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Stability of Finite-Amplitude and Overstable Convection of a Conducting Fluid Through Fixed Porous Bed

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Abstract. The stability of infinitesimal steady and oscillatory motions and finite amplitude steady motions of a conducting fluid through porous media with free boundaries which is heated from below and cooled from above is investigated in the presence of a uniform magnetic field. Infinitesimal steady motions are investigated using Liapunov method and it is shown that the principle of exchange of stability is valid only when $P_m/P_r \leq 1$ with a restricted value of the Hartmann number. It is shown that overstable motions are due to the zonal current induced by the magnetic field. Finite amplitude steady motions are investigated using Veronis [1] analysis and it is shown that for a restricted range of Hartmann numbers and porous parameter P_l, steady finite-amplitude motions can exist for values of the Rayleigh number smaller than that value corresponding to oscillatory motions. Since the Busse number is greater than the wave number the horizontal scale of the steady finite-amplitude motions is larger than that of the overstable motions.

Verschiedene Aspekte der Stabilität eines Fluids in porösen Medien

Zusammenfassung. Die Stabilität eines Fluids mit thermischer und elektrischer Leitfähigkeit wird in einem von unten beheizten und von oben gekühlten Medium behandelt.

Bei überlagertem magnetischen Feld können sich oszillatorische Instabilitäten ausbilden, die sich auf zonale, vom Magnetfeld induzierte Ströme zurückführen lassen.

Andere Formen der Instabilität treten unter anderen Bedingungen auf. Maßgebend dafür sind die Werte der Hartmann- und Rayleighzahlen.

Nomenclature

(x, y, z)	the cartesian Co-ordinates
t	the time
d	the depth of the porous media
$\vec{q}$	(u, v, w) the velocity field
$\vec{H}$	(H_x, H_y, H_z) the magnetic field
P	the pressure
ρ	the density
T	the temperature
Ψ	the velocity stream function
φ	the magnetic stream function
ν	the kinematic viscosity
K	the thermal diffusivity
ν_m	the magnetic viscosity
μ	the magnetic permeability
k	the permeability of porous media
$P_l = d^2/k$	the porous parameter
$Pr = \nu/K$	the Prandtl number
$P_m = \nu/\nu_m$	the magnetic Prandtl number
$S = K/\nu_m$	the Busse number
$R = \alpha g \Delta T d^3/K\nu$	the Rayleigh number
$M = H_0 d \sqrt{\frac{\mu}{\rho_0 \nu \nu_m}}$	the Hartmann number

1 Introduction

When a horizontal layer of conducting flow through porous media is heated uniformly from below and cooled from above it has a tendency towards instability because the hotter fluid is less dense and therefore convects upwards. This is similar to the situation that exists in the geothermal regions where the surface liquid possesses a general upward convective drift due to the buoyancy induced by Joule heat and interior temperature. Since the rising liquid is cooled as it approaches the surface where heat is removed by evaporation, radiation and movement in surface streams an unstable state may be induced and complicated convective motions appear in the layers near the surface. A detailed study of such convections through porous media in the presence of a magnetic field is useful in the extraction of large energy in the geothermal regions.

According to preliminary estimates the heat accumulated in the crust to a depth of 6 to 8 km is fivethousand times greater than the heat that could be liberated from all fuels available on the earth. On the basis of this, it has been estimated that geothermal resources in the United States could supply 395 000 MW of electric power in the year 2000. One of the possible methods of extracting such large geothermal energy is by establishing a very large underground heat exchanger consisting of man made circulation systems with a filtration zone in porous media formed by hydraulic and explosive fracturing of rocks surrounding a geothermal wall. This needs optimising the draw-off rates and operating pressure of filtration zones. To achieve this a detailed understanding of flow process in filtration zones, and in porous media in particular, in the presence a magnetic field will be required. In addition to this geothermal application the results of this investigation are also useful in understanding the dynamo principle of earths core and other planetary interior. This is because convection through porous media is a likely candidate for the origin of the magnetic fields in these planets where the interaction of convective eddies in porous media with magnetic field appears in this case. The present study considers natural convection of a conducting fluid in an idealized porous layer in the presence of a magnetic field and determines the effects of different modes of heating from below having free boundaries.

Although the analysis presented in this paper was originally motivated by the problem of generation of magnetic field by convection through porous material we shall restrict our attention to the case of interaction of two-dimensional convection through porous material with homogeneous magnetic field imposed from outside. At the onset of instability, these convective motions are of two types. If they are independent of time it is usually said that the principle of exchange of stability is valid. If, however, instability manifests itself in the form of oscillatory (with respect to time) motion we have overstability. In the case of overstability, the convective velocity depends on time. After setting up the convection the temperature decreases with decrease in temperature gradient so that the density and velocity decrease till the buoyancy force balances the viscous force. At that stage the uniform heat supplied from below will again increase the buoyancy force which counterbalances the viscous and thermal dissipation and convective motion occurs. This process will continue and we have oscillatory convection as long as temperature and velocity fields are out of phase. This is usually the situation when the Prandtl number Pr is nearly unity. However, when the Prandtl number is sufficiently small the temperature and velocity fields will be in phase and no restoring force exists to execute the oscillatory convective motion. In this paper an externally applied uniform magnetic field will act as a restoring force which sets up the overstable motions when P_m/Pr is sufficiently greater than one, where P_m is the hydromagnetic Prandtl number. It is shown that overstable motions decrease the constraining effect of magnetic field and it is for that reason that convection can arise at a value of the Rayleigh number smaller than that which is required for steady convective motions. The same argument is valid for the existance of instability to finite amplitude motions (Veronis [1]). Hence in this paper the finite amplitude study is also included.

Most of the work on the onset of convection in a porous medium pertains to the hydrodynamic stability (see for example the review in the paper by Nield [2]). The linear hydrodynamic stability of steady convection through porous media has been investigated for the first time, by Horton and Rogers [3] and Lapwood [4] and established the critical Rayleigh number below which convective cellular flow, in an unbounded horizontal porous stratum, cannot take place. Later, Elder [5] has investigated this problem numerically and found that the critical Rayleigh number $4\pi^2$, obtained by Lapwood is indeed correct because the numerical solutions for the Rayleigh number less than $4\pi^2$ are nonconvective and convection occurs only for the Rayleigh number greater than $4\pi^2$. Wooding [6] has extended Lapwood problem to include the effect of variable viscosity and explained the stability properties of the layer qualitatively from physical considerations. Westbrook [7] has extended Lapwood's infinitesimally small perturbation analysis to include arbitrary finite perturbation and investigated the stability analysis using the energy method developed by Joseph [8]. Recently Gupta and Joseph [9] have investigated strongly non-linear heat transport across a porous layer using Howard's [10] variational method. The linear and non-linear stability of a conducting flow through porous media has been investigated for the first time, by Rudraiah [11] and by Prabhamani and Rudraiah [2] in the presence of a uniform poloidal magnetic field. They have investigated the linear theory using the normal mode analysis and the criterical Rayleigh number derived by such a theory gives a necessary condition for stability (or, equivalently, a sufficient condition of instability). The non-linear theory was investigated using the energy method developed by Joseph [8] and they have shown that the critical Rayleigh number provides a sufficient condition for stability of a motionless fluid. Their analysis shows that the critical Rayleigh number is independent of hydrodynamic and hydroenergetic Prandtl numbers. However, it is well known that for the analysis of overstable and finite amplitude motions one should obtain a stability criterion depending on the hydrodynamic and hydromagnetic Prandtl numbers and this is done in this paper.

