PARTIAL DERIVATIVES OF DISPERSION CURVES OF LOVE WAVES IN A LAYERED MEDIUM

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1. INTRODUCTION

In interpreting dispersion curves it is very rare for the theoretical dispersion curve of the model assumed to agree exactly with experimental data. In order to achieve agreement with experimental data, it is usually necessary to change the parameters of the model gradually. To render this procedure effective and sufficiently fast, it is very valuable to know the partial derivatives of the dispersion curve with respect to the parameters of the medium. The knowledge of the partial derivatives extends the possibilities of the interpretation considerably, because it allows for the changes of the dispersion curve, due to the changes of the parameters of the medium, to be determined. Also the objective method of numerical inversion, presented in [3], is based on the computation of the partial derivatives.

The study and applications of partial derivatives of dispersion curves has been the subject of a number of papers. The partial derivatives of the phase velocity, the group velocity and its partial derivatives are usually computed numerically [2, 3, 8, 9]. This procedure is simple, but very time-consuming and also less accurate, especially as regards the partial derivatives of the group velocity [8]. Some of the disadvantages of numerical differentiation were removed successfully by applying energy integrals. Formulas have been derived for computing the group velocity and the partial derivatives of the phase velocity without numerical differentiation [1, 6, 10, 11].

This paper describes another method of computing the group velocity and the partial derivatives of the phase and group velocities without numerical differentiation for a plane Love-wave problem. Thomson-Haskell matrices are used [4]. Only the phase velocity is computed numerically, the group velocity and all the derivatives are obtained by substituting into formulas.

2. FORMULAS FOR COMPUTING THE PARTIAL DERIVATIVES

Let us consider the propagation of Love waves in a medium, which is composed of homogeneous and isotropic, parallel layers, located on a homogeneous and isotropic half-space. Let c represent the phase velocity, U the group velocity, ω the angular velocity, N the number of layers (the index $N + 1$ will denote the half-space). The velocity of transverse waves, the density and the thickness of the m-th layer are denoted by b_m, ϱ_m, and d_m, respectively.

The dispersion equation of Love waves can then formally be written as follows:

$$f(\omega, b_1, \varrho_1, d_1, \ldots, b_N, \varrho_N, d_N, b_{N+1}, \varrho_{N+1}, c) = 0, \tag{1}$$

where the phase velocity c is again a function of the angular velocity ω and of the

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parameters of the medium:

(2) $$c = c(\omega, b_1, \varrho_1, d_1, \ldots, b_{N+1}, \varrho_{N+1}) .$$

The concrete form of the function f can be expressed, e.g., by means of Thomson-Haskell matrices. In particular, for a single-layered medium ($N = 1$) function f is given by

(3) $$f = \tan(\omega d_1 s_1/c) - \varrho_2 b_2^2 s_2'/(\varrho_1 b_1^2 s_1) ,$$

where $s_1 = \sqrt{(c^2/b_1^2 - 1)}$ and $s_2' = \sqrt{(1 - c^2/b_2^2)}$.

The phase velocity c is defined by Eq. (1) in implicit form and it cannot be expressed analytically. However, if the value of the phase velocity is known, its derivatives can be computed analytically using the theorem on implicit functions [5, 7]. Let us first differentiate Eq. (1) with respect to some parameter of the medium, p. As function f is a composite function,

(4) $$\partial f/\partial p + (\partial f/\partial c)(\partial c/\partial p) = 0 .$$

From the latter equation follows the formula for computing the partial derivatives of the phase velocity with respect to the parameter of the medium:

(5) $$\partial c/\partial p = -(\partial f/\partial p) : (\partial f/\partial c) .$$

If function f is expressed analytically, the partial derivatives of the phase velocity can easily be computed by using Eq. (5).

