle d_+^{-2sn}(f^n)^*(R)\otimes(f^n)_*(S)-T_+^s\otimes T_-^s,\widehat{\varphi}\Theta\rangle\\ &\qquad=\langle d_+^{-2sn}(F^n)^*(R\otimes S)-d_+^{-2sm}(F^m)^*(T_+^s\otimes T_-^s),\widehat{\varphi}\Theta\rangle\\ &\qquad=\langle d_+^{-2s(n-m)}(F^{n-m})^*(R\otimes S)-T_+^s\otimes T_-^s,d_+^{-2sm}(F^m)_*(\widehat{\varphi}\Theta)\rangle\\ &\qquad=:\langle d_+^{-2s(n-2m)}(F^{n-2m})^*(T)-T_+^s\otimes T_-^s,\Phi\rangle,\end{aligned}$$

where $T:=d_+^{-2sm}(F^m)^*(R\otimes S)$ and $\Phi:=d_+^{-2sm}(F^m)_*(\widehat{\varphi}\Theta)$.

Hence, T has support in a small neighbourhood $\mathcal{U}$ of the filled Julia set $\mathcal{K}_+^F=\mathcal{K}_+\times\mathcal{K}_-$ of F, and Φ is a smooth form with support in a small neighbourhood $\mathcal{V}$ of $\mathcal{K}_-^F=\mathcal{K}_-\times\mathcal{K}_+$. Moreover, since m is large and φ is plurisubharmonic on $U_2\cap V_2$, we have $dd^c\Phi\geqslant0$ in a neighbourhood $\mathcal{U}'\Supset\mathcal{U}$ of $\mathcal{K}_+^F$. Putting $\widehat{\omega}(z,w):=\omega(z)$, we have $-\widehat{\omega}\leqslant dd^c\widehat{\varphi}\leqslant\widehat{\omega}$ since $\|\varphi\|_{C^2}=1$. It follows that

$$-d_+^{-2sm}(F^m)_*(\widehat{\omega}\wedge\Theta)\leqslant dd^c\Phi\leqslant d_+^{-2sm}(F^m)_*(\widehat{\omega}\wedge\Theta).$$

The positive closed current $d_+^{-2sm}(F^m)_*(\widehat{\omega}\wedge\Theta)$ has mass 1 since Θ is cohomologous to $[\Delta]$. The choice of $\mathcal{W}$, $\mathcal{U}$, $\mathcal{V}$, $\mathcal{U}'$ and m does not depend on φ and n. Lemma 3.3 and Proposition 2.1 applied to F, T and Φ imply that

$$\langle d_+^{-2s(n-2m)}(F^{n-2m})^*(T)-T_+^s\otimes T_-^s,\Phi\rangle\leqslant c'd_+^{-n}$$

and

$$|\langle d_+^{-2s(n-2m)}(F^{n-2m})^*(T) - T_+^s \otimes T_-^s, \Phi\rangle| \leqslant c'_T d_+^{-n}.$$

The desired inequalities of the proposition follow. Since every smooth test function on $\mathbf{P}^k$ can be written as a difference of smooth functions plurisubharmonic on $U_2 \cap V_2$, these inequalities imply that $d_+^{-2sn}(f^n)^*(R) \wedge (f^n)_*(S) \to \mu$. □

COROLLARY 3.4. *The Green measure of F is equal to $\mu \otimes \mu$.*

Proof. Let R and S be as in Proposition 3.1 and such that $\operatorname{supp}(R \otimes S) \cap \{t=0\} \subset I_+^F$ and $\operatorname{supp}(S \otimes R) \cap \{t=0\} \subset I_-^F$. Proposition 3.1 applied to F implies that the Green measure of F is equal to

$$\begin{aligned}
&\lim_{n\to\infty} d_+^{-4sn}(F^n)^*(R \otimes S) \wedge (F^n)_*(S \otimes R) \\
&\qquad = \lim_{n\to\infty} d_+^{-4sn}[(f^n)^*(R) \otimes (f^n)_*(S)] \wedge [(f^n)_*(S) \otimes (f^n)^*(R)] \\
&\qquad = \lim_{n\to\infty} [d_+^{-2sn}(f^n)^*(R) \wedge (f^n)_*(S)] \otimes [d_+^{-2sn}(f^n)^*(R) \wedge (f^n)_*(S)] \\
&\qquad = \mu \otimes \mu.
\end{aligned}$$

