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On some conformally invariant fully nonlinear equations, II. Liouville, Harnack and Yamabe

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1. Introduction

Let (M,g) be an n-dimensional compact smooth Riemannian manifold (without boundary). For $n=2$, we know from the uniformization theorem of Poincaré that there exist metrics that are pointwise conformal to g and have constant Gauss curvature. For $n\geqslant 3$, the well-known Yamabe conjecture states that there exist metrics which are pointwise conformal to g and have constant scalar curvature. The Yamabe conjecture is proved through the work of Yamabe [65], Trudinger [58], Aubin [2] and Schoen [53]. The Yamabe and related problems have attracted much attention in the last 30 years or so, see, e.g., [57], [3] and the references therein. Important methods and techniques in overcoming loss of compactness have been developed in such studies, which also play important roles in the research of other areas of mathematics. For $n\geqslant 3$, let $\hat{g}=u^{4/(n-2)}g$, where u is some positive function on M. The scalar curvature $R_{\hat{g}}$ of $\hat{g}$ can be calculated as

$$R_{\hat{g}}=u^{-(n+2)/(n-2)}\left(R_g u-\frac{4(n-1)}{n-2}\Delta_g u\right),$$

where R_g and Δ_g denote respectively the scalar curvature and the Laplace–Beltrami operator of g. The Yamabe conjecture is therefore equivalent to the existence of a positive solution of

$$-L_g u=\bar{R}u^{(n+2)/(n-2)}\quad \text{on } M,$$

where

$$L_g:=\Delta_g-\frac{n-2}{4(n-1)}R_g$$

is the conformal Laplacian of g, and $\bar{R}=0$ or $\bar{R}=\pm 2(n-1)$. The Yamabe problem can be divided into three cases—the positive case, the zero case and the negative case—according to the sign of the first eigenvalue of $-L_g$. Making a conformal change of metrics $\tilde{g}=\varphi^{4/(n-2)}g$, where φ is a positive eigenfunction of $-L_g$ associated with the first eigenvalue, we are led to the following three cases: $R_g>0$ on M, $R_g\equiv 0$ on M and $R_g<0$ on M. The positive case, i.e. $R_g>0$, is the most difficult.

Let

$$A_g := \frac{1}{n-2}\left(\mathrm{Ric}_g - \frac{R_g}{2(n-1)}g\right)$$

denote the Schouten tensor of g, where Ric_g denotes the Ricci tensor of g. We use $\lambda(A_g)=(\lambda_1(A_g),...,\lambda_n(A_g))$ to denote the eigenvalues of A_g with respect to g. Clearly

$$\sum_{i=1}^{n}\lambda_i(A_g) = \frac{1}{2(n-1)}R_g.$$

Let

$$V_1 = \left\{\lambda\in\mathbf{R}^n \;\middle|\; \sum_{i=1}^{n}\lambda_i > 1\right\},$$

and let

$$\Gamma(V_1) = \{s\lambda \mid s>0,\ \lambda\in V_1\}$$

be the cone with vertex at the origin generated by V_1.

The Yamabe problem in the positive case can be reformulated as follows: Assuming $\lambda(A_g)\in\Gamma(V_1)$, then there exists a Riemannian metric $\hat{g}$ which is pointwise conformal to g and satisfies $\lambda(A_{\hat{g}})\in\partial V_1$ on M.

In general, let V be an open convex subset of $\mathbf{R}^n$ which is symmetric with respect to the coordinates, i.e. $(\lambda_1,...,\lambda_n)\in V$ implies $(\lambda_{i_1},...,\lambda_{i_n})\in V$ for any permutation $(i_1,...,i_n)$ of $(1,...,n)$. We assume that $\varnothing\neq\partial V$ is in $C^{2,\alpha}$ for some $\alpha\in(0,1)$ in the sense that ∂V can be represented as the graph of some $C^{2,\alpha}$-function near every point. For $\lambda\in\partial V$, let $\nu(\lambda)$ denote the inner unit normal of ∂V. We further assume that

$$\nu(\lambda)\in\Gamma_n := \{\lambda\in\mathbf{R}^n \mid \lambda_i>0,\ 1\leqslant i\leqslant n\},\quad \lambda\in\partial V, \tag{1}$$

and

$$\nu(\lambda)\cdot\lambda>0,\quad \lambda\in\partial V. \tag{2}$$

Let

$$\Gamma(V) := \{s\lambda \mid \lambda\in V,\ 0<s<\infty\} \tag{3}$$

be the (open convex) cone with vertex at the origin generated by V.

Our first theorem establishes the existence and compactness of solutions to a fully nonlinear version of the Yamabe problem on locally conformally flat manifolds. A Riemannian manifold (M^n,g) is called *locally conformally flat* if near every point of M the metric can be represented in some local coordinates as $g=e^{\psi(x)}\sum_{i=1}^{n}(dx^i)^2$.

THEOREM 1.1. *For $n \geqslant 3$ and $\alpha \in (0,1)$, we assume that V is a symmetric open convex subset of $\mathbf{R}^n$, with $\varnothing \neq \partial V \in C^{4,\alpha}$, satisfying* (1) *and* (2). *Let (M^n, g) be a compact, smooth, connected, locally conformally flat Riemannian manifold of dimension n satisfying*

$$\lambda(A_g) \in \Gamma(V) \quad \text{on } M^n.$$

Then there exists a positive function $u \in C^{4,\alpha}(M^n)$ such that the conformal metric $\hat{g} = u^{4/(n-2)} g$ satisfies

$$\lambda(A_{\hat{g}}) \in \partial V \quad \text{on } M^n. \tag{4}$$

Moreover, if (M^n, g) is not conformally diffeomorphic to the standard n-sphere, then all positive solutions of (4) *satisfy*

$$\|u\|_{C^{4,\alpha}(M^n,g)} + \|1/u\|_{C^{4,\alpha}(M^n,g)} \leqslant C \quad \text{on } M^n,$$

where C is some positive constant depending only on (M^n, g), V and α.

Remark 1.1. Presumably, the existence of a $C^{2,\alpha}$-solution of (4) should hold under the weaker smoothness hypothesis $\partial V \in C^{2,\alpha}$. We prove this under an additional hypothesis that V is strictly convex, i.e. principal curvatures of ∂V are positive everywhere. See Appendix B.

We make the following conjecture:

Conjecture 1.1. Assume that V is an open symmetric convex subset of $\mathbf{R}^n$, with $\varnothing \neq \partial V \in C^\infty$, satisfying (1) and (2). Let (M^n, g) be a compact smooth Riemannian manifold of dimension $n \geqslant 3$ satisfying

$$\lambda(A_g) \in \Gamma(V) \quad \text{on } M^n.$$

Then there exists a smooth positive function $u \in C^\infty(M^n)$ such that the conformal metric $\hat{g} = u^{4/(n-2)} g$ satisfies

$$\lambda(A_{\hat{g}}) \in \partial V \quad \text{on } M^n. \tag{5}$$

For $V = V_1$, it is the Yamabe problem in the positive case. In general, the equation of u is a fully nonlinear elliptic equation of second order, and therefore the problem can be viewed as a fully nonlinear version of the Yamabe problem.

The fully nonlinear version of the Yamabe problem has the following equivalent formulation. The equivalence of the two formulations is shown in Appendix B.

Assume that

$$\Gamma \subset \mathbf{R}^n \text{ is an open convex symmetric cone with vertex at the origin} \tag{6}$$

satisfying

$$\Gamma_n \subset \Gamma \subset \Gamma_1 := \left\{ \lambda \in \mathbf{R}^n \,\middle|\, \sum_{i=1}^n \lambda_i > 0 \right\}. \tag{7}$$

Naturally, Γ being symmetric means that $(\lambda_1, \lambda_2, ..., \lambda_n) \in \Gamma$ implies $(\lambda_{i_1}, \lambda_{i_2}, ..., \lambda_{i_n}) \in \Gamma$ for any permutation $(i_1, i_2, ..., i_n)$ of $(1, 2, ..., n)$.

For $\alpha \in (0,1)$, assume that

$$f \in C^{4,\alpha}(\Gamma) \cap C^0(\bar{\Gamma}) \text{ is concave and symmetric in } \lambda_i, \tag{8}$$

satisfying

$$f|_{\partial\Gamma} = 0, \quad \nabla f \in \Gamma_n \text{ on } \Gamma \tag{9}$$

and

$$\lim_{s\to\infty} f(s\lambda) = \infty, \quad \lambda \in \Gamma. \tag{10}$$

Conjecture 1.1 is equivalent to the following conjecture:

Conjecture 1.1′. Assume that (f, Γ) satisfies (6)–(10). Let (M^n, g) be a compact smooth Riemannian manifold of dimension $n \geqslant 3$, satisfying $\lambda(A_g) \in \Gamma$ on M^n. Then there exists a smooth positive function $u \in C^\infty(M^n)$ such that the conformal metric $\hat{g} = u^{4/(n-2)} g$ satisfies

$$f(\lambda(A_{\hat{g}})) = 1, \quad \lambda(A_{\hat{g}}) \in \Gamma, \text{ on } M^n. \tag{11}$$

Theorem 1.1 is equivalent to the following theorem:

THEOREM 1.1′. *For $n \geqslant 3$ and $\alpha \in (0,1)$, we assume that (f, Γ) satisfies* (6)–(10). *Let (M^n, g) be a compact, smooth, connected, locally conformally flat Riemannian manifold of dimension n satisfying $\lambda(A_g) \in \Gamma$ on M^n. Then there exists a positive function $u \in C^{4,\alpha}(M^n)$ such that the conformal metric $\hat{g} = u^{4/(n-2)}$ satisfies* (11). *Moreover, if (M^n, g) is not conformally diffeomorphic to the standard n-sphere, all solutions of* (11) *satisfy*

$$\|u\|_{C^{4,\alpha}(M^n,g)} + \|1/u\|_{C^{4,\alpha}(M^n,g)} \leqslant C, \tag{12}$$

where $C > 0$ is some constant depending only on (M^n, g), (f, Γ) and α.

Remark 1.2. C^0- and C^1-bounds of u and $1/u$ do not depend on the concavity of f. This can be seen from the proof.

For $1 \leqslant k \leqslant n$, let

$$\sigma_k(\lambda) = \sum_{1 \leqslant i_1 < ... < i_k \leqslant n} \lambda_{i_1} ... \lambda_{i_k}$$

be the kth symmetric function and let Γ_k be the connected component of $\{\lambda\in\mathbf{R}^n \mid \sigma_k(\lambda)>0\}$ containing the positive cone Γ_n. Then $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$ satisfies the hypothesis of Theorem 1.1′, see [7].

Remark 1.3. For $(f,\Gamma)=(\sigma_1,\Gamma_1)$, it is the Yamabe problem in the positive case on locally conformally flat manifolds, and the result is due to Schoen [53], [55]. For $(f,\Gamma)=(\sigma_2^{1/2},\Gamma_2)$ in dimension $n=4$, the result was proved without the locally conformally flatness by Chang, Gursky and Yang [9]. For $(f,\Gamma)=(\sigma_n^{1/n},\Gamma_n)$, an existence result was established by Viaclovsky [64] on a class of manifolds. For $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$, the result was established in our earlier paper [35], while the existence part for $k\neq\frac{1}{2}n$ was independently established by Guan and Wang in [24] using a different method. Guan, Viaclovsky and Wang [22] subsequently proved the algebraic fact that $\lambda(A_g)\in\Gamma_k$ for $k\geqslant\frac{1}{2}n$ implies the positivity of the Ricci tensor, and therefore (M,g) is conformally covered by $\mathbf{S}^n$, and both the existence and compactness results in this case follow from known results. For $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$, $k=3,4$, on 4-dimensional Riemannian manifolds, as well as for $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$, $k=2,3$, on 3-dimensional Riemannian manifolds which are not simply-connected, the existence and compactness results were established by Gursky and Viaclovsky in [30].

Remark 1.4. If we assume in addition that $f\in C^{k,\alpha}$ for some $k>4$, then, by Schauder theory, (12) can be strengthened as

$$\|u\|_{C^{k,\alpha}(M^n,g)}+\|1/u\|_{C^{k,\alpha}(M^n,g)}\leqslant C,$$

where $C>0$ also depends on k.

Since our C^0- and C^1-estimates for solutions of (4) (or, equivalently, of (11)) do not make use of the convexity of V (or concavity of f), we raise the following question:

Question 1.1. Under the hypotheses of Theorem 1.1′, but without the concavity assumption on f, does there exist a positive Lipschitz function u on M^n such that $\hat{g}=u^{4/(n-2)}g$ satisfies (11) in the viscosity sense?

Equation (11) is a fully nonlinear elliptic equation of u. Fully nonlinear elliptic equations involving $f(\lambda(D^2u))$ have been investigated in the classical and pioneering paper of Caffarelli, Nirenberg and Spruck [7]. Extensive studies and outstanding results on such equations are given by Guan and Spruck [20], Trudinger [59], Trudinger and Wang [60], and many others. Fully nonlinear equations involving $f(\lambda(\nabla_g^2u+g))$ on Riemannian manifolds are studied by Li [43], Urbas [61], and others. Fully nonlinear equations involving the Schouten tensor have been studied by Viaclovsky in [62] and [64], and by Chang, Gursky and Yang in the remarkable papers [9] and [8]. There have been many papers, preprints, expository articles, and works in preparation, on the subject and related ones, see, e.g., [17], [26], [63], [27], [28], [23], [24], [5], [33], [35], [4], [22], [30], [29], [13], [34], [44], [45], [10], [25], [12], [31], [19] and [41]. The approach developed in our earlier work [35] and continued in the present paper makes use of and extends ideas from previous works on the Yamabe equation by Gidas, Ni and Nirenberg [18], Caffarelli, Gidas and Spruck [6], Schoen [54], [55], Li and Zhu [49], and Li and Zhang [46].