To investigate these problems, the basic non-linear equations are developed in section 2. In section 3 the linear stability analysis is investigated using the Liapunov technique with a motive to answer the question whether the Liapunov technique gives more information on the stability analysis than the usual normal made technique or not. The condition for the existence of overstable motions is discussed in section 4 and a simple physical explanation for the oscillatory convective motions is given. The finite amplitude steady convective motions through porous media in the presence of an external magnetic field is investigated in section 5 using Veronis [1] truncated representation of five components. The method of solution is essentially the Galerkin, where the problem reduces, mathematically, to finding a solution to algebraic cubic equation. To simplify the problem, we assume that the induced magnetic field and the current are such that they are small compared to the applied magnetic field, which is usually the situation in most of the practical problems. The final analysis shows that this assumption is equivalent to setting $\alpha^2 = 1/3$. Here the trial function for velocity, temperature and magnetic fields are represented by the marginal stable modes plus the first distortion of these modes by non linear interaction. As in Veronis [1], no other modes are admitted in the representation. The resulting non linear equations for the model amplitudes are then solved on the assumption that the motion is steady. We note that although such an approach involves a drastic oversimplification of the form of velocity, temperature and magnetic fields, especially if the analysis is extended to values of the Rayleigh number far from the critical value, it does represent the simplest nonlinear analysis for which the results can be applied. These results will be useful to discuss the general non-linear problem by considering many more components. The final section contains the general conclusions and it is shown that finite amplitude solutions exist for subcritical values of the Rayleigh number as long as the Hartmann number is smaller than some maximum value (approximately unity for mercury with porous media made up of fibre material) and as long as the Busse number (for mercury the Busse number is approximately 5) is greater than α. This range of Hartmann and Busse numbers is readily accessible in a laboratory experiment with mercury and fibre material for porous media.

2 Mathematical Formulation

The configuration to be considered is shown in Fig. 1 which consists of a horizontal porous layer of permeability k and of infinite extent filled with conducting fluid heated from below and permeated by an externally applied uniform magnetic field H_0. The layer has thickness d and is bounded by two free surfaces. The upper surface is at a constant temperature $T_0 - 1/2\Delta T$ and the lower at $T_0 + 1/2\Delta T$. We write the total temperature as

$$T_{total} = T_0 - \Delta T \cdot (z/d - 1/2) + T(x, z, t) \tag{2.1}$$

where $T(x, z, t)$ is the deviation of the temperature from the linear profile and we assume that all the physical quantities are independent of y. In other words we consider here two-dimensional horizontal rolls.

Then the basic equations are the two-dimensional Darcy-Boussinesq-MHD equations, as set down by Prabhamani and Rudraiah [12], for the conservation of momentum

$$\frac{\partial \vec{q}}{\partial t} + (\vec{q} \cdot \nabla)\vec{q} - \frac{\mu}{\rho_0}(\vec{H} \cdot \nabla)\vec{H} = -\frac{1}{\rho_0}\nabla P - \frac{\rho^1}{\rho_0} - \frac{\nu}{k}\vec{q} \tag{2.2}$$

the conservation of mass

$$\partial u/\partial x + \partial w/\partial z = 0 \tag{2.3}$$

the conservation of magnetic lines

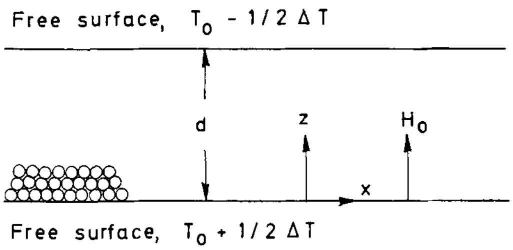

Fig.1. Schematic illustration of a porous layer

$$\frac{\partial \vec{H}}{\partial t} + (\vec{q}\cdot\nabla)\vec{H} = (\vec{H}\cdot\nabla)\vec{q} + \nu_m \nabla^2 \vec{H} \tag{2.4}$$

the linear equation of state for the fluctuation density

$$\rho^1 = -\rho_0 \alpha T \tag{2.5}$$

and the conservation of energy

$$\frac{\partial T}{\partial t} = (\vec{q}\cdot\nabla)T - \left(\frac{\Delta T}{d}\right) W = k\nabla^2 T\ . \tag{2.6}$$

Here $P = p + \mu H^2/2$ is the total pressure, p is the pressure of fluid, $\vec{q}$ is the two-dimensional velocity vector with components (u, w) in the increasing directions of (x, z); $\vec{H}$ is the two-dimensional magnetic field with components $(H_x, H_0 + H_z)$ in the respective directions (x, z), $\vec{g}$ is the gravitational acceleration in the negative z-direction; ρ_0 is the density at temperature T_0, α is the coefficient of thermal expansion, k is the permeability of the porous medium and ν, $\nu_m (= 1/\mu\sigma)$ and K are respectively the coefficients of kinematic viscosity, magnetic viscosity and thermometric diffusivity, σ the electrical conductivity and μ is the magnetic permeability.

We cross-differentiate the momentum Eq.(2.2) and the magnetic induction Eq.(2.4) to obtain

$$\frac{\partial \eta}{\partial t} + (\vec{q}\cdot\nabla)\eta - \frac{\mu}{\rho_0}(\vec{H}\cdot\nabla)\xi = \frac{\mu H_0}{\rho_0}\frac{\partial \xi}{\partial z} - \alpha g \frac{\partial T}{\partial x} - \frac{\nu}{k}\eta \tag{2.7}$$

and

$$\frac{\partial \xi}{\partial t} + (\vec{q}\cdot\nabla)\xi - (\vec{H}\cdot\nabla)\eta = \mu H_0 \frac{\partial \eta}{\partial z} + \nu_m \nabla^2 \xi - 2\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) - 2\left(\frac{\partial H_z}{\partial x} + \frac{\partial H_z}{\partial z}\right)\frac{\partial w}{\partial z} \tag{2.8}$$

where

$$\eta = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \quad \xi = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} \tag{2.9}$$

are the y-components of vorticity and current density respectively.