By differentiating Eq. (1) one can also derive the formula for computing the group velocity and its derivatives. From the formula for the group velocity U,

(6) $$U^{-1} = \partial(\omega c^{-1})/\partial\omega$$

it follows that

(7) $$U = c : [1 - (\omega/c)(\partial c/\partial\omega)] .$$

The partial derivatives of the phase velocity with respect to ω in Eq. (7) can be computed analogously to the partial derivatives in Eq. (5). Let us differentiate Eq. (1) with respect to ω:

(8), (9) $$\frac{\partial f}{\partial\omega} + \frac{\partial f}{\partial c}\frac{\partial c}{\partial\omega} = 0 \Rightarrow \frac{\partial c}{\partial\omega} = -\frac{\partial f}{\partial\omega}\Big/\frac{\partial f}{\partial c} .$$

Formulas (9) and (7) make it possible to compute the group velocity without numerical differentiation. The partial derivatives of the group velocity with respect to some parameter of the medium p can be derived from (7):

(10) $$\frac{\partial U}{\partial p} = \frac{U^2}{c^2}\left[\frac{\partial c}{\partial p}\left(1 - 2\frac{\omega}{c}\frac{\partial c}{\partial\omega}\right) + \omega\frac{\partial^2 c}{\partial\omega\,\partial p}\right].$$

In order to determine the last term in (10), we shall differentiate Eq. (8) with respect to parameter p. The second derivative of c with respect to ω and p is then

$$(11) \qquad \frac{\partial^2 c}{\partial\omega\,\partial p} = -\left[\frac{\partial^2 f}{\partial\omega\,\partial p} + \frac{\partial^2 f}{\partial\omega\,\partial c}\frac{\partial c}{\partial p} + \left(\frac{\partial^2 f}{\partial c\,\partial p} + \frac{\partial^2 f}{\partial c^2}\frac{\partial c}{\partial p}\right)\frac{\partial c}{\partial\omega}\right]\Big/\frac{\partial f}{\partial c}\,.$$

The applicability of these formulas, as well as their use in solving the inverse problem, were first tested on the case of the single-layered medium, where function f is defined by (3).

In order to be able to use the formulas for an N-layered medium, we shall express the function f and its derivatives by means of the Thomson-Haskell matrices. Let us introduce the following notations:

$$(12) \qquad \mu_m = \varrho_m b_m^2\,, \quad r_m = \sqrt{(c^2/b_m^2 - 1)} \quad \text{for} \quad c > b_m\,,$$
$$r_m = -\mathrm{i}\sqrt{(1 - c^2/b_m^2)} \quad \text{for} \quad c < b_m\,, \quad Q_m = (\omega/c)\, d_m r_m\,.$$

The contribution of the m-th layer to the dispersion equation is defined by the matrix

$$(13) \qquad \mathbf{Y}_m = \begin{vmatrix} y_{m1} & y_{m2} \\ y_{m3} & y_{m4} \end{vmatrix}\,,$$

where $y_{m1} = y_{m4} = 1$, $y_{m2} = \mathrm{i}\mu_m^{-1} r_m^{-1} \tan Q_m$, $y_{m3} = \mathrm{i}\mu_m r_m \tan Q_m$. Let us define the matrices $\mathbf{X}_m$ and $\mathbf{Z}_m$ by

$$(14) \qquad \mathbf{X}_1 = \mathbf{Y}_1\,, \quad \mathbf{X}_m = \begin{vmatrix} x_{m1} & x_{m2} \\ x_{m3} & x_{m4} \end{vmatrix} = \mathbf{Y}_m \,.\, \mathbf{Y}_{m-1} \ldots \mathbf{Y}_1 = \mathbf{Y}_m \,.\, \mathbf{X}_{m-1}\,,$$

$$(15) \qquad \mathbf{Z}_N = \mathbf{Y}_N\,, \quad \mathbf{Z}_m = \begin{vmatrix} z_{m1} & z_{m2} \\ z_{m3} & z_{m4} \end{vmatrix} = \mathbf{Y}_N \,.\, \mathbf{Y}_{N-1} \ldots \mathbf{Y}_m\,.$$

The dispersion equation can then be expressed by [4]

$$(16) \qquad f = x_{N3} + \varrho_{N+1} b_{N+1}^2 r_{N+1} x_{N1}\,.$$

It is easy to compute the partial derivatives of f with respect to the parameters of the half-space:

$$(17) \qquad \frac{\partial f}{\partial \varrho_{N+1}} = b_{N+1}^2\, r_{N+1}\, x_{N1}\,, \quad \frac{\partial f}{\partial b_{N+1}} = \varrho_{N+1} \frac{\partial(b_{N+1}^2 r_{N+1})}{\partial b_{N+1}}\, x_{N1}\,.$$