For $\hat{g}=u^{4/(n-2)}g$, we have (see, e.g., [62])

$$A_{\hat{g}}=-\frac{2}{n-2}u^{-1}\nabla^2 u+\frac{2n}{(n-2)^2}u^{-2}\nabla u\otimes\nabla u-\frac{2}{(n-2)^2}u^{-2}|\nabla u|^2 g+A_g,$$

where covariant derivatives on the right-hand side are with respect to g.

Let $g_1=u^{4/(n-2)}g_{\mathrm{flat}}$, where g_{flat} denotes the Euclidean metric on $\mathbf{R}^n$. Then, by the above transformation formula,

$$A_{g_1}=u^{4/(n-2)}A^u_{ij}\,dx^i\,dx^j,$$

where

$$\begin{aligned}A^u:=-\frac{2}{n-2}u^{-(n+2)/(n-2)}\nabla^2 u&+\frac{2n}{(n-2)^2}u^{-2n/(n-2)}\nabla u\otimes\nabla u\\&-\frac{2}{(n-2)^2}u^{-2n/(n-2)}|\nabla u|^2 I,\end{aligned}$$

and I is the identity $n\times n$-matrix. In this case, $\lambda(A_{g_1})=\lambda(A^u)$, where $\lambda(A^u)$ denotes the eigenvalues of the symmetric $n\times n$-matrix A^u.

Let ψ be a Möbius transformation in $\mathbf{R}^n$, i.e. a transformation generated by translation, multiplication by nonzero constants, and the inversion $x\mapsto x/|x|^2$. For any positive C^2-function u, let $u_\psi:=|J_\psi|^{(n-2)/2n}(u\circ\psi)$, where J_ψ denotes the Jacobian of ψ. A calculation shows that A^{u_ψ} and $A^u\circ\psi$ differ only by an orthogonal conjugation, and therefore

$$\lambda(A^{u_\psi})=\lambda(A^u)\circ\psi. \tag{13}$$

Let $\mathcal{S}^{n\times n}$ denote the set of real symmetric $n\times n$-matrices, $O(n)$ denote the set of real orthogonal $n\times n$-matrices, $U\subset\mathcal{S}^{n\times n}$ be an open set satisfying

$$O^{-1}UO=U,\quad O\in O(n), \tag{14}$$

and let $F\in C^1(U)$ satisfy

$$F(O^{-1}MO)=F(M),\quad M\in U,\ O\in O(n). \tag{15}$$

By (13) and (15),

$$F(A^{u_\psi}) \equiv F(A^u)\circ\psi.$$

We proved in [35] that any conformally invariant operator $H(\,\cdot\,,u,\nabla u,\nabla^2 u)$, in the sense

$$H(\,\cdot\,,u_\psi,\nabla u_\psi,\nabla^2 u_\psi) \equiv H(\,\cdot\,,u,\nabla u,\nabla^2 u)\circ\psi,$$

must be of the form $F(A^u)$.

Our next theorem concerns a Harnack-type inequality for general conformally invariant equations on locally conformally flat manifolds. Let $\mathcal{S}_+^{n\times n}\subset\mathcal{S}^{n\times n}$ denote the set of positive definite matrices. We will assume that U and F further satisfy

$$U\cap\{M+tN \mid 0<t<\infty\} \text{ is convex},\quad M\in\mathcal{S}^{n\times n},\ N\in\mathcal{S}_+^{n\times n}, \tag{16}$$

$$(F_{ij}(M))>0,\quad M\in U, \tag{17}$$

where $F_{ij}(M):=(\partial F/\partial M_{ij})(M)$, and, for some $\delta>0$,

$$F(M)\neq 1,\quad M\in U\cap\left\{M\in\mathcal{S}^{n\times n}\,\middle|\, \|M\|:=\left(\sum_{i,j=1}^n M_{ij}^2\right)^{1/2}<\delta\right\}. \tag{18}$$

THEOREM 1.2. *For $n\geqslant 3$, let $U\subset\mathcal{S}^{n\times n}$ satisfy* (14) *and* (16), *and let $F\in C^1(U)$ satisfy* (15), (17) *and* (18). *For $R>0$, let $u\in C^2(B_{3R})$ be a positive solution of*

$$F(A^u)=1,\quad A^u\in U,\ \text{in } B_{3R}, \tag{19}$$

where B_{3R} denotes the ball in $\mathbf{R}^n$ of radius $3R$ and centered at the origin. Then

$$\Big(\sup_{B_R} u\Big)\Big(\inf_{B_{2R}} u\Big)\leqslant C(n)\delta^{(2-n)/2}R^{2-n}, \tag{20}$$

where $C(n)$ is some constant depending only on n.

Let

$$U_k:=\{M\in\mathcal{S}^{n\times n}\mid \lambda(M)\in\Gamma_k\}$$

and

$$F_k(M)=\sigma_k(\lambda(M)),\quad M\in U_k.$$

For $(F,U)=(F_1,U_1)$, (19) takes the form

$$-\Delta u=\tfrac12(n-2)u^{(n+2)/(n-2)}\quad\text{in } B_{3R}.$$

Remark 1.5. The Harnack-type inequality (20) for $(F,U)=(F_1,U_1)$ was obtained by Schoen in [54]. For a class of nonlinearity including $(F,U)=(F_k^{1/k},U_k)$, $1\leqslant k\leqslant n$, the Harnack-type inequality was established in our earlier work [35].

Remark 1.6. In Theorem 1.2, there is no concavity assumption on F, and the constant $C(n)$ is given explicitly in the proof. The Harnack-type inequalities in [54] and [35] are proved by contradiction arguments which do not yield such an explicit constant.

Let g be a smooth Riemannian metric on $B_3\subset\mathbf{R}^n$, $n\geqslant 3$, and let (f,Γ) satisfy our usual hypotheses. Consider

$$f(\lambda(A_{u^{4/(n-2)}g}))=1,\quad \lambda(A_{u^{4/(n-2)}g})\in\Gamma,\ \text{in } B_3. \tag{21}$$

Question 1.2. Are there some positive constants C and δ, depending on (B_3,g) and (f,Γ), such that

$$\big(\sup_{B_\varepsilon} u\big)\big(\inf_{B_{2\varepsilon}} u\big)\leqslant C\varepsilon^{2-n},\quad 0<\varepsilon\leqslant\delta,$$

holds for any positive solution of (21)?

Remark 1.7. The answer to the above question is affirmative for the Yamabe equation (i.e. $(f,\Gamma)=(\sigma_1,\Gamma_1)$) in dimension $n=3,4$, see Li and Zhang [48].

We have avoided the use of Liouville-type theorems in the proofs of Theorems 1.1, 1.1′ and 1.2. However, in order to solve Conjecture 1.1 on general Riemannian manifolds, to answer Question 1.2, or to study many other issues using fully nonlinear elliptic equations involving the Schouten tensor, it is important to establish the corresponding Liouville-type theorems.

For $n\geqslant 3$, consider

$$-\Delta u=\tfrac{1}{2}(n-2)u^{(n+2)/(n-2)}\quad \text{on } \mathbf{R}^n. \tag{22}$$

It was proved by Obata [51] and Gidas, Ni and Nirenberg [18] that any positive C^2-solution of (22) satisfying $\int_{\mathbf{R}^n}u^{2n/(n-2)}<\infty$ must be of the form

$$u(x)=(2n)^{(n-2)/4}\left(\frac{a}{1+a^2|x-\bar{x}|^2}\right)^{(n-2)/2},$$

where $a>0$ and $\bar{x}\in\mathbf{R}^n$. The hypothesis $\int_{\mathbf{R}^n}u^{2n/(n-2)}<\infty$ was removed by Caffarelli, Gidas and Spruck [6]; this is important for applications. The method in [18] is completely different from that of [51]. The method used in our proof of the Liouville-type theorems on general conformally invariant fully nonlinear equations (Theorem 1.3) is in the spirit of [18] rather than that of [51]. As in [6], the superharmonicity of the solution has played an important role in our proof of Theorem 1.3, see Lemma 4.1. On the other hand, under some additional hypothesis on the solution near infinity, the superharmonicity of the solution is not needed, see Theorem 1.4 in [35].

Somewhat different proofs of the result of Caffarelli, Gidas and Spruck were given in [14], [49] and [46]. In particular, the proofs in [49] and [46] fully exploit the conformal invariance of the problem and capture the solutions directly rather than going through the usual procedure of proving radial symmetry of solutions and then classifying radial solutions. A related result of Gidas and Spruck in [20] states that there is no positive solution to the equation $-\Delta u = u^p$ in $\mathbf{R}^n$ when $1<p<(n+2)/(n-2)$.

For $n\geqslant 3$ and $-\infty<p\leqslant(n+2)/(n-2)$, we consider the equation

$$F(A^u)=u^{p-(n+2)/(n-2)}, \quad A^u\in U,\ u>0,\ \text{on } \mathbf{R}^n. \tag{23}$$

For $(F,U)=(F_1,U_1)$, equation (23) takes the form

$$-\Delta u=\tfrac{1}{2}(n-2)u^p, \quad u>0, \text{ on } \mathbf{R}^n.$$

THEOREM 1.3. *For $n\geqslant 3$, let $U\subset\mathcal{S}^{n\times n}$ satisfy* (14) *and* (16), *and let $F\in C^1(U)$ satisfy* (15) *and* (17). *Assume that $u\in C^2(\mathbf{R}^n)$ is a superharmonic solution of* (23) *for some p, $-\infty<p\leqslant(n+2)/(n-2)$. Then either $u\equiv$constant or $p=(n+2)/(n-2)$ and, for some $\bar{x}\in\mathbf{R}^n$ and some positive constants a and b satisfying $2b^2a^{-2}I\in U$ and $F(2b^2a^{-2}I)=1$,*

$$u(x)\equiv\left(\frac{a}{1+b^2|x-\bar{x}|^2}\right)^{(n-2)/2}, \quad x\in\mathbf{R}^n. \tag{24}$$

Remark 1.8. For $(F,U)=(F_k^{1/k},U_k)$, $1\leqslant k\leqslant n$, a solution of (23) is automatically superharmonic.

Remark 1.9. The most difficult case is for $p=(n+2)/(n-2)$. When $(F,U)=(F_1,U_1)$, the result in this case (the rest of this remark also refers to this case), as mentioned earlier, was established by Caffarelli, Gidas and Spruck [6], while under some additional hypothesis the result was proved by Obata [51] and Gidas, Ni and Nirenberg [18]. For $(F,U)=(F_k^{1/k},U_k)$, and under some strong hypothesis on u near infinity, the result was proved by Viaclovsky [62], [63]. For $(F,U)=(F_2^{1/2},U_2)$ in dimension $n=4$, the result is due to Chang, Gursky and Yang [9]. For $(F,U)=(F_k^{1/k},U_k)$, the result was established in our earlier paper [35]; for $(F,U)=(F_2^{1/2},U_2)$ in dimension $n=5$, as well as for $(F,U)=(F_2^{1/2},U_2)$ in dimension $n\geqslant 6$ under the additional hypothesis $\int_{\mathbf{R}^n}u^{2n/(n-2)}<\infty$, the result was independently established by Chang, Gursky and Yang [11, Chapter 3]. Under some fairly strong hypothesis (but weaker than that used in [62] and [63]) on u near infinity, the result was proved in [35] without the superharmonicity assumption on u.

As mentioned earlier, Theorem 1.1′ in the case $(f,\Gamma)=(\sigma_1,\Gamma_1)$ is the Yamabe problem in the positive case on locally conformally flat manifolds, and the result is due to Schoen [53], [55]. The proof in [55] has three main ingredients: The first is the existence of the developing map due to Schoen and Yau [56], the second is the use of the method of moving planes, and the third is the Liouville-type theorem of Caffarelli, Gidas and Spruck [6]. A major difficulty in extending the result for $(f,\Gamma)=(\sigma_1,\Gamma_1)$ to fully nonlinear (f,Γ) was the lack of a corresponding Liouville-type theorem. An important step was taken by Zhang and the second author in [46], which gives a proof of Schoen's Harnack-type inequality for the Yamabe equation without using the Liouville-type theorem in [6]. Adapting this idea, we established in [35, Theorem 1.27] the Harnack-type inequality (20) for a class of nonlinearity including

$$(F,U)=(F_k^{1/k},U_k),\quad 1\leqslant k\leqslant n,$$

under the circumstance that the corresponding Liouville-type theorem was not available. This also made us recognize the possibility of proving Theorem 1.1′ without the corresponding Liouville-type theorem. Indeed, in [35] we have developed an approach, based on the method of moving spheres (i.e. the method of moving planes, together with the conformal invariance of the problem), to prove the existence and compactness results for the fully nonlinear version of the Yamabe problem on locally conformally flat manifolds under the circumstance that the corresponding Liouville-type theorem was not available. Another major difficulty in proving Theorem 1.1′ is the lack of C^0- and C^1-estimates of solutions. We have developed a new approach in [35], again based on the method of moving spheres, to obtain such estimates. We have also introduced in [35] a homotopy which connects the general fully nonlinear version of the Yamabe problem to the Yamabe problem, and used the degree for second-order fully nonlinear elliptic operators in [42] and the result in [55] for the Yamabe problem to prove the existence of solutions to the fully nonlinear ones.