We introduce the stream function Ψ and the magnetic stream function φ, through the definitions

$$u = \frac{\partial \Psi}{\partial z}, \quad w = -\frac{\partial \Psi}{\partial x} \tag{2.10}$$

$$H_x = \frac{\partial \varphi}{\partial z}, \quad H_z = -\frac{\partial \varphi}{\partial x}$$

so that

$$\eta = \nabla^2 \Psi \quad \text{and} \quad \xi = \nabla^2 \varphi\ . \tag{2.11}$$

Our system, using the non-dimensional quantities

$$\vec{q} = k/d\,\vec{q}^*,\ t = d^2/k\,t^*,\ \vec{H} = H_0 \vec{H}^*,\ T = (\Delta T)T^*,\ (x, z) = d(x^*, z^*) \tag{2.12}$$

then becomes

$$\frac{\partial \eta}{\partial t} = J(\Psi, \eta) - \frac{PrM^2}{S} J(\varphi, \xi) - PrR\frac{\partial T}{\partial x} - \frac{\sigma}{P_1}\eta + \frac{PrM^2}{S}\nabla^2\left(\frac{\partial \varphi}{\partial z}\right) \tag{2.13}$$

$$\frac{\partial \xi}{\partial t} = J(\Psi, \xi) - J(\varphi, \eta) + \frac{1}{S}\nabla^2 \xi + \nabla^2\left(\frac{\partial \Psi}{\partial z}\right) - 2\left(\frac{\partial^2 \Psi}{\partial z^2} - \frac{\partial^2 \Psi}{\partial x^2}\right)\frac{\partial^2 \varphi}{\partial x \partial z} + 2\left(\frac{\partial^2 \varphi}{\partial z^2} - \frac{\partial^2 \varphi}{\partial x^2}\right)\frac{\partial^2 \Psi}{\partial x \partial z} \tag{2.14}$$

$$\frac{\partial T}{\partial t} = J(\Psi, T) - \frac{\partial \Psi}{\partial x} + \nabla^2 T \tag{2.15}$$

where, for simplicity, the asterisks (*) are neglected, J stands for the Jacobian,

$Pr = \nu/K$ is the Prandtl number
$P_m = \nu/\nu_m$ the magnetic Prandtl number
$M^2 = \dfrac{\mu H_0^2 d^2}{\rho_0 \nu \nu_m}$ the Hartmann number and
$R = \alpha g \Delta T d^3/K\nu$ the Rayleigh number and $S = K/\nu_m$ is the Busse number (first appeared in the work of Busse [13]).

We also note that the Hartmann number and Pr/S appear in product form as

$$Q = M^2 Pr/S = \mu H_0^2 d^2/\rho_0 k^2$$

so that we could equally well have defined this combination as the non-dimensional number containing the magnetic field. Only in the steady, linear stability problem M^2 appears independent of the Prandtl number.

The boundary conditions for the problem are straight forward. With the boundaries $Z = 0$ and $Z = 1$ taken as flat, magnetic and velocity stress-free, so that

$$\Psi = 0, \ \frac{\partial^2 \Psi}{\partial z^2} = 0, \ T = 0, \ \frac{\partial \varphi}{\partial z} = 0 \quad \text{at} \quad z = 0, 1. \qquad (2.16)$$

Conditions (2.16) represent the simplest kind of boundary conditions for the problem. They are distinguished by the periodic continuation of the convection layer above and below. In the absence of porous media these boundary conditions have been favoured in the analysis of convection problems (Busse [13]). In the absence of magnetic field they have been used by Lapwood [4]. Conditions (2.16) represent natural extension to the porous media case.

3 Steady Linear Stability Analysis Using Liapunov Technique

In this section, we investigate the linear steady stability analysis using the Liapunov technique with the object of knowing whether this method gives more information on convective steady motions than the usual normal mode analysis (Prabhamani and Rudraiah [12]) or not. Before doing this, let us first briefly explain the Liapunov technique.

3.1 The Liapunov Method

It is intuitively clear that if the total energy of a physical system has a local minimum at a certain point, than that system is stable at that point. This idea was generalized by Liapunov [14] into a simple but powerful method for studying stability problems in a broader context. A formal discussion of this method was given by Pritehand [15]. We just summarize the results here briefly.

Let $r(= x, y, z)$ be the spatial coordinate defined over a fixed and finite domain Σ with $\partial\Sigma$ denoting its boundary and t be the time. Our aim is to investigate the stability of the basic state $q_0(r) = 0$ (i.e. a null solution), $\vec{H} = \vec{H}_0$ and $d^2T_0/dZ^2 = 0$. For this, we suppose that a metric space $X(a, B)$ be given such that the metric between any two states a and a' at time t be denoted by $B(a, a')$. Let the basic and perturbed states, both belong to Σ, be denoted by a_0 and a respectively.

The basic state is stable in the sense of Liapunov if for each real number $\varepsilon > 0$ there exists a real number $\delta(\varepsilon) > 0$ such that for every $a \in \Sigma$, $B(a_0, a) < \delta$ at the initial time t_0 implies that $B(a_0, a) < \varepsilon$ for all $t > t_0$. The basic state is asymptotically stable if it is stable and in addition $B \to 0$ as $t \to \infty$. The basic state is unstable if it is not stable. Among unstable systems, we do not consider those that have finite escape time i.e. $B \to \infty$ for finite t.

The following theorem follows by a straight forward extention of the method employed by Pritehand [15].

Theorem: There exist non-decreasing functions $f(B)$ and $g(B)$ such that

$$f(0) = g(0) = 0$$

$$f(B) \to \infty, \ g(B) \to \infty \quad \text{as} \quad B \to \infty$$

and a functional V, called Liapunov functional, such that

$$0 < f(B) \leqslant V \leqslant g(B)$$

and

$$\frac{dV}{dt} \leqslant \delta < 0, \qquad \delta(0) = 0 \ ,$$

then the basic state is uniformly and asymptotically stable in the large with respect to the metric B.

3.2 Liapunov Functional and Stability Criterion

To obtain a suitable Liapunov functional, we consider a small disturbance of the form

$$u = u', \ v = v', \ w = w', \ P = P_0 + P', \ \vec{H} = H_0\hat{k} + \vec{h}$$
$$T = T_0 - \Delta T(z/d - 1/2) + T'(x, y, z, t) \ . \qquad ((3.1)$$

Here Zero suffix quantities denote the basic state, prime quantities and $\vec{h}$ denote the perturbed quantities which are assumed to be small compared to the basic state quantities and k denotes the unit vector in the Z-direction.

Substituting (3.1) into the system (2.2) to (2.6), linearizing, assuming all the perturbed quantities vary of the form

$$f(z)\ \mathrm{Exp}\{i(l_1 x + m_1 y) + P_1 t\} \tag{3.2}$$

and eliminating all the quantities except W and T (Prabhamani and Rudraiah [12]) we get

$$[(n+1/P_1)\{nP_m - (D^2-\alpha^2)\} - M^2D^2](D^2-\alpha^2)w = -\frac{\alpha^2 R^{1/2}}{P_1^{1/2}}[nP_m - (D^2-\alpha^2)]w \tag{(3.3)}$$

$$[nPr - (D^2-\alpha^2)]T = R^{1/2}P_1^{1/2}w \tag{3.4}$$

where $D = d/dZ$, α is the dimensionless wave number and other quantities are define in section 2. The scales for W and T have been chosen such that R appears symmetrically in the two Equations rather than in just one or the other.

This choise is useful to establish a suitable Liapunov functional to the problem. In deriving Eqs. (3.3) and (3.4) we have used the fact that the normal components of current density and vorticity are identically zero due to our boundary conditions (2.16).