In computing the other partial derivatives, it is necessary to differentiate the matrices. By the derivative of a matrix we shall understand a matrix which is created by differentiating all the elements of the matrix. A rule analogous to that for the derivative of a product of functions holds for the product of matrices. By differentiating (14) with respect to c, we obtain

$$(18) \qquad \frac{\partial \mathbf{X}_1}{\partial c} = \frac{\partial \mathbf{Y}_1}{\partial c}\,, \quad \frac{\partial \mathbf{X}_m}{\partial c} = \frac{\partial \mathbf{Y}_m}{\partial c}\,\mathbf{X}_{m-1} + \mathbf{Y}_m \frac{\partial \mathbf{X}_{m-1}}{\partial c}\,.$$

Using the recurrent formulas (18), the partial derivative of $\mathbf{X}_N$ with respect to c can be computed. The partial derivative of f with respect to c is obtained by substituting into

$$(19) \qquad \frac{\partial f}{\partial c} = \frac{\partial x_{N3}}{\partial c} + \varrho_{N+1} b_{N+1}^2 \left(\frac{\partial r_{N+1}}{\partial c} x_{N1} + r_{N+1} \frac{\partial x_{N1}}{\partial c} \right).$$

It is easier to compute the partial derivatives of f with respect to the parameters of the medium. Let us rewrite matrix $\mathbf{X}_N$ as follows:

$$(20) \qquad \mathbf{X}_N = \mathbf{Z}_{m+1} \, . \, \mathbf{Y}_m \, . \, \mathbf{X}_{m-1} \, .$$

As the parameters of the medium of the m-th layer only occur in matrix $\mathbf{Y}_m$, it will hold that

(21), (22)

$$\frac{\partial \mathbf{X}_N}{\partial p_m} = \mathbf{Z}_{m+1} \, . \, \frac{\partial \mathbf{Y}_m}{\partial p_m} \, . \, \mathbf{X}_{m-1} \, , \quad \frac{\partial f}{\partial p_m} = \frac{\partial x_{N3}}{\partial p_m} + \varrho_{N+1} b_{N+1}^2 r_{N+1} \frac{\partial x_{N1}}{\partial p_m} \, ,$$

where p_m represents b_m, ϱ_m, or d_m. Thus, the partial derivative (22) is obtained from Eq. (16) by substituting the elements of matrix $\mathbf{X}_N$ by the elements of matrix (21).

The above represents the description of the computation of all expressions which are required to determine the partial derivatives of the phase velocity according to Eq. (5). In computing the group velocity and its partial derivatives, we need certain other derivatives of function f. The computation of the partial derivative of f with respect to ω is analogous to the computation of the partial derivative of function f with respect to c. In computing the second partial derivatives of the function f it is necessary to differentiate further Eqs. (17) to (19), (21) and (22).

3. SHORT DESCRIPTION OF THE PROGRAMME

Using the formulas, given above, a programme for the MINSK 22 computer was written. The programme consists of four principal parts. For the sake of simplicity, let us first describe the computing of the partial derivatives of the phase velocity.

In the first part of the programme the phase velocity is computed numerically. In the second part the partial derivative of matrix $\mathbf{X}_N$ with respect to c is computed by using the recurrent formulas (18). This part is formed by a cycle which runs through from the first to the N-th layer. The elements of the matrix $\mathbf{X}_m$ (the first column is sufficient) are gradually stored, as well as some other expressions like $r_m \tan Q_m$, etc. In the third part Eq. (19) is computed and the computation of the partial derivatives of the phase velocity with respect to ϱ_{N+1} and b_{N+1} is carried out with the help of Eqs. (17) and (5). The fourth part of the programme is formed by a cycle which has the reversed order, beginning with the N-th layer and ending in the first layer. For each layer Eqs. (21), (22) and (5) are computed, as well as the matrix $\mathbf{Z}_m$ using the formula $\mathbf{Z}_m = \mathbf{Z}_{m+1} \, . \, \mathbf{Y}_m$. Matrix $\mathbf{Z}_m$ will be needed for the subsequent cycle to compute Eq. (21), but index m will be substituted by $m - 1$. After the results have been printed out the whole computation is repeated for the next period.