In [21], Guan, Lin and Wang have also presented a proof of Theorem 1.2, under an additional concavity hypothesis on F, and of Theorem 1.1′. We clarify these overlaps in this paragraph: These results follow immediately from our earlier work [35] and Lemma A.2—a quantitative version of a calculus lemma used repeatedly in [35]. Indeed, the only change one needs to make is to move the four lines below (4.3) on page 1446 of [35] to be right after line 5 of the same page. After making this change, the gradient estimate stated on line 7 of the same page follows from Lemma A.2, and Theorem 1.2, under an additional concavity hypothesis on F, and Theorem 1.1′, as well as our new C^0- and C^1-estimates, follow from the proofs of Theorem 1.25 and Theorem 1.27 in [35]. Theorem 1.2 and Theorem 1.1′, with an emphasis on our new C^0- and C^1-estimates based on the method of moving planes, were presented by the second author in his invited talk at ICM 2002 in Beijing. The proof in [21], following [35] (in particular, following the above-mentioned steps developed there), provides the ingredient beyond [35] which, as explained above, amounts to Lemma A.2. We present the proof of Theorem 1.1′ and Theorem 1.1 in §2 and §3, respectively. The proof of Theorem 1.1′, which appeared in slightly shorter form in [34] and in preprint form in [37], contains one slight simplification of the arguments in [35] which avoids the use of local C^2-estimates (only global C^2-estimates are needed); the proof of Theorem 1.2, which also appeared in slightly shorter form in [34] and in [37], contains one more ingredient to remove the concavity assumption on F, which also yields an explicit constant $C(n)$ in (20).

Due to Theorem 1.1′ (or Theorem 1.1), Conjecture 1.1′ (or Conjecture 1.1) mainly concerns the problem on Riemannian manifolds which are not locally conformally flat. In general, equation (11) does not have a variational formulation. A plausible approach is to establish a priori estimates (12) for all solutions of (11), and to use the homotopy in [35] to connect the problem to the Yamabe problem. For the Yamabe problem (i.e. (11) for $(f,\Gamma)=(\sigma_1,\Gamma_1)$), such estimates were given by Li and Zhu [50] in dimension $n=3$; the estimates in dimension $n=4$ follow from a combination of the results of Li and Zhang [48] and Druet [15]; Li and Zhang have extended the estimate to dimension $n\leqslant 7$, as well as to dimension $n\geqslant 8$, but under an additional hypothesis that the Weyl tensor of g is nowhere vanishing, see [47]. The Liouville-type theorem of Caffarelli, Gidas and Spruck has played an important role in the proof of this result. It is clear that Theorem 1.3 will also play an important role in proving Conjecture 1.1′.

The main difficulty in proving Theorem 1.3 is to remove the possible isolated singularity of u at infinity. By the conformal invariance of the problem, we may assume that the isolated singularity is at 0 instead of at infinity. The following analytical issue is relevant: Let $u\in C^\infty(B_1\setminus\{0\})$ and $v\in C^\infty(B_1)$ be positive solutions of

$$F(A^u)=1, \quad A^u\in U, \text{ in } B_1\setminus\{0\}$$

and

$$F(A^v)=1, \quad A^v\in U, \text{ in } B_1,$$

satisfying

$$u>v \quad \text{in } B_1\setminus\{0\}.$$

Is it true that

$$\liminf_{x\to 0}\,(u(x)-v(x))>0?$$

If the answer to the above question were "yes", then the proof of Theorem 1.4 in [35] would yield a proof of Theorem 1.3 for $p=(n+2)/(n-2)$. So far, the answer to the question is not known even for $(F,U)=(F_k^{1/k},U_k)$, $2\leqslant k\leqslant n$. The answer to the question is "yes" for $(F,U)=(F_1,U_1)$ due to some elementary properties of superharmonic functions in a punctured ball. As far as we know, the isolated singularity issue encountered in the application of the method of moving planes has always been handled by providing an affirmative answer to a local question like the above. Our proof of Theorem 1.3 avoids this local question by exploiting global information of u, through a delicate use of Lemma 4.1. The proof of Theorem 1.3 also fully exploits the conformal invariance of the problem and captures the solutions directly rather than going through the usual procedure of proving radial symmetry of solutions and then classifying radial solutions. Two proofs of Theorem 1.3 appeared in preprint forms in [38] and [39]. We present in §4 the proof in [39]. The present paper is essentially the first part of [40].

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2. Proof of Theorem 1.1 and Theorem 1.1′

In Appendix B, we deduce the equivalence of Theorem 1.1 and Theorem 1.1′. Therefore we only need to prove one of the two theorems.

Proof of Theorem 1.1′. Without loss of generality, we further assume that f is homogeneous of degree 1. Indeed, in Appendix B, we construct a new function $\tilde f$ which is homogeneous of degree 1, and satisfies the same assumptions as f does and $\tilde f^{-1}(1)=f^{-1}(1)$.

We first establish (12). Let $(\widetilde M,\tilde g)$ be the universal cover of (M^n,g), with $i:\widetilde M\to M^n$ being a covering map and $\tilde g=i^*g$. It is well known that there exists a conformal immersion

$$\Phi:(\widetilde M,\tilde g)\longrightarrow(\mathbf{S}^n,g_0),$$

where g_0 denotes the standard metric on $\mathbf{S}^n$. By $\lambda(A_g)\in\Gamma$ and the assumption $\Gamma\subset\Gamma_1$, we have $R_g>0$. Hence by a deep theorem of Schoen and Yau in [56], Φ is injective. Let

$$\Omega=\Phi(\widetilde M).$$

CLAIM 2.1. *We have that*

$$\frac{1}{C} \leqslant u \leqslant C \quad \text{and} \quad |\nabla_g u| \leqslant C \quad \text{on } M^n,$$

where $u \in C^2(M^n)$ *is an arbitrary positive solution of* (11) *with* $\hat{g}=u^{4/(n-2)}g$ *and* $C>0$ *is some constant depending only on* (M^n,g) *and* (f,Γ).

For convenience, we introduce

$$U=\{A\in \mathcal{S}^{n\times n} \mid \lambda(A)\in\Gamma\}$$

and

$$F(A)=f(\lambda(A)), \quad A\in U.$$

We distinguish two cases:

Case 1. $\Omega=\mathbf{S}^n$;

Case 2. $\Omega\neq\mathbf{S}^n$.

In Case 1, $(\Phi^{-1})^*\tilde{g}=\eta^{4/(n-2)}g_0$ on $\mathbf{S}^n$, where η is a positive smooth function on $\mathbf{S}^n$. Let $\tilde{u}=u\circ i$. Since $F(A_{\tilde{u}^{4/(n-2)}\tilde{g}})=1$ on $\widetilde{M}$, we have

$$F\big(A_{[(\tilde{u}\circ\Phi^{-1})\eta]^{4/(n-2)}g_0}\big)=1 \quad \text{on } \mathbf{S}^n.$$

By Corollary 1.6 in [35], $(\tilde{u}\circ\Phi^{-1})\eta=a|J_\varphi|^{(n-2)/2n}$ for some positive constant a and some conformal diffeomorphism $\varphi\colon\mathbf{S}^n\to\mathbf{S}^n$. Since $\varphi^*g_0=|J_\varphi|^{2/n}g_0$, we have, by the above equation, that

$$f(a^{-4/(n-2)}(n-1)e)=f(a^{-4/(n-2)}\lambda(A_{g_0}))=1,$$

where $e=(1,\dots,1)$. By (10) and the concavity of f, we know that $\nabla f(\lambda)\cdot\lambda>0$ for any $\lambda\in\Gamma$. Thus $f|_{\partial\Gamma}=0$, and (10) implies that a is a constant uniquely determined by (f,Γ).

Fix a compact subset E of $\widetilde{M}$ such that $i(E)=M^n$. Since (M^n,g) is not conformally diffeomorphic to $(\mathbf{S}^n,g_0)$, $\pi_1(M^n)$ is nontrivial. Let $\tilde{x}^{(1)}\in E$ and $\tilde{x}^{(2)}\in\widetilde{M}$ be two distinct points satisfying $\tilde{u}(\tilde{x}^{(1)})=\tilde{u}(\tilde{x}^{(2)})=\max_{M^n}u$. Then

$$\operatorname{dist}_{g_0}(\Phi(\tilde{x}^{(1)}),\Phi(\tilde{x}^{(2)}))\geqslant\frac{1}{C}.$$

Consequently,

$$\min\{|J_\varphi(\Phi(\tilde{x}^{(1)}))|,|J_\varphi(\Phi(\tilde{x}^{(2)}))|\}\leqslant C,$$

from which we deduce that

$$\min\{\tilde{u}(\tilde{x}^{(1)})\eta(\Phi(\tilde{x}^{(1)})),\tilde{u}(\tilde{x}^{(2)})\eta(\Phi(\tilde{x}^{(2)}))\}\leqslant C.$$

It follows that

$$\max_{M^n} u = \tilde{u}(\tilde{x}^{(1)}) = \tilde{u}(\tilde{x}^{(2)}) \leqslant C.$$

Moreover, we also know from the above and the formula of $\tilde{u}$ that

$$|J_\varphi| \leqslant C \quad \text{on } \mathbf{S}^n,$$

from which we deduce that

$$\| |J_\varphi| \|_{C^m(\mathbf{S}^n, g_0)} + \| 1/|J_\varphi| \|_{C^m(\mathbf{S}^n, g_0)} \leqslant C(m)$$

and therefore

$$\|u\|_{C^m(M^n,g)} + \|1/u\|_{C^m(M^n,g)} \leqslant C$$

for some C depending only on (M,g), (f,Γ) and m. The estimates in (12) are established in this case.

In Case 2, by the result in [56], $\Omega = \Phi(\widetilde{M})$ is an open and dense subset of $\mathbf{S}^n$, and $(\Phi^{-1})^* \tilde{g} = \eta^{4/(n-2)} g_0$ on Ω, where η is a positive smooth function in Ω satisfying $\lim_{z \to \partial\Omega} \eta(z) = \infty$. Let $u(x) = \max_{M^n} u$ for some $x \in M^n$, and let $i(\tilde{x}) = x$ for some $\tilde{x} \in E$. By composing with a rotation of $\mathbf{S}^n$, we may assume without loss of generality that $\Phi(\tilde{x}) = S$, the south pole of $\mathbf{S}^n$. Let $P: \mathbf{S}^n \to \mathbf{R}^n$ be the stereographic projection, and let v be the positive function on the open subset $P(\Omega)$ of $\mathbf{R}^n$ determined by

$$(P^{-1})^* (\eta^{4/(n-2)} g_0) = v^{4/(n-2)} g_{\text{flat}},$$

where g_{flat} denotes the Euclidean metric on $\mathbf{R}^n$. Then for some $\varepsilon > 0$, depending only on (M^n, g), we have

$$B_{9\varepsilon} := \{ x \in \mathbf{R}^n \mid |x| < 9\varepsilon \} \subset P(\Omega)$$

and

$$\operatorname{dist}_{\text{flat}}(P(\Phi(E)), \partial P(\Omega)) > 9\varepsilon.$$

On $P(\Omega)$,

$$F(A^{\hat{u}}) = 1 \quad \text{and} \quad \lambda(A^{\hat{u}}) \in \Gamma,$$

where $\hat{u} = (\tilde{u} \circ \Phi^{-1} \circ P^{-1}) v$.

By the property of η, we know that

$$\lim_{P(\Omega) \ni y \to \bar{y} \in \partial P(\Omega)} \hat{u}(y) = \infty \tag{25}$$

and, if the north pole of $\mathbf{S}^n$ does not belong to Ω,

$$\lim_{\substack{|y| \to \infty \\ y \in P(\Omega)}} (|y|^{n-2} \hat{u}(y)) = \infty. \tag{26}$$

For every $x\in\mathbf{R}^n$ satisfying $\mathrm{dist}_{\mathrm{flat}}(x, P(\Phi(E)))<2\varepsilon$, we can perform a moving sphere argument as in the corresponding part in [35] (for w_j there) to show that, for $0<\lambda<4\varepsilon$, $|y-x|\geqslant\lambda$ and $y\in P(\Omega)$,

$$\hat{u}_{x,\lambda}(y) := \frac{\lambda^{n-2}}{|y-x|^{n-2}}\,\hat{u}\left(\frac{\lambda^2(y-x)}{|y-x|^2}\right) \leqslant \hat{u}(y). \tag{27}$$

When proving the above, there is a minor difference between the cases $N\in\Omega$ and $N\notin\Omega$, where N is the north pole of $\mathbf{S}^n$. If $N\notin\Omega$, then by (26), there is no worry about "touching at infinity" in the moving sphere procedure. If $N\in\Omega$, then ∞ is a regular point of $\hat{u}$ (i.e. $|z|^{2-n}\hat{u}(z/|z|^2)$ can be extended as a positive C^2-function near $z=0$), and therefore by the strong maximum principle argument as in [35], if "touching at infinity" occurs, $\hat{u}_{x,\lambda}$ would coincide with $\hat{u}$ in the unbounded connected component of $P(\Omega)$ for some $0<\lambda<4\varepsilon$, which violates (25) since $\hat{u}_{x,\lambda}$ is apparently bounded near any point of $\partial P(\Omega)$.