□

4. Speed of mixing

In this section, we give the proof of the main theorem. We first consider the case of smooth observables. Assume that $\alpha=\beta=2$ and that φ and ψ are $\mathcal{C}^2$-observables. Fix a bounded domain D in $\mathbf{C}^k$ containing $\mathcal{K} := \mathcal{K}_+ \cap \mathcal{K}_-$. Observe that φ and ψ can be written as differences of smooth functions strictly plurisubharmonic on a neighbourhood of $\overline{D}$. Hence, we can assume that $dd^c\varphi \geqslant \omega$ and $dd^c\psi \geqslant \omega$ on D, and that $\|\varphi\|_{\mathcal{C}^2} \leqslant M$ and $\|\psi\|_{\mathcal{C}^2} \leqslant M$ for some fixed constant $M>0$. The constants c, A and c' below do not depend on φ and ψ.

It is sufficient to prove the theorem for n even. Since

$$\langle \mu, (\varphi \circ f^{2n})\psi\rangle = \langle \mu, (\varphi \circ f^n)(\psi \circ f^{-n})\rangle$$

we have to prove that

$$|\langle \mu, (\varphi \circ f^n)(\psi \circ f^{-n})\rangle - \langle \mu, \varphi\rangle\langle \mu, \psi\rangle| \leqslant c d_+^{-n}. \tag{4}$$

Observe, since μ is invariant, that the left-hand side of (4) does not change if we add a constant to φ and/or ψ. Consequently, it suffices to show that there is a constant A such that

$$\langle \mu, (\varphi \circ f^n + A)(\psi \circ f^{-n} + A)\rangle - \langle \mu, \varphi + A\rangle\langle \mu, \psi + A\rangle \leqslant c d_+^{-n} \tag{5}$$

and

$$\langle \mu, (\varphi\circ f^n - A)(-\psi\circ f^{-n}+A)\rangle - \langle \mu, \varphi - A\rangle\langle \mu, -\psi+A\rangle \leqslant c d_+^{-n}. \tag{6}$$

We choose $A>0$ large enough so that $\phi(z,w):=(\varphi(z)+A)(\psi(w)+A)$ and $\phi'(z,w):=(\varphi(z)-A)(-\psi(w)+A)$ are plurisubharmonic on $D\times D$. This allows us to apply Proposition 3.1 to the automorphism F and to the test functions ϕ and ϕ'. We will check (5). The estimate (6) can be proved in the same way.

Fix a sufficiently large integer m. Define $T_1:=T_+^s\otimes T_-^s$ and $T_2:=d_+^{-2sm}(F^m)_*[\Delta]$. Since $F^*(T_1)=d_+^{2s}T_1$, and $T_\pm$ have continuous potentials in $\mathbf{C}^k$, we get the identities

$$\begin{aligned}\langle \mu, (\varphi\circ f^n+A)(\psi\circ f^{-n}+A)\rangle &= \langle T_+^s\wedge T_-^s, (\varphi\circ f^n+A)(\psi\circ f^{-n}+A)\rangle\\ &= \langle T_1\wedge[\Delta], \phi\circ F^n\rangle\\ &= \langle d_+^{-4sn+2sm}(F^{2n-m})^*(T_1)\wedge[\Delta], \phi\circ F^n\rangle\\ &= \langle d_+^{-4sn+2sm}(F^{n-m})^*(T_1)\wedge(F^n)_*[\Delta], \phi\rangle\\ &=: \langle d_+^{-4sn+4sm}(F^{n-m})^*(T_1)\wedge(F^{n-m})_*T_2, \phi\rangle.\end{aligned}$$

By Lemma 3.2, T_2 has support in a small neighbouhood $\mathcal{V}$ of $\mathcal{K}_-^F$.