In computing the partial derivatives of the group velocity, apart from all the computations of the partial derivatives of the phase velocity, it is also necessary to compute some other expressions. The first part of the programme remains the same. Moreover, in the second part the following partial derivatives are computed:

$$\partial \mathbf{X}_m/\partial \omega\,, \quad \partial^2 \mathbf{X}_m/\partial c^2\,, \quad \partial^2 \mathbf{X}_m/\partial c\,\partial \omega\,. \tag{23}$$

The partial derivatives of matrix $\mathbf{X}_m$ with respect to c and the first expression in (23) are stored. In the third part also the following expressions are computed:

$$\partial f/\partial \omega\,, \quad \partial^2 f/\partial c^2\,, \quad \partial^2 f/\partial c\,\partial \omega\,, \tag{24}$$

as well as formulas (9) and (7) and the partial derivatives of the group velocity with respect to the parameters of the half-space. In the fourth part of the programme also the partial derivative of (21) with respect to c and ω, expressions (11), (10) and the partial derivatives of matrix $\mathbf{Z}_m$ with respect to c and ω are computed.

Table 1. Parameters of the model of the Canadian shield CANSD. The quantities i, b_i, ϱ_i and d_i represent the number of the layer, the velocity of the transverse waves, the density and the thickness of the i-th layer, respectively.

i	b_i	ϱ_i	d_i
1	3·47	2·70	6·0
2	3·64	2·80	10·5
3	3·85	2·85	18·7
4	4·72	3·30	80·0
5	4·54	3·44	100·0
6	4·51	3·53	100·0
7	4·76	3·60	80·0
8	5·12	3·76	∞

Table 2a. Dispersion curves for the fundamental mode of Love waves in the CANSD model.

T (s)	c (km/s)	U (km/s)
10·0	3·74107	3·51010
20·0	4·00710	3·52732
30·0	4·25059	3·72166
40·0	4·40209	4·01515
50·0	4·47858	4·22071

The properties of the programme were tested using computations for the model of the Canadian shield CANSD [2]. The parameters of the model are given in Tab. 1, an example of the computations are in Tab. 2. The following numerical data in this section concern the programme alterna-

Table 2b. Computation for the fundamental mode of Love waves in the CANSD model. Partial derivatives for the individual layers are printed under one another.

T	$\partial c/\partial b_i$	$\partial c/\partial \varrho_i$	$\partial c/\partial d_i$	$\partial U/\partial b_i$	$\partial U/\partial \varrho_i$	$\partial U/\partial d_i$
20	0·25048	−0·05192	−0·02455	0·39610	−0·03708	−0·01867
	0·40458	−0·03721	−0·01807	0·56217	0·01013	−0·00798
	0·40666	0·03642	−0·01113	0·29970	0·06946	0·00043
	0·13636	0·04255	0·00000	−0·17980	−0·03776	−0·00002
	0·00022	0·00005	0·00000	−0·00205	−0·00046	0·00000
	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000
	0·00000	0·00000	0·00000	0·00000	0·00000	0·00000
	0·00000	0·00000		0·00000	0·00000	
40	0·09234	−0·03562	−0·01633	0·26627	−0·08695	−0·04030
	0·16802	−0·04359	−0·01415	0·47105	−0·09121	−0·03377
	0·26171	−0·02070	−0·01100	0·64765	0·01359	−0·02506
	0·48998	0·07374	0·00037	0·37427	0·14947	−0·00050
	0·10746	0·00813	0·00000	−0·28059	−0·00354	0·00006
	0·01742	0·00136	−0·00001	−0·10107	−0·00573	0·00011
	0·00147	0·00027	0·00000	−0·01280	−0·00223	0·00002
	0·00011	0·00003		−0·00132	−0·00036	

tive which was written for the MINSK 22 computer. If a different computer or programme is used some of the data below may differ slightly.