By Lemma A.2 in Appendix A, we deduce from (27) that

$$|\nabla(\log \hat{u})(y)| \leqslant C(\varepsilon), \quad \text{for } y \text{ such that } \mathrm{dist}_{\mathrm{flat}}(y, P(\Phi(E))) < \varepsilon.$$

It follows, for some C depending only on (M^n, g), that

$$|\nabla_g \log u| \leqslant C \quad \text{on } M^n.$$

Hence Claim 2.1 follows directly from the bounds

$$\min_{M^n} u \leqslant C \quad \text{and} \quad \max_{M^n} u \geqslant \frac{1}{C} \quad \text{for some universal constant } C. \tag{28}$$

To establish (28), let $u(\bar{x})=\min_{M^n} u$. At $\bar{x}$, by $\nabla u(\bar{x})=0$, $(\nabla^2 u(\bar{x}))\geqslant 0$ and (9), we have

$$1 = f(\lambda(A_{\hat{g}})) \leqslant f(u^{-4/(n-2)}\lambda(A_g)),$$

which implies, by $f|_{\partial\Gamma}=0$ and $f\in C^0(\bar{\Gamma})$, that $u^{-4/(n-2)}(\bar{x})\geqslant C$, i.e. $u(\bar{x})\leqslant C$. Similarly, by the properties of f (in particular (10)), we can establish $\max_{M^n} u\geqslant 1/C$. The C^2-estimate of u has been established in [35] (see also [64] for the estimates for $(f,\Gamma)=(\sigma_k^{1/k},\Gamma_k)$). The C^2-estimate of $1/u$ follows in view of Claim 2.1.

Thus when (M^n, g) is not conformally diffeomorphic to a standard sphere, we have proved that any positive solution of (11) satisfies, for some constant C depending only on (M^n, g) and (f,Γ),

$$\|u\|_{C^2(M^n,g)} + \|1/u\|_{C^2(M^n,g)} \leqslant C.$$

Since f is concave in Γ, $C^{2,\alpha}$- and higher-order derivative estimates follow from a theorem of Evans [16] and Krylov [32], and the Schauder estimate.

To establish the existence part of Theorem 1.1′, we only need to treat the case that (M^n, g) is not conformally diffeomorphic to a standard sphere, since it is obvious otherwise. We use the following homotopy introduced in [35]. For $0 \leqslant t \leqslant 1$, let

$$f_t(\lambda) = f(t\lambda + (1-t)\sigma_1(\lambda)e)$$

be defined on

$$\Gamma_t := \{\lambda \in \mathbf{R}^n \mid t\lambda + (1-t)\sigma_1(\lambda)e \in \Gamma\},$$

where $e = (1, 1, ..., 1)$.

Consider, for $0 \leqslant t \leqslant 1$,

$$f_t(\lambda(A_{\hat{g}})) = 1, \quad \lambda(A_{\hat{g}}) \in \Gamma_t, \text{ on } M^n. \tag{29}$$

Here and below $\hat{g} = u^{4/(n-2)} g$.

By the a priori estimates that we have just established, there exists some constant $C > 0$ independent of $t \in [0, 1]$ such that for all solutions u of (29),

$$\|u\|_{C^{4,\alpha}}(M^n, g) + \|1/u\|_{C^{4,\alpha}}(M^n, g) \leqslant C. \tag{30}$$

By (30) and the assumption $f|_{\partial\Gamma} = 0$, there exists $\delta > 0$ independent of $t \in [0, 1]$ such that all solutions u of (29) satisfy

$$\mathrm{dist}(\lambda(A_{\hat{g}}), \partial\Gamma_t) \geqslant 2\delta.$$

Define, for $0 \leqslant t \leqslant 1$,

$$O_t^* = \{u \in C^{4,\alpha}(M^n) \mid \lambda(A_{\hat{g}}) \in \Gamma_t, \ \mathrm{dist}(\lambda(A_{\hat{g}}), \partial\Gamma_t) > \delta, \\ u > 0, \ \|u\|_{C^{4,\alpha}(M^n, g)} + \|1/u\|_{C^{4,\alpha}(M^n, g)} < 2C\},$$

where C is the constant in (30). By [42],

$$d_t := \deg(F_t - 1, O_t^*, 0), \quad 0 \leqslant t \leqslant 1,$$

is well-defined, where $F_t[u] := f_t(\lambda(A_{\hat{g}})) - 1$, and

$$d_t \equiv d_0, \quad 0 \leqslant t \leqslant 1.$$

In particular,

$$d_1 = d_0.$$

The equation (29) for $t = 0$ is the Yamabe equation. By the result of Schoen in [55] for the Yamabe problem, $d_0 = -1$. Thus $d_1 \neq 0$ and equation (11) has a solution. Theorem 1.1′ is established. □

3. Proof of Theorem 1.2

Proof of Theorem 1.2. Part of the proof of this theorem is taken from [35], which we include here for the reader's convenience. We only need to prove the theorem for $R=\delta=1$. Indeed, let

$$\widetilde{F}(M):=F(\delta M),\quad \widetilde{U}:=\delta^{-1}U\quad\text{and}\quad \tilde{u}(x):=\delta^{(n-2)/4}R^{(n-2)/2}u(Rx).$$

Then

$$\widetilde{F}(A^{\tilde{u}})=1,\quad A^{\tilde{u}}\in\widetilde{U},\ \text{ in } B_3,$$

and $(\widetilde{F},\widetilde{U})$ satisfies the hypothesis of Theorem 1.2 with $R=\delta=1$. Thus, once we have established the theorem in the case $R=\delta=1$, we have

$$\Bigl(\sup_{B_R}u\Bigr)\Bigl(\inf_{B_{2R}}u\Bigr)=\delta^{(2-n)/2}R^{2-n}\Bigl(\sup_{B_1}\tilde{u}\Bigr)\Bigl(\inf_{B_2}\tilde{u}\Bigr)\leqslant C\delta^{(2-n)/2}R^{2-n}.$$

Thus we assume in the following that $R=\delta=1$. Let $u(\bar{x})=\max_{\bar{B}_1}u$. As in the proof of Theorem 1.27 in [35], we can find $\tilde{x}\in B_{1/2}(\bar{x})$ such that

$$u(\tilde{x})\geqslant 2^{(2-n)/2}\sup_{B_\sigma(\tilde{x})}u$$

and

$$\gamma:=u(\tilde{x})^{2/(n-2)}\sigma\geqslant\tfrac{1}{2}u(\bar{x})^{2/(n-2)},\tag{31}$$

where $\sigma=\frac{1}{2}(1-|\tilde{x}-\bar{x}|)\leqslant\frac{1}{2}$.

If

$$\gamma\leqslant 2^{n+8}n^4,$$

then

$$\Bigl(\sup_{B_1}u\Bigr)\Bigl(\inf_{B_2}u\Bigr)\leqslant u(\bar{x})^2\leqslant(2\gamma)^{(n-2)/2}\leqslant C(n),$$

and we are done. So we always assume that

$$\gamma>2^{n+8}n^4.$$

Let $\Gamma:=u(\tilde{x})^{2/(n-2)}\geqslant 2\gamma$, and consider

$$w(y):=\frac{1}{u(\tilde{x})}u\biggl(\tilde{x}+\frac{y}{u(\tilde{x})^{2/(n-2)}}\biggr),\quad |y|<\Gamma.$$

Clearly

$$\min_{\partial B_\Gamma}w\geqslant\frac{1}{u(\tilde{x})}\inf_{B_2}u,\tag{32}$$

$$1=w(0)\geqslant 2^{(2-n)/2}\sup_{B_\gamma}w.\tag{33}$$

By the conformal invariance of the equation satisfied by u,

$$F(A^w)=1, \quad w>0, \text{ on } B_\Gamma.$$

Fix

$$r=2^{n+6}n^4<\tfrac{1}{4}\gamma.$$

For all $|x|<r$, consider

$$w_{x,\lambda}(y):=\left(\frac{\lambda}{|y-x|}\right)^{n-2} w\left(x+\frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

By the conformal invariance of the equation, we have

$$F(A^{w_{x,\lambda}})=1, \quad w_{x,\lambda}>0, \text{ on } B_\Gamma\setminus B_\lambda(x), \ 0<\lambda<\tfrac{3}{4}\gamma.$$

As in [35], there exists $0<\lambda_x<r$ such that we have

$$w_{x,\lambda}(y)\leqslant w(y), \quad 0<\lambda<\lambda_x, \ y\in B_\Gamma\setminus B_\lambda(x),$$

and

$$w_{x,\lambda}(y)<w(y), \quad 0<\lambda<\lambda_x, \ y\in \partial B_\Gamma.$$

By the moving sphere argument as in [35], we only need to consider the following two cases:

Case 1. For some $|x|<r$ and some $\lambda\in(0,r)$, $w_{x,\lambda}$ touches w on ∂B_Γ.

Case 2. For all $|x|<r$ and all $\lambda\in(0,r)$, we have

$$w_{x,\lambda}(y)\leqslant w(y), \quad |y-x|\geqslant\lambda, \ y\in B_\Gamma.$$

In Case 1, let $\lambda\in(0,r)$ be the smallest number for which $w_{x,\lambda}$ touches w on ∂B_Γ. By (32), we have, for some $|y_0|=\Gamma$,

$$\frac{1}{u(\tilde{x})}\inf_{B_2} u\leqslant \min_{\partial B_\Gamma} w=w_{x,\lambda}(y_0).$$

Recall (33),

$$w_{x,\lambda}(y_0)\leqslant\left(\frac{\lambda}{|y_0-x|}\right)^{n-2}\sup_{B_\gamma} w\leqslant 2^{(n-2)/2}\left(\frac{\lambda}{|y_0-x|}\right)^{n-2}\leqslant 2^{(n-2)/2}\left(\frac{r}{\Gamma-r}\right)^{n-2}.$$

Therefore

$$\sigma^{(n-2)/2}u(\tilde{x})\inf_{B_2} u\leqslant 2^{(n-2)/2}\sigma^{(n-2)/2}u(\tilde{x})^2\left(\frac{r}{\Gamma-r}\right)^{n-2}.$$

Since $4r<\gamma\leqslant\frac{1}{2}\Gamma$ and $\sigma\leqslant\frac{1}{2}$,

$$\begin{aligned}\sigma^{(n-2)/2}u(\tilde{x})\inf_{B_2}u &\leqslant 2^{(n-2)/2}\sigma^{(n-2)/2}u(\tilde{x})^2\frac{r^{n-2}}{\left(\frac{1}{2}\Gamma\right)^{n-2}}\\ &=2^{3(n-2)/2}\sigma^{(n-2)/2}r^{n-2}\leqslant 2^{n-2}r^{n-2}.\end{aligned}\tag{34}$$

We deduce from (31) and (34) that

$$\left(\sup_{B_1}u\right)\left(\inf_{B_2}u\right)\leqslant 4^{n-2}r^{n-2}\leqslant C(n).$$

In Case 2, we have, by Lemma A.2 and (33), that

$$|\nabla w(y)|\leqslant 2(n-2)r^{-1}w(y)\leqslant (n-2)\,2^{n/2}r^{-1},\quad |y|\leqslant r.$$

Let ε be the number such that

$$\xi(y):=\frac{1-\varepsilon}{r}(r-|y|^2),\quad |y|<\sqrt{r},$$

satisfies

$$w\geqslant\xi\quad\text{on } B_{\sqrt{r}}$$

and, for some $|\bar{y}|<\sqrt{r}$,

$$w(\bar{y})=\xi(\bar{y}).$$

Since $1=w(0)\geqslant\xi(0)=1-\varepsilon$ and $w(\bar{y})>0$, we have $0\leqslant\varepsilon<1$.

By the estimates of $|\nabla w|$ and the mean value theorem,

$$|w(y)-1|=|w(y)-w(0)|\leqslant(n-2)\,2^{n/2}r^{-1/2},\quad |y|\leqslant\sqrt{r}.$$

So

$$1-(n-2)\,2^{n/2}r^{-1/2}\leqslant w(\bar{y})=\xi(\bar{y})\leqslant 1-\varepsilon,$$

and therefore

$$0\leqslant\varepsilon\leqslant(n-2)\,2^{n/2}r^{-1/2}.$$

Clearly,

$$\nabla w(\bar{y})=\nabla\xi(\bar{y}),\quad |\nabla\xi(\bar{y})|\leqslant\frac{2}{\sqrt{r}}\quad\text{and}\quad D^2w(\bar{y})\geqslant D^2\xi(\bar{y})=-2(1-\varepsilon)r^{-1}I.$$

It follows that

$$A^w(\bar{y})\leqslant A^\xi(\bar{y})\leqslant\frac{10n+4}{(n-2)^2}2^{2n/(n-2)}r^{-1}I.$$

Since $F(A^w(\bar{y}))=1$, we have, by (18) (recall that $\delta=1$),

$$\frac{10n+4}{(n-2)^2}2^{2n/(n-2)}r^{-1}\geqslant 1,$$

violating the choice of r. Thus we have shown that Case 2 can never occur. Theorem 1.2 is established. □

4. Proof of Theorem 1.3

LEMMA 4.1. *For $n\geqslant 2$ and $B_1\subset\mathbf{R}^n$, let $u\in L^1_{\mathrm{loc}}(B_1\setminus\{0\})$ be the solution of*

$$\Delta u\leqslant 0 \quad \text{in } B_1\setminus\{0\}$$

in the distribution sense. Assume that there exist $a\in\mathbf{R}$ and $p\neq q\in\mathbf{R}^n$ such that

$$u(x)\geqslant \max\{a+p\cdot x-\delta(x), a+q\cdot x-\delta(x)\}, \quad x\in B_1\setminus\{0\},$$

where $\delta(x)\geqslant 0$ satisfies $\lim_{x\to 0}(\delta(x)/|x|)=0$. Then

$$\lim_{r\to 0}\inf_{B_r} u > a.$$

Proof. Let

$$v(x):=a+p\cdot x-\delta(x) \quad\text{and}\quad w(x):=a+q\cdot x-\delta(x), \qquad x\in B_1.$$

By subtracting $a+p\cdot x$ from u, v and w, respectively, we can assume that $a=0$ and $p=0$. After a rotation and a dilation of the coordinates, we can also assume that $\nabla w(0)=e_1$.