Using a regularization of currents, we may find smooth currents T_1' and T_2' of mass 1 with support in small neighbourhoods $\mathcal{U}$ of $\mathcal{K}_+^F$ and $\mathcal{V}$ of $\mathcal{K}_-^F$, respectively, so that

$$\begin{aligned}&\langle d_+^{-4sn+4sm}(F^{n-m})^*(T_1)\wedge(F^{n-m})_*T_2, \phi\rangle\\ &\qquad\qquad - \langle d_+^{-4sn+4sm}(F^{n-m})^*(T_1')\wedge(F^{n-m})_*T_2', \phi\rangle \leqslant d_+^{-n}.\end{aligned}$$

The currents T_1' and T_2' depend on n. The choice of m, $\mathcal{U}$ and $\mathcal{V}$ depends only on D and f with $\mathcal{U}\cap\mathcal{V}\Subset D\times D$.

Since $\langle \mu, \varphi+A\rangle\langle \mu, \psi+A\rangle = \langle \mu\otimes\mu, \phi\rangle$, we only have to check that

$$\langle d_+^{-4sn+4sm}(F^{n-m})^*(T_1')\wedge(F^{n-m})_*T_2' - \mu\otimes\mu, \phi\rangle \leqslant c' d_+^{-n}.$$

This inequality follows directly from Corollary 3.4 and Proposition 3.1 applied to F and ϕ. This concludes the proof of the theorem in the case of $\mathcal{C}^2$-observables.

We complete the proof of the main theorem by passing to test functions of Hölder class. For this we use a special case of an argument obtained in collaboration with Nessim Sibony, see also Dolgopyat [12, p. 358]. Fix a test function ψ of class $\mathcal{C}^2$. Observe that the correlations $I_n(\,\cdot\,,\psi)$ define continuous linear forms on the space $\mathcal{C}^0$ of continuous functions and that we have

$$|I_n(\varphi,\psi)| \leqslant c\|\varphi\|_\infty\|\psi\|_{\mathcal{C}^2} \quad \text{for } \varphi \text{ continuous,}$$

where $c>0$ is a constant independent of n.

On the other hand, we have proved that

$$|I_n(\varphi,\psi)| \leqslant c d_+^{-n/2} \|\varphi\|_{\mathcal{C}^2} \|\psi\|_{\mathcal{C}^2} \quad \text{for } \varphi \text{ of class } \mathcal{C}^2.$$

The theory of interpolation between the Banach spaces $\mathcal{C}^0$ and $\mathcal{C}^2$ [22, p. 201] implies that

$$|I_n(\varphi,\psi)| \leqslant c' d_+^{-n\alpha/4} \|\varphi\|_{\mathcal{C}^\alpha} \|\psi\|_{\mathcal{C}^2} \quad \text{for } \varphi \text{ of class } \mathcal{C}^\alpha,$$

with $c'>0$ independent of n.

Now fix a function φ of class $\mathcal{C}^\alpha$. Applying the same argument to $I_n(\varphi,\cdot\,)$, we have

$$|I_n(\varphi,\psi)| \leqslant c'' d_+^{-n\alpha\beta/8} \|\varphi\|_{\mathcal{C}^\alpha} \|\psi\|_{\mathcal{C}^\beta} \quad \text{for } \psi \text{ of class } \mathcal{C}^\beta.$$

This completes the proof. □

Remark 4.1. In order to have $|I_n(\varphi,\psi)| \lesssim d^{-n/2}$, it suffices that ϕ and ϕ' are plurisubharmonic on $D\times D$. This holds in particular for $\varphi=-\log(-\varphi')$ and $\psi=-\log(-\psi')$ with φ' and ψ' strictly negative and strictly plurisubharmonic on a neighbourhood of $\overline{D}$. Indeed, one checks easily that $i\partial\varphi\wedge\bar\partial\varphi \leqslant i\partial\bar\partial\varphi$ and $i\partial\psi\wedge\bar\partial\psi \leqslant i\partial\bar\partial\psi$, and one can bound $\partial\varphi\wedge\bar\partial\psi$ by $i\partial\varphi\wedge\bar\partial\varphi + i\partial\psi\wedge\bar\partial\psi$ using the Cauchy–Schwarz inequality. Such functions φ and ψ can be nowhere continuous.