One of the most important advantages of the programme described is the large saving of machine time. For the sake of comparison, let it be assumed that the partial derivatives of the dispersion curves are to be computed numerically and that the computation of a single difference would be considered adequate. Let it also be assumed that each computation of the phase velocity would require, e.g., 15 iterations. It then appears tha t for the CANSD model the computation of all 23 partial derivatives of the phase velocity by means of the programme mentioned, using a MINSK 22 computer, would be at least 15 times faster and the computing of all partial derivatives of the group velocity at least 20 times faster then computations made by numerical differentiation. At the same time one may expect that the computations with the help of the said programme will be more accurate. For example, the partial derivatives of the phase and group velocities with respect to b_1 (velocity of transverse waves in the first layer) in Tab. 2 have been computed with an error smaller than 1×10^{-4}, which is a higher accuracy than with numerical differentiation [8]. In order to achieve this accuracy, it would be sufficient to compute the phase velocity with an accuracy of only 1×10^{-5} km/s. The accuracy of the computations, however, decreases considerably if the phase velocity approaches the velocity of the transverse waves in any of the layers. If the phase velocity is equal to the velocity of the transverse waves in any of the layers, some of the expressions become indefinite because the denominators are equal to zero. For example for $T = 14.2$ s the phase velocity is $c = 3.84887$ km/s, which is close to velocity $b_3 = 3.85$ km/s and in computing the expressions

$$\partial y_{32}/\partial c\,, \quad \partial^2 y_{32}/\partial c^2\,, \quad \partial y_{32}/\partial b_3\,, \quad \partial^2 y_{32}/\partial c\,\partial b_3 \tag{25}$$

three valid decimal places are lost. This disadvantage could be removed by using Taylor's expansions for indefinite expresions (this was not carried out in the programme).

In computing dispersion curves, matrices are frequently used, which are created by multiplying all the elements of matrix (13) by cos Q_m. On the basis of these matrices a programme was written for computing the derivatives of dispersion curves in the Algol language for the Elliott 503 computer. The formulas in this programme required special modifications for short periods. The author is of the opinion that it is more suitable to use formulas (13) for the considered problem, as they make it possible to obtain more accurate results.

4. SOME APPLICATIONS

The programme, described above, was used to investigate the partial derivatives of dispersion curves for the fundamental mode of Love waves on a model of the Canadian shield CANSD (Tab. 1 and 2). The effect of the Earth's curvature on the dispersion curves was not considered. Figures 1 to 3 show the partial derivatives of the

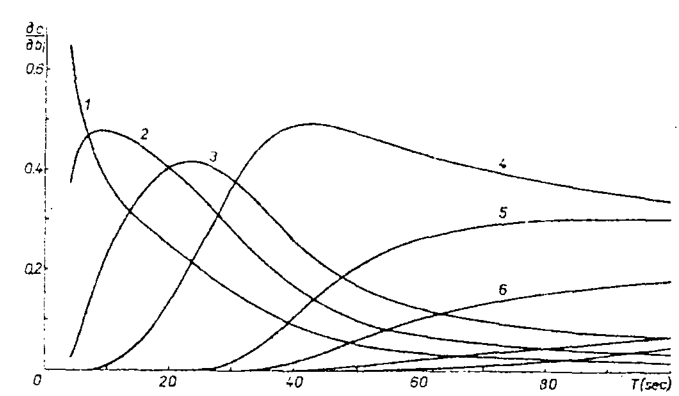


Fig. 1. Partial derivatives of the phase velocity of Love waves with respect to the velocities of transverse waves for the CANSD model. The numbers with the curves represent the numbers of the layers in Tab. 1.

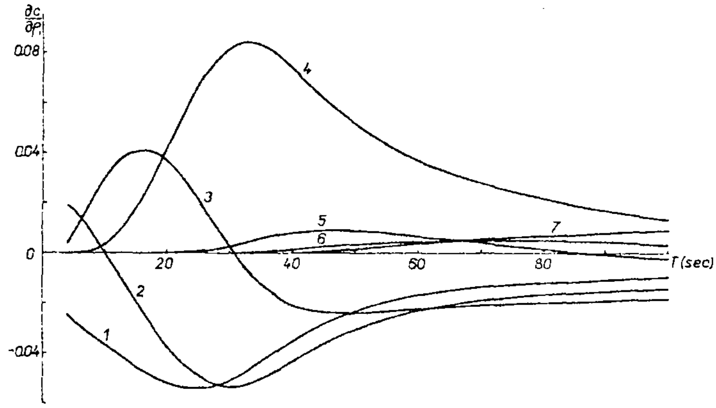


Fig. 2. Partial derivatives of the phase velocity with respect to the densities.