From (3.3) and (3.4) we have to obtain the Liapunov functional, V, a positive definite functional such that

$$0 < \alpha_0(B) \leqslant V \leqslant \alpha,\ (B)$$

and (3.5)

$$\frac{dV}{dt} \leqslant \delta(B) < 0$$

where V and B involve the volume integrals of linear function of $|W|^2$, $|DW|^2$ and soon, and can be expressed, using (3.2), as

$$V = \bar{V}\ \mathrm{Exp}(2n_r t) \quad \ldots\ldots \qquad B^2 = \bar{B}^2\ \mathrm{Exp}(2n_r t) \tag{3.6}$$

where $\bar{V}$ and $\bar{B}$ are volume integrals of linear functions of $|W|^2$, $|DW|^2$ and so on. We assume that $\alpha_0(B)$, $\alpha_1(B)$ and $\delta(B)$ take the values $\bar{\alpha}_0 B^2$, $\bar{\alpha}_1 B^2$ and $\bar{\delta}B^2$ respectively. Then (3.5) becomes

$$0 < \bar{\alpha}_0\,\bar{B}^2 \leqslant \bar{V} \leqslant \bar{\alpha}_1\,\bar{B}^2 \qquad 2n_r\bar{V} \leqslant \bar{\delta}\bar{B}^2 \leqslant 0\ . \tag{3.7}$$

Eliminating n between (3.3) and (3.4), multiplying the resulting equation by W*, the complex conjugate of W, integrating over the volume of fluid (Rudraiah [16, 17, 18]) using the boundary conditions (2.16) and considering only the real part, since n is complex, we get

$$2n_r\bar{V}_1 = -P_m(n_r^2 - n_i^2) < |DW|^2 + \alpha^2|W|^2 > - \frac{1}{P_1} <(1+M^2P_1)|D^2W|^2 + \alpha^2\left(\alpha^2 - \frac{P_1P_mR}{P_r}\right)|W|^2 - - \alpha^2R^{1/2}P_1^{1/2}(P_m/Pr - 1)R_1(W^*\Psi) > \tag{3.8}$$

where

$$\Psi = (D^2 - \alpha^2)\Theta \tag{3.9}$$

$$\bar{V}_1 = \frac{1}{2}\langle |DW|^2 + (2\alpha^2 + P_m/P_1)|DW|^2 + + \alpha^2(\alpha^2 + P_m/P_1)|W|^2\rangle$$

and R_1 denotes the real part of the quantity. Similarly from (3.4), we get

$$P_r n_r \langle |D\Theta|^2 + \alpha^2|\Theta|^2\rangle = = -\langle |\Psi|^2 + R^{1/2}P_1^{1/2}R_1(W^*\Psi)\rangle \ \ldots \tag{3.10}$$

We note that the form of $\bar{V}_1$ given by (3.9) is such that we cannot be sure of the sign of dV_1/dt and hence we seek a new Liapunov functional. For this, we multyply (3.10) by an arbitrary positive constant λ and adding the resulting equation to (3.8), we obtain

$$2n_r\bar{V} = -P_m(n_r^2 - n_i^2)\langle |DW|^2 + + \alpha^2|W|^2\rangle - \frac{1}{P_1}\langle (1+M^2P_1)|D^2W|^2 + + \alpha^2(2+M^2P_1)|DW|^2 + \alpha^2\left(\alpha^2 - \frac{P_1P_mR}{Pr}\right)|W|^2$$

$$+ \lambda\alpha^2|\Psi|^2 + \alpha^2 R^{1/2} P_1^{1/2} (1 + \lambda - P_m/Pr) R_1 (W*\Psi) \rangle \quad (3.11)$$

where

$$\overline{V} = \overline{V}_1 + \lambda\overline{V}_2, \quad \overline{V}_2 = \frac{\alpha^2 Pr}{2P_1} \langle |D\Theta|^2 + \alpha^2|\Theta|^2 \rangle .$$

This new Liapunov functional makes the integral to be a negative definite quadratic form. Since our aim in this section is to consider the marginal state, we set $n_i = 0$ in (3.11) and obtain

$$2n_r\overline{V} < -\frac{1}{P_1} \langle (1 + M^2P_1)|D^2W|^2 + \alpha^2(2 + M^2P_1)|DW|^2 + \alpha^2\{\alpha^2 - \alpha^2\frac{P_1P_mR}{Pr} - \alpha^2RP_1/4\lambda(\lambda + 1 - P_m/Pr)^2\}|W|^2 \rangle \quad (3.13)$$

We introduce a quantity A such that the following inequality is true:

$$\langle (1 + M^2P_1)|D^2W|^2 + \alpha^2(2 + M^2P_1)|DW|^2 + \alpha^4|W|^2 \rangle > A \langle |W|^2 \rangle \quad (3.14)$$

By using calculus of variation, we find

$$(1 + M^2P_1)D^4W - \alpha^2(2 + M^2P_1)D^2W + (\alpha^4 - A)W = 0 \quad (3.15)$$

A solution of this equation, satisfying the boundary conditions (3.11), is

$$W = W_0 \operatorname{Sin} \pi z \quad (3.16)$$

where w_0 is an arbitrary constant. From (3.15), using (3.16), we obtain

$$A = \pi^4(\alpha^2 + 1)^2 + \pi^4M^2P_1(\alpha^2 + 1) . \quad (3.17)$$

Also from (3.13) and (3.14), we obtain

$$2n_r\overline{V} < -\frac{1}{P_1}\left\{A - \alpha^2\frac{P_mP_1R}{Pr} - \alpha^2\frac{(\lambda + 1 - P_m/Pr)^2}{4\lambda}RP_1\right\} \langle |W|^2 \rangle . \quad (3.18)$$

Here we choose RP_1 in such a way that

$$A - \alpha^2RP_1\left\{\frac{P_m}{Pr} + \frac{(\lambda + 1 - P_m/Pr)^2}{4\lambda}\right\} > 0 . \quad (3.19)$$

From this, using (3.17), we get

$$R < \frac{\pi^2(\alpha^2 + 1)}{\alpha^2P_1}\left[\frac{1 + M^2P_1 + \alpha^2}{P_m/Pr + (\lambda + 1 - P_m/Pr)/4\lambda}\right] = \overline{R} . \quad (3.20)$$

Since λ is an arbitrary positive constant, we choose it in such a way that R becomes maximum. Therefore, we let

$$f(\lambda) = (\lambda + 1 - P_m/Pr)^2$$

then

$$\frac{df}{d\lambda} = (\lambda + P_m/Pr - 1)(\lambda - P_m/Pr + 1)/\lambda^2 ,$$

$$\frac{d^2f}{d\lambda^2} = 2(P_m/Pr - 1)^2/\lambda^3$$

we choose

$$\lambda = \pm(1 - P_m/Pr) \quad (3.21)$$

where we choose positive sign when $P_m/Pr < 1$ and negative sign when the opposite is true. Therefore, for $P_m/Pr < 1$, the maximum value of $\overline{R}$ is given by

$$\overline{R}_{1c} = \frac{\pi^2(\alpha^2 + 1)}{\alpha^2P_1}[M^2P_1 + \alpha^2 + 1] . \quad (3.22)$$

However when $P_m/Pr > 1$, the maximum value of $\overline{R}$ is given by

$$\overline{R}_{2c} = \frac{Pr}{P_m}\overline{R}_{1c} . \quad (3.23)$$

In particular, when $P_m/Pr = 1$, $\lambda = 0$ and in that case $\overline{R}_{1c} = \overline{R}_{2c}$. Knowing λ, we use (3.12) to get the required Liapunov functional

$$\overline{V} = \frac{1}{2} < |D^2W|^2 + (2\alpha^2 + P_m/P_1)|DW|^2 + \alpha^2(\alpha^2 + P_m/P_1)|W|^2 + |Pr - P_m|\frac{\alpha^2}{P_1}(|D\Theta|^2 + \alpha^2|\Theta|^2) \quad (3.24)$$

we suppose that the matric $\overline{B}$ is given by

$$\overline{B}^2 = \frac{1}{2}\langle |D^2W|^2 + |DW|^2 + |W|^2 + |D\Theta|^2 + |\Theta|^2\rangle . \tag{3.25}$$