Let $u_\varepsilon:=u(\varepsilon\,\cdot)/\varepsilon$, $v_\varepsilon:=v(\varepsilon\,\cdot)/\varepsilon$ and $w_\varepsilon:=w(\varepsilon\,\cdot)/\varepsilon$. We have

$$v_\varepsilon(x)=o(1) \quad\text{and}\quad w_\varepsilon(x)=x_1+o(1),$$

where $o(1)\to 0$ uniformly on $\overline{B}_1$ as $\varepsilon\to 0$. For all $\bar\delta>0$, since $u_\varepsilon\geqslant v_\varepsilon$, there exists $\varepsilon_0>0$ such that

$$u_\varepsilon(x)\geqslant -\bar\delta \quad \text{in } B_1, \text{ for all } \varepsilon\leqslant\varepsilon_0.$$

By $u_\varepsilon\geqslant w_\varepsilon$, we have $u_\varepsilon\geqslant c_0>0$ on $\Omega:=B_{1/4}\left(\frac{1}{2}e_1\right)$ for some universal constant c_0 independent of $\bar\delta$ and ε.

Let $\xi^{\bar\delta}$ be the solution of

$$\begin{cases} \Delta\xi^{\bar\delta}=0 & \text{in } B_1\setminus\overline{\Omega},\\ \xi^{\bar\delta}=\frac{1}{2}c_0 & \text{on } \partial\Omega,\\ \xi^{\bar\delta}=-2\bar\delta & \text{on } \partial B_1.\end{cases}$$

Since $\xi^{\bar\delta}\to\xi^0$ in $C^\infty(\overline{B}_1)$, we have, for small $\bar\delta$,

$$\xi^{\bar\delta}(0)>\tfrac{1}{2}\xi^0(0)>0, \tag{35}$$

where ξ^0 is the solution of

$$\begin{cases} \Delta\xi^{0}=0 & \text{in } B_1\setminus\overline{\Omega},\\ \xi^{0}=\frac{1}{2}c_0 & \text{on } \partial\Omega,\\ \xi^{0}=0 & \text{on } \partial B_1.\end{cases}$$

In the following, we fix some $\bar{\delta}>0$ such that (35) holds.

Let G be the solution of

$$\begin{cases} -\Delta G = \delta_0 & \text{in } B_1, \\ G = 0 & \text{on } \partial B_1, \\ G(x) \to \infty & \text{as } x \to 0, \end{cases}$$

where δ_0 is the Dirac mass at 0.

Let $A>1$ be chosen later. For $0<\delta<\frac{1}{10}$, consider

$$\eta_\varepsilon := u_\varepsilon + \frac{A}{G(\delta)} G - \xi^{\bar{\delta}} \quad \text{on } B_1 \setminus (B_\delta \cup \Omega).$$

We have

$$\Delta \eta_\varepsilon \leqslant 0 \quad \text{in } B_1 \setminus \overline{(B_\delta \cup \Omega)}.$$

Near ∂B_δ,

$$\eta_\varepsilon \geqslant -\bar{\delta} + \tfrac{1}{2} A - \tfrac{1}{2} c_0 > 0 \quad \text{for large } A,$$

and near ∂B_1, $\eta_\varepsilon \geqslant -\bar{\delta}+\frac{3}{2}\bar{\delta}>0$. Hence

$$\eta_\varepsilon > 0 \quad \text{in } B_1 \setminus \overline{(B_\delta \cup \Omega)}. \tag{36}$$

For any fixed $x\in B_1\setminus\{0\}$, for all δ, with $0<\delta<|x|$, and all $\varepsilon>0$ small, sending $\delta\to 0$ in (36) leads to $u_\varepsilon(x)\geqslant \xi^{\bar{\delta}}(x)$. Therefore, for all $\varepsilon\leqslant\varepsilon_0$,

$$\liminf_{r\to 0} \inf_{B_r} u = \liminf_{r\to 0} \inf_{B_r} u_\varepsilon \geqslant \xi^{\bar{\delta}}(0) > \tfrac{1}{2}\xi_0(0) > 0. \qquad \square$$

Lemma 4.1 is sufficient for our use. Such a result holds for more general linear elliptic operators of second order. For example, we have the following lemma:

LEMMA 4.2. *For $n\geqslant 2$ and $B_1\subset\mathbf{R}^n$, let $u\in C^2(B_1\setminus\{0\})$ satisfy*

$$Lu := a^{ij} u_{ij} + b^i u_i + cu \leqslant f \quad \text{in } B_1 \setminus \{0\},$$

where $(a^{ij})>0$, $a^{ij}\in C^\alpha(B_1)$ for some α, $0<\alpha<1$, and $f, b_i, c\in L^\infty(B_1)$. Assume that there exist $a\in\mathbf{R}$ and $p\neq q\in\mathbf{R}^n$ such that

$$u(x) \geqslant \max\{a+p\cdot x-\delta(x), a+q\cdot x-\delta(x)\}, \quad x\in B_1\setminus\{0\},$$

where $\delta>0$ satisfies $\lim_{x\to 0}\delta(x)/|x|=0$. Then

$$\liminf_{x\to 0} u(x) > a.$$

Proof. Let

$$v(x):=a+p\cdot x-\delta(x) \quad \text{and} \quad w(x):=a+q\cdot x-\delta(x), \qquad x\in B_1.$$

By subtracting $a+p\cdot x$ from u, v and w, respectively, and replacing $f(x)$ by

$$f(x)-b^i(x)v_i(0)-c(x)\nabla v(0)\cdot x-c(x)v(0),$$

we can assume that

$$a=0, \quad p=0 \quad \text{and} \quad Lu\leqslant f \quad \text{in } B_1\setminus\{0\}.$$

Let $Q\in GL(n)$ satisfy $Q(a^{ij}(0))Q^t=I_{n\times n}$. Replacing u, v and w by

$$u(Q^{-1}\cdot), \quad v(Q^{-1}\cdot) \quad \text{and} \quad w(Q^{-1}\cdot),$$

and $a^{ij}(x)$, $b^i(x)$, $c(x)$ and $f(x)$ by

$$Q(a^{ij}(Q^{-1}x))Q^t, \quad Q^t(b^i(Q^{-1}x)), \quad c(Q^{-1}x) \quad \text{and} \quad \bar f(Q^{-1}x),$$

respectively, we can assume that $(a^{ij})(0)=I_{n\times n}$.

Let $u_\varepsilon:=u(\varepsilon\cdot)/\varepsilon$, $v_\varepsilon:=v(\varepsilon\cdot)/\varepsilon$ and $w_\varepsilon:=w(\varepsilon\cdot)/\varepsilon$. We have

$$v_\varepsilon(x)=o(1) \quad \text{and} \quad w_\varepsilon(x)=\nabla w(0)\cdot x+o(1) \quad \text{on } \overline{B}_1.$$

We may also assume that $|\nabla w(0)|=1$ by a dilation. Hence, since $u_\varepsilon\geqslant v_\varepsilon$ and $u_\varepsilon\geqslant w_\varepsilon$, for all $\bar\delta>0$, there exists $\varepsilon_0>0$ such that for all $\varepsilon\leqslant\varepsilon_0$,

$$u_\varepsilon(x)\geqslant-\bar\delta \quad \text{on } B_1 \qquad \text{and} \qquad u_\varepsilon\geqslant c_0 \quad \text{on } \Omega:=B_{1/4}\big(\tfrac12\nabla w(0)\big),$$

where $c_0>0$ is some universal constant independent of $\bar\delta$ and ε. Moreover, u_ε satisfies the equation

$$L^\varepsilon u_\varepsilon(x):=a^{ij}(\varepsilon x)(u_\varepsilon)_{ij}(x)+\varepsilon b^i(\varepsilon x)(u_\varepsilon)_i(x)+\varepsilon^2c(\varepsilon x)u_\varepsilon(x)\leqslant\varepsilon f(\varepsilon x) \quad \text{in } B_1.$$

Let $\xi_{\bar\delta}$ be the solution of

$$\begin{cases} L^\varepsilon\xi^{\bar\delta}(x)=\varepsilon f(\varepsilon x) & \text{in } B_1\setminus\overline{\Omega}, \\ \xi^{\bar\delta}=\frac12 c_0 & \text{on } \partial\Omega, \\ \xi^{\bar\delta}=-2\bar\delta & \text{on } \partial B_1. \end{cases}$$

We have $\xi^{\bar\delta}\to\xi^0$ in $C^1(B_1\setminus\bar\Omega)$, where ξ^0 is the solution of

$$\begin{cases} \Delta\xi^0=0 & \text{in } B_1\setminus\bar\Omega,\\ \xi^0=\frac{1}{2}c_0 & \text{on } \partial\Omega,\\ \xi^0=0 & \text{on } \partial B_1.\end{cases}$$

Hence we can initially pick some $\bar\delta>0$ such that $\xi^{\bar\delta}(0)>\xi^0(0)>0$.

Let G be the solution of

$$\begin{cases} -L^\varepsilon G=\delta_0 & \text{in } B_1,\\ G=0 & \text{on } \partial B_1,\\ G(x)\to\infty & \text{as } x\to 0.\end{cases}$$

We know that G is asymptotically radial as $\varepsilon\to 0$.

Let $A>1$ be chosen later. For $0<\delta<\frac{1}{10}$, consider

$$\eta_\varepsilon:=u_\varepsilon+\frac{A}{\min_{\partial B_\delta}G}G-\xi^{\bar\delta}\quad \text{on } B_1\setminus(B_\delta\cup\Omega).$$

We have

$$L^\varepsilon\eta_\varepsilon\leqslant 0\quad \text{in } B_1\setminus\overline{(B_\delta\cup\Omega)}.$$

On ∂B_δ,

$$\eta_\varepsilon\geqslant -\bar\delta+A-\tfrac{1}{2}c_0>0,$$

and on ∂B_1, $-\bar\delta+2\bar\delta=\bar\delta>0$. Hence

$$\eta_\varepsilon>0\quad \text{in } B_1\setminus\overline{(B_\delta\cup\Omega)}. \tag{37}$$

For any fixed $x\in B_1\setminus\{0\}$, for all δ with $0<\delta<|x|$, and all $\varepsilon\leqslant\varepsilon_0$, sending $\delta\to 0$ in (37) leads to $u_\varepsilon(x)\geqslant\xi^{\bar\delta}(x)$. Therefore

$$\liminf_{x\to 0}u(x)=\liminf_{x\to 0}u_\varepsilon(x)\geqslant\xi^{\bar\delta}(0)>\tfrac{1}{2}\xi^0(0)>0. \qquad\square$$

Proof of Theorem 1.3 *for* $p=(n+2)/(n-2)$. Since u is a positive superharmonic function, we have, by the maximum principle, that

$$u(x)\geqslant\frac{\min_{\partial B_1}u}{|x|^{n-2}},\quad |x|\geqslant 1.$$

In particular,

$$\liminf_{|x|\to\infty}|x|^{n-2}u(x)>0. \tag{38}$$

LEMMA 4.3. *For any $x\in\mathbf{R}^n$, there exists $\lambda_0(x)>0$ such that*

$$u_{x,\lambda}(y):=\left(\frac{\lambda}{|y-x|}\right)^{n-2}u\left(x+\frac{\lambda^2(y-x)}{|y-x|^2}\right)\leqslant u(y),\quad |y-x|\geqslant\lambda,\ 0<\lambda<\lambda_0(x).$$

Proof. This follows from the proof of Lemma 2.1 in [46]. □

For any $x\in\mathbf{R}^n$, set

$$\bar\lambda(x):=\sup\{\mu\mid u_{x,\lambda}(y)\leqslant u(y)\text{ for }|y-x|\geqslant\lambda\text{ and }0<\lambda<\mu\}.$$

Let

$$\alpha:=\liminf_{|x|\to\infty}(|x|^{n-2}u(x)).\tag{39}$$

Because of (38),

$$0<\alpha\leqslant\infty.\tag{40}$$

If $\alpha=\infty$, then the moving sphere procedure will never stop, and therefore $\bar\lambda(x)=\infty$ for any $x\in\mathbf{R}^n$. This follows from arguments in [46] and [35] (see also [36]). By the definition of $\bar\lambda(x)$ and the fact that $\bar\lambda(x)=\infty$, we have

$$u_{x,\lambda}(y)\leqslant u(y),\quad |y-x|\geqslant\lambda>0.$$

By a calculus lemma (see, e.g., Lemma 11.2 in [46]), $u\equiv$constant, and Theorem 1.3 for $p=(n+2)/(n-2)$ is proved in this case (i.e. $\alpha=\infty$). So, from now on, we assume that

$$0<\alpha<\infty.\tag{41}$$

By the definition of $\bar\lambda(x)$,

$$u_{x,\lambda}(y)\leqslant u(y),\quad |y-x|\geqslant\lambda,\ 0<\lambda<\bar\lambda(x).$$

Multiplying the above by $|y|^{n-2}$ and sending $|y|\to\infty$, we have

$$\alpha\geqslant\lambda^{n-2}u(x),\quad 0<\lambda<\bar\lambda(x).$$

Sending $\lambda\to\bar\lambda(x)$, we have (using (41))

$$\infty>\alpha\geqslant\bar\lambda(x)^{n-2}u(x),\quad x\in\mathbf{R}^n.\tag{42}$$

Since the moving sphere procedure stops at $\bar\lambda(x)$, we must have, by using the arguments in [46] and [35] (see also [36]),

$$\liminf_{|y|\to\infty}(u(y)-u_{x,\bar\lambda(x)}(y))\,|y|^{n-2}=0,\tag{43}$$

i.e.

$$\alpha = \bar{\lambda}(x)^{n-2}u(x), \quad x \in \mathbf{R}^n. \tag{44}$$

Let us switch to some more convenient notation. For a Möbius transformation ϕ, we use the notation

$$u_\phi := |J_\phi|^{(n-2)/2n}(u\circ\phi),$$

where J_ϕ denotes the Jacobian of ϕ.