phase velocity with respect to the transverse-wave velocities, the densities and the thicknesses of the individual layers and of the half-space. These curves agree with the curves in [2]. The appropriate partial derivatives of the group velocity with respect

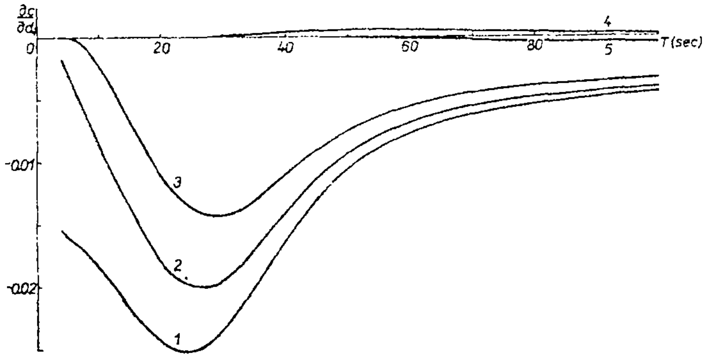


Fig. 3. Partial derivatives of the phase velocity with respect to the thicknesses of the layers.

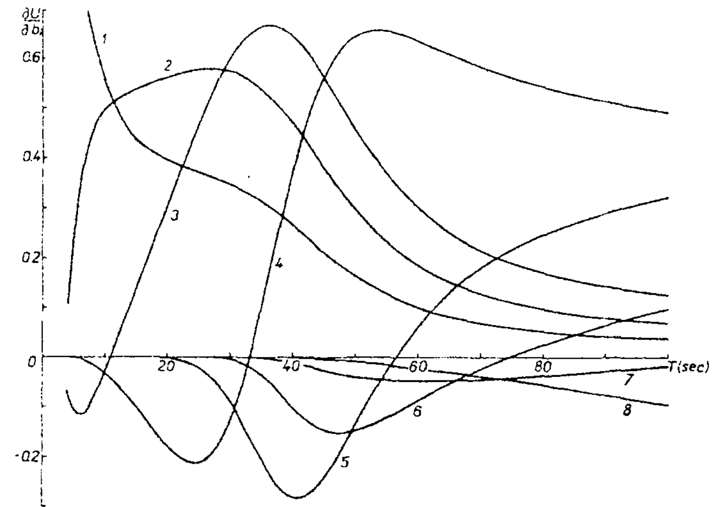


Fig. 4. Partial derivatives of the group velocity with respect to the velocities of transverse waves.

to the parameters of the medium are shown in Figs. 4 to 6. The numbers with the curves in Figs. 1 to 6 represent the numbers of the layers going down, and number 8 represents the half-space. The velocities are given in km/s, the densities in g/cm^3, and the thicknesses and depths in km. The partial derivatives, which have not been included in some of the figures, were too small in absolute value.

The analysis of the figures and their mutual comparison yields a number of conclusions concerning the properties of the partial derivatives of the dispersion curves.

The partial derivatives of the phase velocity with respect to the velocities of the transverse waves (Fig. 1) are only positive, with the exception of curves 1 and 8 they have a similar shape, and they display a single maximum. The different character of curves 1 and 8 also in the other figures is a result of the exceptional location of the first layer (it forms the boundary with the surface of the medium) and of the half-space.

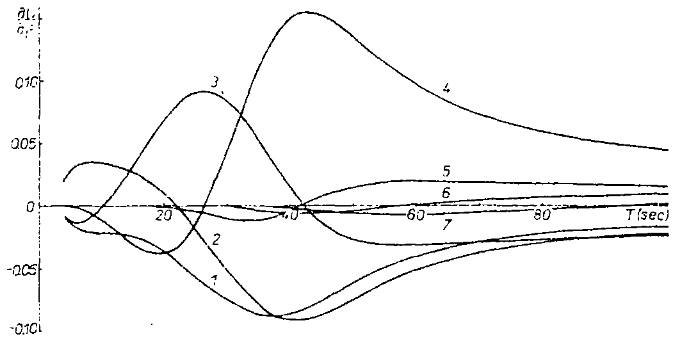


Fig. 5. Partial derivatives of the group velocity with respect to the densities.