Then the conditions (2.16) are satisfied if we choose

$$\bar{\alpha}_0 = \mathrm{Min}\left\{1,\ 2\alpha^2 + P_m/P_1,\ \alpha^2(\alpha^2 + P_m/P_1),\ \alpha^2/P_1|P_m - Pr|,\ \frac{\alpha^4}{P_1}|P_m - Pr|\right\} \tag{3.26}$$

and

$$\bar{\alpha}_1 = \mathrm{Max}\left\{1,\ 2\alpha^2 + P_m/P_1,\ \alpha^2(\alpha^2 + P_m/P_1),\ \frac{\alpha^2}{P_1}|P_m - Pr|,\ \frac{\alpha^4}{P_1}|P_m - Pr|\right\} . \tag{3.27}$$

In order to obtain R_c, the critical Rayleigh number, we minimize R with respect to α and obtain the minimum wave number α_c given by

$$\alpha_c = (1 + M^2P_1)^{1/4} . \tag{3.28}$$

Substituting this α_c in (3.22) and (3.23), we get

$$\overline{R}_{1c} = \frac{\pi^2}{P_1}\left[1 + \sqrt{1 + M^2P_1}\right]^2 \quad \text{for} \quad \frac{P_m}{Pr} < 1 \tag{3.29}$$

and

$$R_{2c} = \frac{Pr}{P_m} R_{1c} \quad \text{for} \quad \frac{P_m}{Pr} > 1 . \tag{3.30}$$

4 Stability of Overstable Motions

The analysis of section 3 reveals that there exist two Rayleigh numbers, one, R_{1c} for $P_m/Pr < 1$ and the other, R_{2c} for $P_m/Pr > 1$. The former coincides with the Rayleigh number given by Prabhamani and Rudraiah [12] for steady marginal state, whereas the latter decreases R_{1c} by an amount of P_m/Pr. In other words in the case of $P_m/Pr > 1$ the principle of exchange of stability is not valid and we have a situation analogous to the overstable motions. To substantiate this we discuss in this section, following the analysis of Chandrasekhar [19], the stability of overstable motions and try to answer the question whether or not instability can arise as oscillations of increasing amplitude.

For this we obtain, from (3.3) and (3.4) after eliminating Θ, the stability Equation

$$(D^2 - r_1^2)(D^2 - r_1^2 - nPr)[(n + 1/P_1)(D^2 - r_1^2 - nP_m) + M^2D^2]W = Rr_1^2(D^2 - r_1^2 - nP_m)W . \tag{4.1}$$

The trial solution of (4.1), satisfying the boundary conditions (2.16), is

$$w = w_0 \sin \pi z , \tag{4.2}$$

where w_0 is an arbitrary constant. Substituting (4.2) into (4.1), we obtain the eigenvalue equation

$$\pi^2(\alpha^2 + 1)(\alpha^2 + 1 + nPr/\pi^2)[(n + 1/Pr)(\alpha^2 + 1 + nP_m/\pi^2) + M^2] = \alpha^2R(\alpha^2 + 1 + nP_m/\pi^2) \tag{4.3}$$

where n is complex. Defining $\alpha^2 = r_1^2/\pi^2$ and $n = in_1$, we can rewrite (4.3) in the form

$$R = \pi^2\frac{(\alpha^2 + 1)}{\alpha^2}\left[(\alpha^2 + 1 + in_1Pr/\pi^2)(in_1 + 1/P_1) + \frac{M^2(\alpha^2 + 1 + in_1Pr/\pi^2)}{(\alpha^2 + 1 + in_1P_m/\pi^2)}\right] . \tag{4.4}$$

From (4.4) it is clear that for an arbitrary assigned n_1, R will be complex. For overstable motions we denote this R by R_0. The physical meaning of R_0 requires it to be real. Consequently, the condition that R_0 be real implies a relation between the real M and the imaginary part of n_1. Since our interest is in oscillatory motion, it is enough if we seek solutions of (4.4) for which n_1 is real. Assuming, then, that n_1 is real and equating the real and imaginary parts of (4.4) and simplifying we get

$$n_1^2 = \frac{\alpha^2 + 1}{P_m\Delta}\left[\frac{M^2}{\pi^2}(P_m - Pr) - (1 + \alpha^2)^2 - \frac{(1 + \alpha^2)Pr}{\pi^2P_1}\right] \tag{4.5}$$

$$R_0 = \pi^2\frac{(\alpha^2 + 1)}{\alpha^2}\left[\frac{(\alpha^2 + 1)(P_m + Pr)}{\pi^2\Delta} + \frac{(\alpha^2 + 1)P_mPr}{P_1\Delta} + \frac{Pr(P_m + Pr)(1 + \alpha^2)^2}{P_1\Delta} + \frac{\alpha^2 + 1}{P_1}(1 + Pr^2/\Delta) + \right.$$

$$+ M^2 \left\{ 1 - \frac{P_m Pr(P_m^2 - Pr^2)(\alpha^2+1)}{\pi^2 P_1 \Delta} + \frac{Pr}{\pi^2 P_1} \right\} \Bigg] \Bigg] \quad (4.6)$$

where

$$\Delta = \alpha^2 + 1 + Pr/\pi^2 P_1 .$$

Equations (4.5) and (4.6) are the Eqs. which must be satisfied for overstability is to occur for a wave number corresponding to α^2 and the magnetic field corresponding to M^2. From (4.5) it is clear that overstable motions occur if

$$P_m > Pr, \text{ and } M^2 > \frac{\alpha^2+1}{P_m - Pr}(1+\alpha^2+Pr/\pi^2 P_1) \quad (4.7)$$

otherwise n^2 would be negative contrary to hypothesis. For a given M^2 overstable solutions are, therefore, possible only for $\alpha^2 < \alpha_*^2$ where α_*^2 is such that

$$\alpha_*^2\left(\alpha_*^2 + 2 - \frac{P_m}{\pi^2 P_1}\right) = M^2(P_m - Pr) - (1+Pr/\pi^2 P_1) \quad (4.8)$$

when $\alpha^2 = \alpha_*^2$, $n_1^2 = 0$ and from (4.4) we have

$$R_c = \pi^2 \frac{(\alpha^2+1)}{\alpha^2}\left[M^2 + \frac{\alpha^2+1}{P_1}\right] \quad (4.9)$$

which, as one would expect, is the same as the one given by (3.35) at which the steady convection will occur for a wave number corresponding to α_*. For $\alpha > \alpha_*$, overstability is not possible and the principle of exchange of stability is valid. In the (α, R_c)-plane the curve given by (4.9) defines the locus of states which are marginal with respect to stationary convection and the effect of permeability of the material is to increase the value of R_c compared to that given by Chandrasekhar [19] for impermeable material. The overstable solutions branch off from the locus of marginal steady state at the point α_* and for $\alpha < \alpha_*$ they are described by

$$R_0 = R_c - n_1^2 \frac{\alpha^2+1}{\alpha^2}\left[P_m + Pr + \frac{P_m Pr}{\pi^2(\alpha^2+1)P_1}\right] \quad (4.10)$$

which shows that the Rayleigh number for overstable motions is always less than R_c. Depending on P_m, Pr, M^2 and P_1 the branch point α_*^c can occur either before or after the point α_{min}^c at which R_c attains its minimum. If $\alpha_* > \alpha_{min}^c$, then it is clear that for all $\alpha < \alpha_*$ overstability is the preferred manner of instability.