For $x\in\mathbf{R}^n$, let

$$\phi^{(x)}(y) := x + \frac{\bar{\lambda}(x)^2(y-x)}{|y-x|^2}.$$

We know that $u_{\phi^{(x)}} = u_{x,\bar{\lambda}(x)}$.

Let $\psi(y) := y/|y|^2$, and let

$$w^{(x)} := (u_{\phi^{(x)}})_\psi = u_{\phi^{(x)}\circ\psi}.$$

For $x\in\mathbf{R}^n$, the only possible singularity for $w^{(x)}$ (on $\mathbf{R}^n\cup\{\infty\}$) is $x/|x|^2$. In particular, $y=0$ is a regular point of $w^{(x)}$. A direct calculation yields

$$w^{(x)}(0) = \bar{\lambda}(x)^{n-2}u(x),$$

and therefore, by (44),

$$w^{(x)}(0) = \alpha, \quad x\in\mathbf{R}^n.$$

Clearly, $u_\psi \in C^2(\mathbf{R}^n\setminus\{0\})$ and $\Delta u_\psi \leqslant 0$ in $\mathbf{R}^n\setminus\{0\}$, $\liminf_{y\to 0} u_\psi(y) = \alpha$ and, for some $\delta(x)>0$,

$$w^{(x)} \in C^2(B_{\delta(x)}), \quad x\in\mathbf{R}^n,$$
$$u_\psi \geqslant w^{(x)} \quad \text{in } B_{\delta(x)}\setminus\{0\},\ x\in\mathbf{R}^n.$$

LEMMA 4.4. *$\nabla w^{(x)}(0) = \nabla w^{(0)}(0)$, i.e. $\nabla w^{(x)}(0)$ is independent of $x\in\mathbf{R}^n$.*

Proof. This follows from Lemma 4.1. Indeed, for any $x, \tilde{x}\in\mathbf{R}^n$, let

$$v := w^{(x)}, \quad w := w^{(\tilde{x})} \quad \text{and} \quad u := u_\psi.$$

We know that $w(0)=v(0)$, $u_\psi \geqslant w$ and $u_\psi \geqslant v$ near the origin, and we also know that $\liminf_{y\to 0} u_\psi(y) = w(0)$, so, by Lemma 4.1, we must have $\nabla v(0) = \nabla w(0)$, i.e. $\nabla w^{(x)}(0) = \nabla w^{(\tilde{x})}(0)$. Lemma 4.4 is established. □

For $x\in\mathbf{R}^n$,

$$w^{(x)}(y)=\frac{1}{|y|^{n-2}}\left(\frac{\bar\lambda(x)}{|y/|y|^2-x|}\right)^{n-2}u\left(x+\frac{\bar\lambda(x)^2(y/|y|^2-x)}{|y/|y|^2-x|^2}\right)$$
$$=\left(\frac{\bar\lambda(x)}{|y/|y|-|y|x|}\right)^{n-2}u\left(x+\frac{\bar\lambda(x)^2(y-|y|^2x)}{|y/|y|-|y|x|^2}\right)$$
$$=\left(\frac{\bar\lambda(x)^2}{1-2x\cdot y+|y|^2|x|^2}\right)^{(n-2)/2}u\left(x+\frac{\bar\lambda(x)^2(y-|y|^2x)}{1-2x\cdot y+|y|^2|x|^2}\right).$$

So, for $|y|$ small,

$$w^{(x)}(y)=\bar\lambda(x)^{n-2}(1+(n-2)x\cdot y)\,u(x+\bar\lambda(x)^2y)+O(|y|^2),$$

and, using (44),

$$\nabla w^{(x)}(0)=(n-2)\,\bar\lambda(x)^{n-2}u(x)x+\bar\lambda(x)^n\nabla u(x)=(n-2)\alpha x+\alpha^{n/(n-2)}u(x)^{n/(2-n)}\nabla u(x).$$

By Lemma 4.4, $\vec V:=\nabla w^{(x)}(0)$ is a constant vector in $\mathbf{R}^n$, so we have

$$\nabla_x\left(\tfrac12(n-2)\alpha^{n/(n-2)}u(x)^{-2/(n-2)}-\tfrac12(n-2)\alpha|x|^2+\vec V\cdot x\right)\equiv 0.$$

Consequently, for some $\bar x\in\mathbf{R}^n$ and $d\in\mathbf{R}$,

$$u(x)^{-2/(n-2)}\equiv\alpha^{-2/(n-2)}|x-\bar x|^2+d\alpha^{-2/(n-2)}.$$

Since $u>0$, we must have $d>0$. Thus

$$u(x)\equiv\left(\frac{\alpha^{2/(n-2)}}{d+|x-\bar x|^2}\right)^{(n-2)/2}.$$

Let $a=\alpha^{2/(n-2)}d^{-1}$ and $b=d^{-1/2}$. Then u is of the form (24). Clearly $A^u(0)=2b^2a^{-2}I$, so $2b^2a^{-2}I\in U$ and $F(2b^2a^{-2}I)=1$. Theorem 1.3 in the case $p=(n+2)/(n-2)$ is established. $\square$

Proof of Theorem 1.3 *for* $-\infty<p<(n+2)/(n-2)$. In this case, the equation satisfied by u is no longer conformally invariant, but it transforms to our advantage when making reflections with respect to spheres, i.e. the inequalities have the right direction so that the strong maximum principle and the Hopf lemma can still be applied.

First, we still have (38) since this only requires the superharmonicity and the positivity of u. Lemma 4.3 still holds since it only uses (38) and the C^1-regularity of u in $\mathbf{R}^n$. For $x\in\mathbf{R}^n$, we still define $\bar\lambda(x)$ in the same way. We also define α as in (39), and we still have (40).

For $x\in\mathbf{R}^n$ and $\lambda>0$, the equation of $u_{x,\lambda}$ now takes the form

$$F(A^{u_{x,\lambda}}(y))=\left(\frac{\lambda}{|y-x|}\right)^{(n-2)((n+2)/(n-2)-p)}u_{x,\lambda}(y)^{p-(n+2)/(n-2)},\qquad A^{u_{x,\lambda}}(y)\in U,\quad\text{for all } y\neq x. \tag{45}$$

LEMMA 4.5. *If $\alpha=\infty$, then $\bar{\lambda}(x)=\infty$ for all $x\in\mathbf{R}^n$.*

Proof. Suppose on the contrary that $\bar{\lambda}(\bar{x})<\infty$ for some $\bar{x}\in\mathbf{R}^n$. Without loss of generality, we may assume that $\bar{x}=0$, and we use the notation

$$\bar{\lambda}:=\bar{\lambda}(0), \quad u_\lambda:=u_{0,\lambda} \quad \text{and} \quad B_\lambda:=B_\lambda(0).$$

By the definition of $\bar{\lambda}$,

$$u_{\bar{\lambda}}\leqslant u \quad \text{on } \mathbf{R}^n\setminus B_{\bar{\lambda}}.$$

By (45),

$$F(A^{u_{\bar{\lambda}}})\leqslant u_{\bar{\lambda}}^{p-(n+2)(n-2)}, \quad A^{u_{\bar{\lambda}}}\in U, \text{ on } \mathbf{R}^n\setminus B_{\bar{\lambda}}. \tag{46}$$

Recall that u satisfies

$$F(A^u)=u^{p-(n+2)/(n-2)}, \quad A^u\in U, \text{ on } \mathbf{R}^n\setminus B_{\bar{\lambda}}. \tag{47}$$

By (46) and (47),

$$\begin{gathered} F(A^{u_{\bar{\lambda}}})-F(A^u)-\big(u_{\bar{\lambda}}^{p-(n+2)/(n-2)}-u^{p-(n+2)/(n-2)}\big)\leqslant 0, \\ A^{u_{\bar{\lambda}}}\in U,\ A^u\in U, \quad \text{on } \mathbf{R}^n\setminus B_{\bar{\lambda}}. \end{gathered} \tag{48}$$

Since $\alpha=\infty$, we have

$$\liminf_{|y|\to\infty}\,(|y|^{n-2}(u-u_{\bar{\lambda}})(y))>0. \tag{49}$$

The inequality in (48) goes in the right direction. Thus, with (49), the arguments for $p=(n+2)/(n-2)$ work essentially in the same way here, and we obtain a contradiction by continuing the moving sphere procedure a little bit further. This deserves some explanations. Because of (49), and using arguments in [35] (see also [36]), we only need to show that

$$u_{\bar{\lambda}}(y)<u(y), \quad |y|>\bar{\lambda}, \tag{50}$$

and

$$\frac{d}{dr}(u-u_{\bar{\lambda}})\Big|_{\partial B_{\bar{\lambda}}}>0, \tag{51}$$

where d/dr denotes the differentiation in the outer normal direction with respect to $\partial B_{\bar{\lambda}}$.

If $u_{\bar{\lambda}}(\bar{y})=u(\bar{y})$ for some $|\bar{y}|>\bar{\lambda}$, then, using (48) as in the proof of Lemma 2.1 in [35], we know that $u_{\bar{\lambda}}-u$ satisfies

$$L(u_{\bar{\lambda}}-u)\leqslant 0,$$

where $L=-a_{ij}(x)\partial_{ij}+b_i(x)\partial_i+c(x)$ with $(a_{ij})>0$ continuous, and b_i and c continuous.

Since $u_{\bar{\lambda}}-u\leqslant 0$ near $\bar{y}$, we have, by the strong maximum principle, $u_{\bar{\lambda}}\equiv u$ near $\bar{y}$. For the same reason, $u_{\bar{\lambda}}(y)\equiv u(y)$ for any $|y|\geqslant\bar{\lambda}$, violating (49). Estimate (50) has been checked. Estimate (51) can be established in a similar way by using the Hopf lemma (see the proof of Lemma 2.1 in [35]). Thus Lemma 4.5 is established. □

By Lemma 4.5 and the usual arguments, we know that if $\alpha=\infty$, u must be a constant, and Theorem 1.3 for $-\infty<p<(n+2)/(n-2)$ is also proved in this case.

From now on, we always assume (41). As before, we obtain (42). Since the inequality in (46) goes in the right direction, the arguments for $p=(n+2)/(n-2)$ (see also the arguments in the proof of Lemma 4.5) essentially apply, and we still have (43) and (44). Applying the rest of the arguments for $p=(n+2)/(n-2)$, we have that u is of the form (24) with some positive constants a and b. However, we know that, for u of the form (24), $A^u\equiv 2b^2a^{-2}I$ and $F(A^u)\equiv$constant. This violates (23) since $u^{p-(n+2)/(n-2)}$ is not a constant when $p<(n+2)/(n-2)$. Theorem 1.3 for $-\infty<p<(n+2)/(n-2)$ is established. □

Appendix A

LEMMA A.1. *Let $a>0$ be a positive number and α be a real number. Assume that $h\in C^1[-4a,4a]$ satisfies, for $|\tau|<2a$, $|s|\leqslant 4a$, $0<\lambda<a$ and $\lambda<|s-\tau|$,*

$$\left(\frac{\lambda}{|s-\tau|}\right)^{\alpha} h\left(\tau+\frac{\lambda^2(s-\tau)}{|s-\tau|^2}\right)\leqslant h(s). \tag{52}$$

Then

$$|h'(s)|\leqslant \frac{\alpha}{2a}h(s), \quad |s|\leqslant a.$$

Proof. By considering $h(as)$, we only need to prove the lemma for $a=1$. If $\alpha=0$, it is easy to see that h is identically equal to a constant on $[-1,1]$. So we always assume that $\alpha\neq 0$. We only need to show that

$$-h'(s)\leqslant \tfrac{1}{2}\alpha h(s), \quad |s|<1, \tag{53}$$

since the estimate for $h'(s)$ can be obtained by applying the above $h(-s)$.