Acknowledgment. This article was partially written during my visit to the Humboldt University of Berlin. I would like to thank the Alexander von Humboldt foundation for its support and Professor Jürgen Leiterer for his hospitality. I am also grateful to the referee for helpful suggestions.

References

[1] Bedford, E., Lyubich, M. & Smillie, J., Polynomial diffeomorphisms of $\mathbf{C}^2$, IV. *Invent. Math.*, 112 (1993), 77–125.
[2] Bedford, E. & Smillie, J., Polynomial diffeomorphisms of $\mathbf{C}^2$, III. *Math. Ann.*, 294 (1992), 395–420.
[3] — Real polynomial diffeomorphisms with maximal entropy: Tangencies. *Ann. of Math.*, 160 (2004), 1–26.
[4] Benedicks, M. & Young, L.-S., Sinaĭ–Bowen–Ruelle measures for certain Hénon maps. *Invent. Math.*, 112 (1993), 541–576.
[5] — Markov extensions and decay of correlations for certain Hénon maps. *Astérisque*, 261 (2000), 13–56.
[6] Bost, J.-B., Gillet, H. & Soulé, C., Heights of projective varieties and positive Green forms. *J. Amer. Math. Soc.*, 7 (1994), 903–1027.
[7] Demailly, J.-P., Monge–Ampère operators, Lelong numbers and intersection theory, in *Complex Analysis and Geometry*, pp. 115–193. Plenum, New York, 1993.

[8] DINH, T.-C. & SIBONY, N., Dynamique des applications d'allure polynomiale. *J. Math. Pures Appl.*, 82 (2003), 367–423.
[9] — Distribution des valeurs de transformations méromorphes et applications. To appear in *Comment. Math. Helv.*, 81 (2006). arXiv:math.DS/0306095.
[10] — Green currents for holomorphic automorphisms of compact Kähler manifolds. *J. Amer. Math. Soc.*, 18 (2005), 291–312.
[11] — Dynamics of regular birational maps in $\mathbf{P}^k$. *J. Funct. Anal.*, 222 (2005), 202–216.
[12] DOLGOPYAT, D., On decay of correlations in Anosov flows. *Ann. of Math.*, 147 (1998), 357–390.
[13] FORNÆSS, J. E. & SIBONY, N., Complex Hénon mappings in $\mathbf{C}^2$ and Fatou–Bieberbach domains. *Duke Math. J.*, 65 (1992), 345–380.
[14] — Complex dynamics in higher dimension, in *Complex Potential Theory* (Montreal, QC, 1993), pp. 131–186. NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., 439. Kluwer, Dordrecht, 1994.
[15] FRIEDLAND, S. & MILNOR, J., Dynamical properties of plane polynomial automorphisms. *Ergodic Theory Dynam. Systems*, 9 (1989), 67–99.
[16] GILLET, H. & SOULÉ, C., Arithmetic intersection theory. *Publ. Math. Inst. Hautes Études Sci.*, 72 (1990), 93–174.
[17] GRIFFITHS, P. & HARRIS, J., *Principles of Algebraic Geometry*. Wiley, New York, 1994.
[18] GUEDJ, V., Courants extrémaux et dynamique complexe. *Ann. Sci. École Norm. Sup.*, 38 (2005), 407–426.
[19] HARVEY, R. & POLKING, J., Extending analytic objects. *Comm. Pure Appl. Math.*, 28 (1975), 701–727.
[20] SIBONY, N., Dynamique des applications rationnelles de $\mathbf{P}^k$, in *Dynamique et géométrie complexes* (Lyon, 1997), pp. 97–185. Panor. Synthèses, 8. Soc. Math. France, Paris, 1999.
[21] SKODA, H., Prolongement des courants positifs, fermés de masse finie. *Invent. Math.*, 66 (1982), 361–376.
[22] TRIEBEL, H., *Interpolation Theory, Function Spaces, Differential Operators*. North-Holland Math. Library, 18. North-Holland, Amsterdam–New York, 1978.

TIEN-CUONG DINH
Institut de Mathématiques de Jussieu
Plateau 7D, Analyse Complexe
Université Paris VI
175, rue du Chevaleret
FR-75013 Paris
France
dinh@math.jussieu.fr

Received July 7, 2005