In the remaining part of this section a simple physical explanation (Veronis [1], Rudraiah and Srimani [20]) is given for overstable motions.

In the hydrodynamic stability problem of Lapwood (1948) the horizontal temperature gradient of the perturbed field releases potential energy which is supplemented by the Darcy resistance of the motion. This can be seen from the equation,

$$\frac{\partial \eta}{\partial t} + (\vec{q}\cdot\nabla)\eta = -g\beta\frac{\partial T}{\partial x} - (\nu/k)\eta \quad (4.11)$$

which can be obtained from (2.7) by letting the magnetic field tend to zero. Thermally, the upward convection of warm fluid is balanced by the diffusion of excess temperature. In these simple balances the velocity and temperature fields are in phase and no restoring force exists; and hence no overstable motions are possible. In other words, the principle of exchange of stability is always possible in that case. However, in the present case (2.7) shows that the magnetic field introduces a Lorentz force with the result that a "magnetic wind" component, $\frac{\mu H_o}{\rho_o}\frac{\partial \xi}{\partial z}$ is generated. In the linear analysis the magnetic field describes a balance between a horizontal temperature gradient supplemented by the Darcy resistance and the vertical shear of the current density (we shall call, in analogy with the zonal velocity (Veronis [1]), the zonal current) normal to the temperature gradient and Darcy resistance term. In (2.1), neglecting the non-linear terms, this balance is given by the terms $\frac{\mu H_o}{\rho_0}\partial\xi/\partial z$ and $\alpha g\frac{\partial T}{\partial x} + (\nu/k)\eta$. The inhibition of convection by magnetic field is clearly traceable to the magnetic constraint which is energetically inactive. The larger the strength of the magnetic field, the larger the zonal current. Hence, less potential energy is released for a given horizontal temperature gradient and Darcy resistance. Thus oscillatory motions are possible in a conducting fluid in the presence of a magnetic field because the Lorentz force can act as a restoring mechanism. This motion also involves a balance between the local acceleration and the Lorentz force. Therefore, in the steady convection the Lorentz force balances the force which releases the horizontal temperature gradient and Darcy resistance. Whereas, in the transient motion, a part of the Lorentz force can be balanced by the local acceleration so that less of magnetic constraint is available to offset the horizontal temperature gradient. Consequently the cell must be distorted (i.e. shrunk) less by the magnetic field and there is less dissipation associated with the somewhat larger cell. Convection can, therefore, be maintained for a smaller imposed temperature difference (i.e. smaller Rayleigh number).

In the process of balance, we note that a time dependent temperature field involves, as in the hydrodynamic case (Veronis [1]), a perturbation temperature which is out of phase with the vertical velocity and hence is less efficient than the steady motion for convecting heat upward. Therefore, the overstable motions are prefered only when the effects of these out-of-phase temperature fluctuations are smaller than the effects of the time-dependent motions in the dynamical process because the latter enhance convection by offsetting the constraining force of magnetic field. In this process the ratio P_m/Pr, which measures the relative role of thermal conductivity to magnetic viscosity, plays a significant role. If P_m/Pr is greater than unity, magnetic diffusive processes are relatively less important than thermal diffusive processes and time-dependent motions are more important in the dynamical balance than in the thermal balance. Hence, the onset of convection as overstable motions for larger P_m/Pr.

The same kind of qualitative argument can be used in connection with non-linear analysis including inertial process of velocity, temperature and magnetic fields. Thus, it is possible that motions with finite amplitude may exist at subcritical values of the Rayleigh number because the inertial process $((\vec{q}\cdot\nabla)\eta)$ in (2.7) may balance the constraining effect of magnetic field namely

$$\frac{\mu}{\rho_0}(\vec{H}\cdot\nabla)\xi + \frac{\mu H_0}{\rho_0}\frac{\partial\xi}{\partial z}.$$

This is considered in the next section.

5 Finite Amplitude Analysis with a Limited Representation

In this section we discuss, following Veronis [1], the finite amplitude analysis by considering a truncated representation of velocity, magnetic and temperature fields and try to deduce certain general physical results with a minimum amount of mathematical analysis. We note that the results obtained from such a simple analysis can be used as a starting value in solving a fully non-linear convection problem.

In section 3, we have seen that the linear stability problem has a steady state solution whose form is given by (3.30) for velocity. The first effect of nonlinearity is to distort the temperature field through the interaction of Ψ and T and the zonal current field through the interaction of Ψ and φ. The distortion of temperature field will correspond to a change in the horizontal mean, i.e. a component of the form $\sin 2\pi Z$ will be generated. Similarly, the zonal current field will be distorted by a component of the form $\sin 2\pi\alpha x$. Thus, a minimal system which describes finite amplitude convection is given by

$$\Psi = A(t)\sin\pi\alpha x \sin\pi Z \tag{5.1}$$

$$T = B(t)\cos\pi\alpha x\,\mathrm{Sin}\,\pi Z + C(t)\sin 2\pi Z \tag{5.2}$$

$$\varphi = D(t)\sin\pi\alpha x\cos\pi Z + E(t)\sin 2\pi\alpha x \tag{5.3}$$

where the amplitudes A, B, C, D and E are generally functions of time and are to be determined by the dynamics of the system. Substituting (5.1) to (5.3) into (2.13) to (2.15) and equating the like terms, we get

$$\dot{A} = \pi^2\alpha Q\,\frac{(3\alpha^2-1)}{\alpha^2+1}\,DE - \pi QD - \frac{Pr}{P_1}A - \frac{\alpha PrRB}{\pi(\alpha^2+1)} \tag{5.4}$$

$$\dot{B} = -\pi^2(\alpha^2+1)B - \pi\alpha A - \pi^2\alpha AC \tag{5.5}$$

$$\dot{C} = \frac{\pi^2}{2}\alpha AB - 4\pi^2 C \tag{5.6}$$

$$\dot{D} = \pi^2\alpha AE + \pi A - \frac{\pi^2}{S}(\alpha^2+1)D \tag{5.7}$$

$$\dot{E} = -\frac{\pi^2}{2}\alpha AD - 4\pi^2\frac{\alpha^2}{S}E \tag{5.8}$$

where the dot over corresponds to a time derivative.

This set of non-linear ordinary differential equations is not amenable to analytical treatment for the general time-dependent variables and we have to solve it using a numerical method. However, in the case of steady motions these equations can be solved analytically. Such solutions are very useful because they show that a finite amplitude steady solution to the system is possible for subcritical values of the Rayleigh number and that the minimum values of R for which a steady solution is possible lies below the critical values for instability to either a steady infinitesimal disturbance or an overstable infinitesimal disturbance.