Now for $|\tau|<2$, let $h_\tau(s):=h(\tau+s)$. Then (52) is equivalent to

$$\left(\frac{\lambda}{|s-\tau|}\right)^{\alpha} h_\tau\left(\frac{\lambda^2(s-\tau)}{|s-\tau|^2}\right)\leqslant h_\tau(s-\tau), \quad |\tau|<2,\ |s|\leqslant 4,\ 0<\lambda<1,\ \lambda<|s-\tau|,$$

which, by setting $x=s-\tau$, implies that

$$\left(\frac{\lambda}{|x|}\right)^{\alpha} h_\tau\left(\frac{\lambda^2 x}{|x|^2}\right)\leqslant h_\tau(x), \quad |\tau|<2,\ 0<\lambda<1,\ \lambda<x<2.$$

Letting $y=\lambda^2 x/|x|^2=\lambda^2/x$ above, we have

$$y^{\alpha/2}h_\tau(y)\leqslant x^{\alpha/2}h_\tau(x),\quad 0<y<x<1.$$

Thus

$$0\leqslant\frac{d}{dx}(x^{\alpha/2}h_\tau(x))=\tfrac{1}{2}\alpha x^{\alpha/2-1}h_\tau(x)+x^{\alpha/2}h'_\tau(x),\quad 0<x<1,$$

i.e.

$$\tfrac{1}{2}\alpha h_\tau(x)+xh'_\tau(x)\geqslant 0,\quad 0<x<1.$$

Letting $x\to 1$ above, we have

$$\tfrac{1}{2}\alpha h_\tau(1)\geqslant -h'_\tau(1),$$

i.e.

$$\tfrac{1}{2}\alpha h(\tau+1)\geqslant -h'(\tau+1),\quad |\tau|<2.$$

Estimate (53) follows from the above. □

LEMMA A.2. *Let $a>0$ be a constant, and let $B_{8a}\subset\mathbf{R}^n$ be the ball of radius $8a$ and centered at the origin, $n\geqslant 3$. Assume that $u\in C^1(B_{8a})$ is a nonnegative function satisfying*

$$u_{x,\lambda}(y)\leqslant u(y),\quad x\in B_{4a},\ y\in B_{8a},\ 0<\lambda<2a,\ \lambda<|y-x|,$$

where

$$u_{x,\lambda}(y):=\left(\frac{\lambda}{|y|}\right)^{n-2}u\left(x+\frac{\lambda^2(y-x)}{|y-x|^2}\right).$$

Then there exists $C(n)>0$ such that

$$|\nabla u(x)|\leqslant\frac{n-2}{2a}u(x),\quad |x|<a.$$

Proof. For $x\in B_a$ and $e\in\mathbf{R}^n$, $|e|=1$, let $h(s):=u(x+se)$. Then, by the hypothesis on u, h satisfies the hypothesis of Lemma A.1. Thus we have

$$|h'(0)|\leqslant\frac{n-2}{2a}h(0),$$

i.e.

$$|\nabla u(x)\cdot e|\leqslant\frac{n-2}{2a}u(x).$$

Lemma A.2 follows from the above. □

Appendix B

We first show that we may assume without loss of generality that the f in Theorem 1.1′ is in addition homogeneous of degree 1. We achieve this by constructing the $\tilde{f}$ which is homogeneous of degree 1 satisfying $\tilde{f}^{-1}(1)=f^{-1}(1)$ and the hypotheses of Theorem 1.1′.

By the cone structure of Γ, the ray $\{s\lambda \mid s>0\}$ belongs to Γ for every $\lambda\in\Gamma$. By the concavity of f, we deduce from (10) that

$$\sum_{i=1}^{n} f_{\lambda_i}(\lambda)\lambda_i > 0, \quad \lambda\in\Gamma. \tag{54}$$

Since $f(0)=0$, f satisfies (10) and (54), and $f\in C^{4,\alpha}(\Gamma)$, the equation

$$f(\varphi(\lambda)\lambda) = 1, \quad \lambda\in\Gamma, \tag{55}$$

defines, using the implicit function theorem, a positive function $\varphi\in C^{4,\alpha}(\Gamma)$. It is easy to see from the definition of φ that $\varphi(s\lambda)=s^{-1}\varphi(\lambda)$ for all $\lambda\in\Gamma$ and $0<s<\infty$. Set

$$\tilde{f} = \frac{1}{\varphi} \quad \text{on } \Gamma.$$

By the homogeneity of φ, $\tilde{f}$ is homogeneous of degree 1. We will show that $\tilde{f}$ has the desired properties. Clearly, $\tilde{f}$ is symmetric, (10) is satisfied and $\tilde{f}^{-1}(1)=f^{-1}(1)$.

To prove that $\nabla\tilde{f}\in\Gamma_n$, applying $\partial/\partial\lambda_i$ to (55), we have

$$0 = f_{\mu_i}(\mu)\varphi(\lambda) + \frac{\varphi_{\lambda_i}(\lambda)}{\varphi(\lambda)}\sum_{j=1}^{n} f_{\mu_j}(\mu)\mu_j,$$

where $\mu=\varphi(\lambda)\lambda$. Since $f_{\mu_i}(\mu)>0$ and $\sum_{j=1}^{n} f_{\mu_j}(\mu)\mu_j>0$, we have $\varphi_{\lambda_i}(\lambda)<0$, i.e.

$$\tilde{f}_{\lambda_i} > 0 \quad \text{on } \Gamma,\ 1\leqslant i\leqslant n.$$

Next we prove the concavity of $\tilde{f}$. For $\lambda, \bar{\lambda}\in\Gamma$, we have, by the concavity of f, that

$$\begin{aligned}
f&\left(\frac{\varphi(\lambda)\varphi(\bar{\lambda})}{t\varphi(\bar{\lambda})+(1-t)\varphi(\lambda)}[t\lambda+(1-t)\bar{\lambda}]\right)\\
&= f\left(\frac{t\varphi(\bar{\lambda})}{t\varphi(\bar{\lambda})+(1-t)\varphi(\lambda)}\varphi(\lambda)\lambda + \frac{(1-t)\varphi(\lambda)}{t\varphi(\bar{\lambda})+(1-t)\varphi(\lambda)}\varphi(\bar{\lambda})\bar{\lambda}\right)\\
&\geqslant \frac{t\varphi(\bar{\lambda})}{t\varphi(\bar{\lambda})+(1-t)\varphi(\lambda)} f(\varphi(\lambda)\lambda) + \frac{(1-t)\varphi(\lambda)}{t\varphi(\bar{\lambda})+(1-t)\varphi(\lambda)} f(\varphi(\bar{\lambda})\bar{\lambda})\\
&= 1 = f(\varphi(t\lambda+(1-t)\bar{\lambda})[t\lambda+(1-t)\bar{\lambda}]).
\end{aligned}$$

By (54), f is strictly increasing along any ray in Γ starting from the origin. Therefore we deduce from the above that

$$\frac{\varphi(\lambda)\varphi(\bar\lambda)}{t\varphi(\bar\lambda)+(1-t)\varphi(\lambda)} \geqslant \varphi(t\lambda+(1-t)\bar\lambda),$$

i.e.

$$t\tilde f(\lambda)+(1-t)\tilde f(\bar\lambda) \leqslant \tilde f(t\lambda+(1-t)\bar\lambda).$$

We have showed that $\tilde f$ is a concave function in Γ.

To check that $\tilde f\in C^0(\bar\Gamma)$ and $\tilde f=0$ on $\partial\Gamma$, we only need to show that

$$\lim_{\substack{\lambda\to\bar\lambda\\ \lambda\in\Gamma}} \tilde f(\lambda)=0, \quad \bar\lambda\in\partial\Gamma.$$

We show the above by a contradiction argument. Suppose the contrary. Then for some $\bar\lambda\in\partial\Gamma$ there exists a sequence $\lambda^i\in\Gamma$, $\lambda^i\to\bar\lambda$, such that $\lim_{i\to\infty}\tilde f(\lambda^i)>0$. It follows that $\varphi(\lambda^i)\to a$ for some $a\in[0,\infty)$. By the continuity of f on $\bar\Gamma$, we have $1=f(\varphi(\lambda^i)\lambda^i)\to f(a\bar\lambda)$. Since $f=0$ on $\partial\Gamma$, we have $a>0$ and $\bar\lambda\in\Gamma$, a contradiction. We have proved that $\tilde f$ has the desired properties.

PROPOSITION B.1. *Let V be an open symmetric convex subset of $\mathbf{R}^n$ with $\partial V\neq\varnothing$. Assume that*

$$\nu(\lambda)\in\Gamma_n, \quad \lambda\in\partial V, \tag{56}$$

and

$$\nu(\lambda)\cdot\lambda>0, \quad \lambda\in\partial V, \tag{57}$$

where $\nu(\lambda)$ denotes the unit inner normal of a supporting plane of V at λ. Then $\Gamma(V)$ as defined in (3) *is an open symmetric convex cone with vertex at the origin. Moreover,*

$$\Gamma_n\subset\Gamma(V)\subset\Gamma_1 \tag{58}$$

and

$$\Gamma(V)=\{s\lambda \mid \lambda\in\partial V,\ s>0\}. \tag{59}$$

Remark B.1. No regularity assumption on ∂V is needed.

To prove Proposition B.1, we need the following lemma:

LEMMA B.1. *Let V be as in Proposition* B.1.
(i) *If $\lambda\in V$, then $\{s\lambda \mid s\geqslant1\}\subset V$;*
(ii) $0\notin\bar V$;
(iii) *If $\lambda\in\partial V$, then $\{s\lambda \mid -\infty<s<1\}\cap\bar V=\varnothing$ and $\{s\lambda \mid s>1\}\subset V$.*

Proof. If (i) does not hold, then there exists some $\lambda\in V$ and $\bar{s}>1$ such that $\bar{s}\lambda\in\partial V$. By the convexity of V, we have

$$(\lambda-\bar{s}\lambda)\cdot\nu(\bar{s}\lambda)\geqslant 0.$$

From this we deduce, by $\bar{s}>1$, that $\bar{s}\lambda\cdot\nu(\bar{s}\lambda)\leqslant 0$, contradicting (57). (i) is established.

If $0\in\bar{V}$, then $0\notin\partial V$ by (57). Hence $0\in V$. Since V is open, an open neighborhood of 0 belongs to V, and therefore, by (i), $V=\mathbf{R}^n$, contradicting the fact that $\partial V\neq\varnothing$. (ii) is established.

Let $\lambda\in\partial V$. For $-\infty<s<1$, we have, by (57), that $\nu(\lambda)\cdot(s\lambda-\lambda)=(s-1)\nu(\lambda)\cdot\lambda<0$. Since $\nu(\lambda)$ is an inner normal, $s\lambda\notin\bar{V}$. Thus we have proved the first statement in (iii). For the second statement in (iii), let $\lambda\in\partial V$. We know from the first statement of (iii) that $\{s\lambda\mid s>1\}\cap\partial V=\varnothing$. So either $\{s\lambda\mid s>1\}\subset V$ or $\{s\lambda\mid s>1\}\cap V=\varnothing$. Noticing that the first case is what we want to prove, we can assume the second case. Then, in view of the first statement of (iii), the line $\{s\lambda\mid s\in\mathbf{R}\}$ has no intersection with V. It follows from [52, Theorem 11.2] that there is a supporting plane of V containing the line $\{s\lambda\mid s\in\mathbf{R}\}$, and therefore $\nu(\lambda)\cdot\lambda=0$, where $\nu(\lambda)$ denotes the unit inner normal of the supporting plane, contradicting (57). (iii) is established. $\square$

Proof of Proposition B.1. It is easy to see that $\Gamma(V)$ is an open symmetric convex cone with vertex at the origin. Now we prove that $\Gamma(V)\subset\Gamma_1$.

For any $\lambda=(\lambda_1,...,\lambda_n)\in\Gamma(V)$, let

$$\begin{cases}\lambda^1=\lambda=(\lambda_1,...,\lambda_n),\\ \lambda^2=(\lambda_2,...,\lambda_n,\lambda_1),\\ \quad\vdots\\ \lambda^n=(\lambda_n,\lambda_1,...,\lambda_{n-1}).\end{cases}$$

Since $\Gamma(V)$ is symmetric, $\lambda^i\in\Gamma(V)$, $1\leqslant i\leqslant n$. By the convexity of $\Gamma(V)$,

$$\bar{\lambda}:=\frac{1}{n}\sum_{i=1}^n\lambda^i=\frac{\sigma_1(\lambda)}{n}e\in\Gamma(V),$$

where $e=(1,...,1)$ and $\sigma_1(\lambda)=\sum_{i=1}^n\lambda_i$.

Let

$$\bar{s}:=\inf\{s>0\mid s\bar{\lambda}\in V\}.$$

By (ii) in Lemma B.1, $\bar{s}>0$ and $\bar{s}\bar{\lambda}\in\partial V$. Let $\nu(\bar{s}\bar{\lambda})$ be the unit inner normal of a supporting plane of V at $\bar{s}\bar{\lambda}$. We have, by (57),

$$0<\nu(\bar{s}\bar{\lambda})\cdot(\bar{s}\bar{\lambda})=\frac{\bar{s}}{n}\sigma_1(\nu(\bar{s}\bar{\lambda}))\sigma_1(\lambda).$$

By (56), $\sigma_1(\nu(\bar{s}\bar{\lambda}))>0$, and thus $\sigma_1(\lambda)>0$, i.e. $\Gamma(V)\subset\Gamma_1$.