Thus, if the system is steady, Eqs. (5.4) to (5.8) take the form

$$\frac{\pi^2}{P_1}(\alpha^2+1)A + \pi\alpha RB + \frac{\pi^3}{Pr}Q(\alpha^2+1)D - \frac{\pi^4 Q\alpha(3\alpha^2-1)}{Pr}DE = 0 \qquad (5.9)$$

$$\pi^2\alpha AC + \pi\alpha A + \pi^2(\alpha^2+1)B = 0 \qquad (5.10)$$

$$\frac{\pi^2}{2}\alpha AB - 4\pi^2 C = 0 \qquad (5.11)$$

$$\pi^2\alpha AE + \pi A - \frac{\pi^2}{S}(\alpha^2+1)D = 0 \qquad (5\text{-}12)$$

$$\frac{\pi^2}{2}\alpha AD + 4\pi^2\frac{\alpha^2}{S}E = 0\,. \qquad (5.13)$$

Equations (5.10) to (5.13) can be re-written in the form

$$B = -\pi\alpha A/\pi^2(\alpha^2+1+\tfrac{1}{8}\alpha^2A^2) \qquad (5.14)$$

$$C = \tfrac{1}{8}\alpha AB \qquad (5.15)$$

$$D = \pi SA/\pi^2(\alpha^2+1+\tfrac{1}{8}S^2A^2) \qquad (5.16)$$

$$E = -SAD/8\alpha\,. \qquad (5.17)$$

Substituting (5.14) to (5.17) into (5.9) and after some simplification, we get

$$\frac{\pi^2S^2\alpha^2(\alpha^2+1)}{P_1}(A^2/8)^3 + [\pi^2S^2(\alpha^2+1)^2(1+2\alpha^2/S^2)/P_1 - \alpha^2S^2R + 4\pi^2\alpha^4M^2]\left(\frac{A^2}{8}\right)^2$$
$$+\left[\pi^2(\alpha^2+1)^3(2+\alpha^2/S^2)/P_1 - 2\alpha^2R(\alpha^2+1) + M^2\pi^2\alpha^2(\alpha^2+1)\left\{4+\frac{(\alpha^2+1)}{S^2}\right\}\right]\frac{A^2}{8} + \frac{\pi^2(\alpha^2+1)^2}{P_1}\cdot$$
$$+ M^2\pi^2(\alpha^2+1) - \alpha^2R = 0 \qquad (5.18)$$

We have to look for real positive roots of this cubic equation in $A^2/8$, otherwise the amplitude of the streamfunction becomes imaginary. We know that (Abramowitz and Segun [21]) the roots of (5.18) are real if $q^2 + r^2 \leqslant 0$ where $q = \frac{1}{3}a_1 - \frac{1}{q}a_2^2$,

$$r = \frac{1}{6}(a_1a_2 - 3a_0) - \frac{1}{27}a_2^3,$$

$$a_0 = \frac{P_1}{\pi^2S^2\alpha^2(\alpha^2+1)}\left[\frac{\pi^2(\alpha^2+1)^2}{P_1} + M^2\pi^2(\alpha^2+1) - \alpha^2R\right],\quad a_1 = \frac{P_1}{\pi^2S^2(\alpha^2+1)}[\pi^2(\alpha^2+1)^3(2+\alpha^2/S^2)/P_1$$
$$-2R\alpha^2(\alpha^2+1) + M^2\pi^2\alpha^2(\alpha^2+1)\{4+(\alpha^2+1)/S^2\}],$$

$$a_2 = \frac{P_1}{\pi^2S^2(\alpha^2+1)\alpha^2}\left[\frac{\pi^2S^2(\alpha^2+1)^2(1+2\alpha^2/S^2)}{P_1} - \alpha^2S^2R + 4\pi^2\alpha^4M^2\right].$$

The expression for $q^2 + r^2$ is very complicated and difficult to arrive at definite conclusions on the Rayleigh number. However, we note that if $a_0 = 0$, one root of (5.17) is zero corresponding to pure conduction, which we know to be a possible solution although it is unstable for large R. The remaining solutions are given by

$$\frac{A^2}{8} = \frac{P_1}{2\pi^2S^2\alpha^2(\alpha^2+1)}[\alpha^2S^2R^2 - \pi^2(\alpha^2+1)^2(1+2\alpha^2/S^2) - 4\pi^2\alpha^4M^2] \pm \{[\alpha^2S^2R - \pi^2(\alpha^2+1)^2(1+2\alpha^2/S^2)/P_1 -$$
$$-4\pi^2\alpha^4M^2]^2 + \frac{4\pi^2S^2\alpha^2(\alpha^2+1)}{P_1}[2R\alpha^2(\alpha^2+1) - (\alpha^2+1)^3(2+\alpha^2/S^2)/P_1 - M^2\pi^2\alpha^2(\alpha^2+1)\{4 + S^2(\alpha^2+1)\}\}^{1/2}]\,. \qquad (5.19)$$

Only the solution with the positive sign in front of the root of the discriminant is admissible since otherwise A^2 is negative implying the amplitude of the streamfunction is imaginary. Now, consider the case where finite solutions exist for $R_f < R_c$, R_f being the Rayleigh number for finite amplitude solution. The minimum value of R_f for which solutions exist is that value of R_f which makes the discriminent zero provided that the first form on the right-hand-side of (5.19) be non-negative. The discriminant is zero if

$$R_f = \frac{\pi^2}{\alpha^2 P_1}\left[(\alpha^2+1)(1-2\alpha^2/S^2) + 4M^2P_1\alpha^4/S^2 + \right.$$
$$+ \frac{2\alpha^2(\alpha^2+1)}{S^2}\left\{(\alpha^2+1) + M^2S^2P_1(4-7\alpha^2/S^2 + \right.$$
$$\left.\left. + \frac{1}{S^2}\right\}^{1/2}\right]. \tag{5.20}$$

With this value of R_f the amplitudes are real provided that the first term on the right-hand side of (5.19) be non-negative or equivalently

$$\frac{1}{P_1} > \frac{1}{S^2}\left[1 + \frac{2\alpha^2}{S^2} + 14\alpha^4M^2/(\alpha^2+1)\right]. \tag{5.21}$$

This condition on porous parameter, $1/P_1 = d^2/k$, is readily accessible in the laboratory experiment with sand ($k \approx 0(10^{-4})\mathrm{cm}^2$) as porous media and with mercury (for which $P_m = 0{,}17$, $Pr = 0{,}025$, $S^2 = 46.277$). The numerical values of (5.20) shows that R_f is less than R_c.