Next we prove that $\Gamma_n\subset\Gamma(V)$ by a contradiction argument. Suppose that there exists $\mu\in\Gamma_n\setminus\Gamma(V)$. Take any $\lambda\in\Gamma(V)\subset\Gamma_1$. Consider the 2-dimensional plane $\mathbf{P}$ generated by μ and λ. We know that $\Gamma(V)\cap\mathbf{P}$ lies on one side of the line $\partial\Gamma_1\cap\mathbf{P}$. So $\{s\mu\,|\,s\in\mathbf{R}\}\cap\Gamma(V)=\varnothing$, and therefore $\Gamma(V)\cap\mathbf{P}$ stays on one side of $\{s\mu\,|\,s\in\mathbf{R}\}$ in $\mathbf{P}$, i.e.

$$\left\{\tilde{\mu}\in\mathbf{P}\,\middle|\,\tilde{\mu}\cdot\left[\lambda-\left(\lambda\cdot\frac{\mu}{|\mu|}\right)\frac{\mu}{|\mu|}\right]<0\right\}\cap\Gamma(V)=\varnothing.$$

Fix some $\tilde{\mu}\in\Gamma_n\cap\mathbf{P}$ such that

$$\tilde{\mu}\cdot\left[\lambda-\left(\lambda\cdot\frac{\mu}{|\mu|}\right)\frac{\mu}{|\mu|}\right]<0.$$

Then the line $l:=\{s\tilde{\mu}\,|\,s\in\mathbf{R}\}$ has no intersection with $\overline{V}\cap\mathbf{P}$. Now moving l parallelly towards $\overline{V}\cap\mathbf{P}$, then a first touching of the moving line and $\overline{V}\cap\mathbf{P}$ must occur. Let $\bar{l}$ denote the first touching line, and let $\bar{\lambda}\in\bar{l}\cap(\overline{V}\cap\mathbf{P})$. Clearly $\bar{\lambda}\in\partial V$ and $\bar{l}\cap V=\varnothing$. So there exists a supporting plane of V at $\bar{\lambda}$ which contains $\bar{l}$. Let $\nu(\bar{\lambda})$ denote the unit inner normal of the supporting plane. Then $\nu(\bar{\lambda})\cdot\tilde{\mu}=0$, a contradiction to $\tilde{\mu}\in\Gamma_n$ and $\nu(\bar{\lambda})\in\Gamma_n$ by (56). Thus $\Gamma_n\subset\Gamma(V)$, and (58) is established.

Let

$$\widetilde{\Gamma}(V):=\{s\lambda\,|\,\lambda\in\partial V,\ s>0\}.$$

Next we show that $\Gamma(V)=\widetilde{\Gamma}(V)$. For $\lambda\in V$, consider the ray $\{s\lambda\,|\,s>0\}$. Since $0\notin\overline{V}$, we know that

$$\bar{s}:=\inf\{s\,|\,s\lambda\in V\}>0.$$

By the openness of V and the definition of $\bar{s}$, $\bar{s}\lambda\in\partial V$. So $\lambda\in\widetilde{\Gamma}(V)$. We have showed that $\Gamma(V)\subset\widetilde{\Gamma}(V)$. On the other hand, by (ii) and (iii) of Lemma B.1, $\widetilde{\Gamma}(V)\subset\Gamma(V)$. We have established (59), and so Proposition B.1. $\square$

In the following, we deduce the equivalence of Theorem 1.1 and Theorem 1.1′.

Theorem 1.1 $\Rightarrow$ Theorem 1.1′. Let $V:=\{\lambda\in\Gamma\,|\,f(\lambda)>1\}$. By (9) and (10), $\Gamma(V)=\Gamma$. By the concavity and symmetry of f, V is open, symmetric and convex. Clearly $\partial V=\{\lambda\in\Gamma\,|\,f(\lambda)=1\}\neq\varnothing$ is $C^{4,\alpha}$ and ∇f is an inner normal to ∂V. Therefore $\nabla f\in\Gamma_n$ implies (1). Above we have proved the concavity of f, and (10) forces $\nabla f(\lambda)\cdot\lambda>0$. Restricted onto ∂V, we have (2). Hence Theorem 1.1′ follows from Theorem 1.1.

Theorem 1.1′ $\Rightarrow$ Theorem 1.1. We only need to construct a pair (f,Γ) satisfying all the assumptions in Theorem 1.1′ and $\{\lambda\,|\,f(\lambda)=1\}=\partial V$. Let $\Gamma:=\Gamma(V)$ as defined in (3). By Proposition B.1, (6) and (7) hold for Γ. Let $f(s\lambda):=s$ for $s\geqslant 0$ and $\lambda\in\partial V$. By (59),

$\Gamma=\{s\lambda \mid \lambda\in\partial V,\ s>0\}$. So f is well-defined, symmetric and $C^{4,\alpha}$ on Γ. It is easy to see from the definition that f is homogeneous of degree 1, and therefore (10) follows directly. To prove that f is concave, take any two points $a\lambda$ and $b\mu$ in Γ, where $\lambda,\mu\in\partial V$ and $a,b>0$. For any $0\leqslant t\leqslant 1$, since $\lambda,\mu\in\partial V$ and $\overline{V}$ is convex, we have

$$\bar\lambda := \frac{ta}{ta+(1-t)b}\lambda+\frac{(1-t)b}{ta+(1-t)b}\mu\in\overline{V}.$$

Recalling the definition of f, we have $\bar\lambda/f(\bar\lambda)\in\partial V$. However, by (i) and (iii) in Lemma B.1, we know that $s\bar\lambda\in V$ for any $s>1$. Therefore $1/f(\bar\lambda)\leqslant 1$, i.e. $f(\bar\lambda)\geqslant 1$. From this we deduce that f is concave, since

$$\begin{aligned} f(ta\lambda+(1-t)b\mu) &= (ta+(1-t)b)\,f\biggl(\frac{ta}{ta+(1-t)b}\lambda+\frac{(1-t)b}{ta+(1-t)b}\mu\biggr)\\ &= (ta+(1-t)b)f(\bar\lambda)\\ &\geqslant ta+(1-t)b\\ &= tf(a\lambda)+(1-t)f(b\mu). \end{aligned}$$

Now the only assumption left to check is that f can be continuously extended to $\partial\Gamma$ and vanishes on $\partial\Gamma$. To see this, take any sequence $\{\lambda^i\}_{i=1}^\infty$ in Γ with $\lambda^i\to\bar\lambda\in\partial\Gamma$. We need to show that $\lim_{i\to\infty} f(\lambda^i)=0$. Suppose the contrary. Then there exists a subsequence of $\{\lambda^i\}_{i=1}^\infty$, still denoted by $\{\lambda^i\}_{i=1}^\infty$, such that $f(\lambda^i)\geqslant\delta$ for some constant $\delta>0$. By the definition of f, $\lambda^i/f(\lambda^i)\in\partial V$. On the other hand, $\lambda^i\to\bar\lambda$ and $f(\lambda^i)\geqslant\delta>0$ implies that $\{\lambda^i/f(\lambda^i)\}_{i=1}^\infty$ stays in a bounded set of $\mathbf{R}^n$. Hence $\lambda^i/f(\lambda^i)\to\mu$ for some $\mu\in\mathbf{R}^n$. Noticing that ∂V is closed, $\mu\in\partial V$. By (59), $\{s\mu\mid s>0\}\subset\Gamma$. Recalling that $0\notin\partial V$ and $\lambda^i\to\bar\lambda$, we have that $f(\lambda^i)$ is uniformly bounded. Without loss of generality, we can assume that $f(\lambda^i)\to c_0>0$. It follows that

$$\partial\Gamma\ni\bar\lambda\leftarrow\lambda^i=f(\lambda^i)\frac{\lambda^i}{f(\lambda^i)}\to c_0\mu\in\Gamma,$$

a contradiction. Theorem 1.1 follows from Theorem 1.1′.

In the rest of this section, we address Remark 1.1. We assume that $\partial V\in C^{2,\alpha}$, but that the principle curvatures of ∂V are positive. Let $P_1:=\partial\Gamma_1$. After a rotation of the axis system, ∂V can be represented as the graph of a $C^{2,\alpha}$-function $\bar\phi$ defined on $P_1\equiv\mathbf{R}^{n-1}$ and satisfying

$$(\nabla^2\bar\phi)>0 \quad \text{on } \mathbf{R}^{n-1}. \tag{60}$$

The cone Γ_n in the new axis system is still an open convex cone, denoted by $\widetilde\Gamma_n$. The assumptions (56) and (57) are translated into

$$(-\nabla\bar\phi(y'),1)\in\widetilde\Gamma_n \quad\text{and}\quad (-\nabla\bar\phi(y'),1)\cdot(y',\bar\phi(y'))>0,\qquad y'\in\mathbf{R}^{n-1}. \tag{61}$$

In the following, all the functions are defined on $\mathbf{R}^{n-1}$ if not specified. For $R>0$ and $\varepsilon>0$, consider

$$\phi_R^\varepsilon := \varrho_R \bar{\phi}^\varepsilon + (1-\varrho_R)\bar{\phi},$$

where $\bar{\phi}^\varepsilon$ is a smooth mollifier of $\bar{\phi}$ and ϱ_R is a radially symmetric cut-off function having value 1 in B_R and 0 outside B_{2R}. Let V_R^ε be the set above the graph of ϕ_R^ε. For any $R>0$, ϕ_R^ε is identically equal to $\bar{\phi}$ outside B_{2R}, and $\phi_r^\varepsilon \to \bar{\phi}$ in $C^{2,\alpha}_{\mathrm{loc}}$ as $\varepsilon\to 0$. So, for some small $\varepsilon=\varepsilon(R)>0$, (56) and (57) hold for ϕ_R^ε. Noticing that ∂V_R^ε coincides with ∂V when $|y'|\geqslant 2R$, we can assume, for the same small ε, that $\Gamma(V_R^\varepsilon)=\Gamma(V)$. Back to $\Gamma(V)$, V and V_R^ε define f and f_R^ε as homogeneous functions of degree 1 in $\Gamma(V_R^\varepsilon)=\Gamma(V)$ taking value 1 on ∂V and ∂V_R^ε, respectively. The function f_R^ε satisfies all the assumptions of f assumed in Theorem 1.1$'$. Now take a sequence $R_i\to\infty$ and take $\varepsilon_i>0$ such that (60) and (61) hold for $\phi_{R_i}^{\varepsilon_i}$. Let $f_{R_i}^{\varepsilon_i}$ be the corresponding function on $\Gamma(V)$. We know that $f_{R_i}^{\varepsilon_i}$ satisfies all the assumptions of f in Theorem 1.1$'$, that it is smooth in any compact subset of Γ and that $f_{R_i}^{\varepsilon_i}\to f$ in $C^{2,\alpha}_{\mathrm{loc}}(\Gamma)$.

Consider the equation

$$f_{R_i}^{\varepsilon_i}(\lambda(A_{u^{4/(n-2)}g}))=1, \quad \lambda(A_{u^{4/(n-2)}g})\in\Gamma, \text{ on } M^n. \tag{62}$$

Applying Theorem 1.1$'$ to $(f_{R_i}^{\varepsilon_i},\Gamma)$, for any solution u_i of the equation (62), we have that

$$\|u_i\|_{C^{2,\alpha}(M^n,g)}+\|1/u_i\|_{C^{2,\alpha}(M^n,g)}\leqslant C \tag{63}$$

for some constant C independent of i—this is clear from the proof of Theorem 1.1$'$. This implies that $\lambda(A_{u_i^{4/(n-2)}g})$ stays in a compact subset of Γ independent of i. Hence for i large enough, $f_{R_i}^{\varepsilon_i}$ is $C^{4,\alpha}$ in this compact subset, and we have, by (63) and Schauder theory, that

$$\|u_i\|_{C^{4,\alpha}(M^n,g)}\leqslant C_i,$$

where C_i is some constant that may depend on i.

Following the degree arguments at the end of the proof of Theorem 1.1$'$ and replacing O_t^* by

$$\begin{aligned} O_t^i := \{u\in C^{4,\alpha}(M^n,g) \mid \|u\|_{C^{2,\alpha}(M^n,g)}+\|1/u\|_{C^{2,\alpha}(M^n,g)}\leqslant 2C,& \\ \|u\|_{C^{4,\alpha}(M^n,g)}\leqslant 2C_i,\ \lambda(A_{u^{4/(n-2)}g})\in\Gamma\}&, \end{aligned}$$

we can find a solution u_i of (62). Since u_i is uniformly bounded in $C^{2,\alpha}(M^n,g)$, after passing to a subsequence, u_i converges in $C^2(M^n,g)$ to some function u in $C^{2,\alpha}(M^n,g)$. Sending $i\to\infty$ in (62), we have

$$f(\lambda(A_{u^{4/(n-2)}g}))=1, \quad \lambda(A_{u^{4/(n-2)}g})\in\Gamma, \text{ on } M^n.$$

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