We note that setting the constant term in the cubic equation (5.18) equal to zero, is equivalent to restricting the value of R very near to R_c. However, this can be avoided by assuming that the terms involving induced magnetic field and induced current are small compared to the applied magnetic field; which is usually the case in many practical problems. In this case (2.7) takes the form

$$\frac{\partial\eta}{\partial t} + (\vec{q}\cdot\nabla)\eta = \frac{\mu H_0}{\rho_0}\frac{\partial\xi}{\partial z} - \frac{\alpha g}{\rho_0}\frac{\partial T}{\partial x} - \frac{\nu}{k}\eta \tag{5.22}$$

and all other equations remain the same. Now, substituting (5.1) to (5.3) into these Eqs. and assuming, as before, the steady case we get

$$\frac{\pi^2}{P_1}(\alpha^2+1)A + \pi\alpha RB + \pi^3\frac{M^2}{S}(\alpha^2+1)D = 0 \tag{5.23}$$

and the other Eqs. are the same as (5.10) to (5.13). As before, substituting (5.14) to (5.16) into (5.23), we get

$$A\left[\frac{\pi^2}{P_1}S^2\alpha^2(\alpha^2+1)(A^2/8)^2 + \left\{\frac{\pi^2}{P_1}S^2(\alpha^2+1)^2 + \alpha^4R_c - \right.\right.$$
$$\left.\left. - \alpha^2S^2R\right\}\frac{A^2}{8} + \alpha^2(\alpha^2+1)(R_c - R)\right] = 0. \tag{5.24}$$

The solution $A = 0$ corresponds to the pure conduction and the other solutions are given by

$$\frac{A^2}{8} = \frac{P_1}{2\pi^2\alpha^2S^2(\alpha^2+1)}[\alpha^2S^2(R-R_c) - \alpha^4R_c +$$
$$+ \pi^2S^2(\alpha^2+1)M^2 \pm \{[\alpha^2S^2(R-R_c) - \alpha^4R_c +$$
$$+ \pi^2S^2(\alpha^2+1)M^2]^2 + 4S^2\alpha^4[\alpha^2R_c - \pi^2(\alpha^2 + 1M^2]\}^{1/2}]. \tag{5.25}$$

To ensure the amplitude of the stream function to be real, we have to take the positive sign in front of the root of the discriminant in (5.25).

Consider the case where finite solutions exist for $R < R_c$. We know that (Veronis [1]) the minimum value of R for which solutions exist is that value of R which makes the discriminant zero provided that the first term on the right-hand side of (5.25) be non-negative. The discriminant is zero provided that

$$R_f = \frac{\pi^2(\alpha^2+1)}{S^2}\left[\left\{\frac{(S^2-\alpha^2)(\alpha^2+1)}{\alpha^2P_1}\right\}^{1/2} + M\right]^2 \tag{5.26}$$

where $R = R_f$ denotes the Rayleigh number for finite amplitude solutions. With this value of R_f, the amplitudes are real provided that the first term on the right-hand side of (5.25) be non-negative; or equivalently

$$M^2 > \frac{\alpha^2(\alpha^2+1)}{P_1(S^2-\alpha^2)}. \tag{5.27}$$

Conditions (5.26) and (5.27) are meaningful only when

$$S > \alpha$$

We note that conditions (5.27) and (5.28) are readily accessible in a laboratory experiment with fibre material for porous media and with mercury flowing through it. For this model (5.21) is satisfied if M is greater than one when $\alpha = 1.2$ and $P_1 = 0.1$. In other words finite amplitude steady convective motions are possible at small Hartmann numbers and small porous parameters. Whereas condition (4.7) shows that infinitesimal oscillatory motions are possible only at moderately high Hartmann numbers (for example $M > 14$ when $P_l = .001$, $M > 10$ when $P_l = .01$ with $\alpha = 0.5$, $Pr = .025$, $P_m = 0.17$ i.e. for mercury).

The physical reason for this is discussed in the next section.

6 Conclusions

Infinitesimal steady and oscillatory convective motions and steady finite-amplitude convective motions are investigated. It is shown that the existence of these convections will depend on the parameters M^2, P_l and α, and the following conclusions are drawn:

I. The Liapunov technique, applied to investigate the infinitesimal steady motions, gives two critical Rayleigh numbers, R_{1c} and R_{2c} for $P_m/Pr \leqslant 1$ and $P_m/Pr > 1$ respectively. The former coincides with the Rayleigh number given by normal mode technique (Prabhamani and Rudraiah [12]) and the latter is always less than R_{1c}. Therefore, we conclude that the principle of exchange of stability is valid only when $P_m/Pr \leqslant 1$ and $P_m/Pr > 1$ gives the results similar to the overstable motions.

II. To substantiate the result of $P_m/Pr > 1$ given by Liapunov technique, the stability of infinitesimal oscillatory motions through porous media is investigated using Chandrasekhar [19] analysis. The results show that overstable motions are possible only when $P_m/Pr > 1$ with M^2 satisfying Eq.(4.7). This condition (4.7) imposes the condition on P_l and α and it shows that overstable motions are possible only for moderately large values of M (i.e. $M > 14$ for $P_l = .01$ and $M > 10$ for $P_l = .001$ for mercury). In other words, for smaller values of M^2 the motion should be steady, infinitesimal or finite, rather than overstable and the scale of the motion should be comparable to that which occurs for hydrodynamic convection through porous media (Lapwood [4]). We find that the generation of zonal current by the magnetic field is responsible for oscillatory motions which exist at a Rayleigh number smaller than that of infinitesimal steady motions. This zonal current is also responsible for the existence of steady finite amplitude motions.

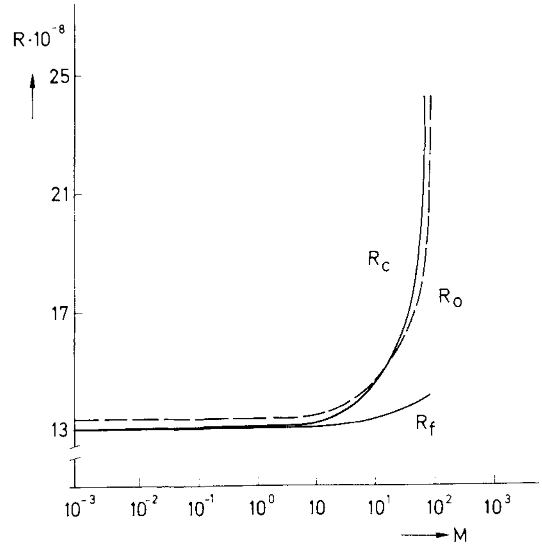

Fig.2. Curves of R_c, R_0 and R_f as function of the Hartmann number M for $P_l = 10^{-3}$ and $\alpha = 0,3$

III. Therefore, finite amplitude steady convection is investigated using Veronis [1] truncated representation. We find that for large values of the Hartmann number M finite amplitude steady convections are not possible because they occur only in a subcritical range of R as long as they can reduce the constraint of magnetic field. When M is large, the motions must have large amplitudes to offset the effect of magnetic field. This is because greater amplitudes require more release of potential energy which in turn requires a larger value of R. Hence when M becomes large enough to offset the constraint the motion must have an amplitude which cannot be achieved for $R < R_0$. We also find that no finite amplitude motions are possible when the Busse number is smaller than α. Since finite amplitude motions are possible for $S > \alpha$ (i.e. small α), the preferred scale of finite amplitude motions is larger than the corresponding scale for infinitesimal oscillatory motions. These general results are shown in Fig.2, for mercury with $P_m = 0.025$, $Pr = .17$, $S^2 = 46.277$ and $P_l = 0.001$. From this figure it is clear that the oscillatory motions are possible only for large Hartmann numbers, while the marginal state and the steady finite amplitude motions are possible even at the low Hartmann numbers. Similar behaviour is also observed for other values of P_1 where the Rayleigh number increases with increasing M and decreases with decreasing P_1.

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