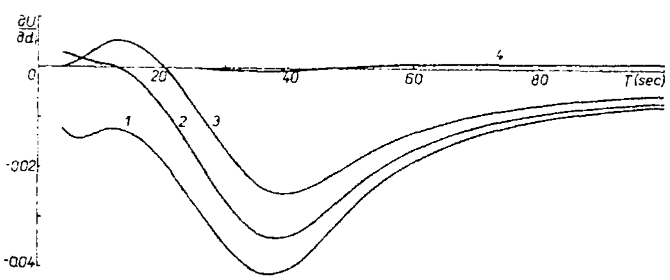


Fig. 6. Partial derivatives of the group velocity with respect to the thicknesses of the layers.

The partial derivatives of the phase velocity with respect to the densities (Fig. 2) have a more complicated character, the explanation of which is given in [2]. The partial derivatives of the group velocity with respect to the velocities of the transverse waves (Fig. 4), with the exception of curves 1 and 8, are negative for short periods and positive for long periods. In the negative sections of these curves there is an interesting phenomenon, i.e. the increase in the velocity of the transverse waves causes a decrease in the group velocity. The partial derivatives of the group velocity with respect to the densities (Fig. 5) have a very complicated character. It should be pointed out that the graphs in Figs. 3 and 6 represent partial derivatives with respect to the thicknesses of the layers, and not partial derivatives with respect to the depths of the interfaces. The comparison of Figs. 1 to 3 and 4 to 6 indicates that the partial derivatives of the phase velocity with respect to the parameters of the medium are smaller in absolute value than the corresponding partial derivatives of the group velocity. This means that the group velocities are more sensitive to changes of the parameters of the medium than the phase velocities. The regions of the extremes on curves in Figs. 4 to 6 are connected with the points of inflection on curves in Figs. 1 to 3. The partial derivative of the group velocity has its smallest (or largest) values for periods in the neighbourhood of the point of inflection on the ascending (or descending) part of the curve of the partial derivative of the phase velocity, respectively. For the interpretation of the dispersion data it is very important that the partial derivatives of the phase and group velocity with respect to the velocities of the transverse waves (Figs. 1 and 4) are considerably larger than the partial derivatives with respect to the densities (Figs. 2 and 5).

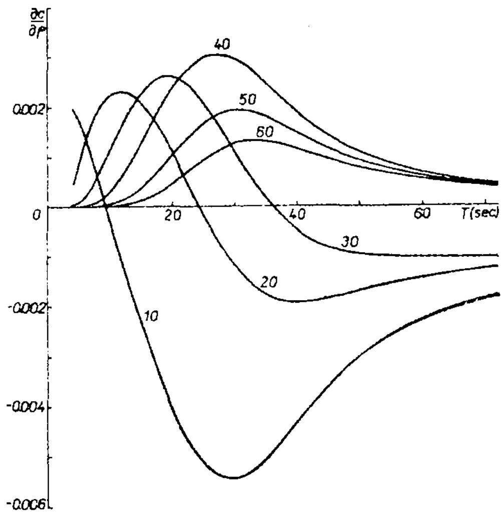


Fig. 8. Partial derivatives of the phase velocity with respect to the density for various depths of the layers.

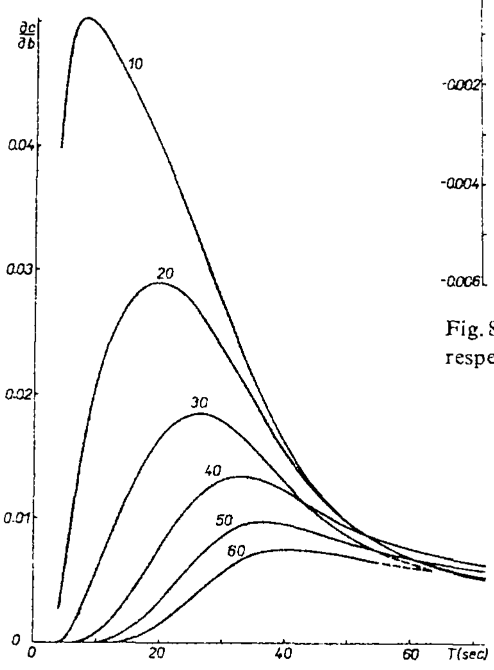
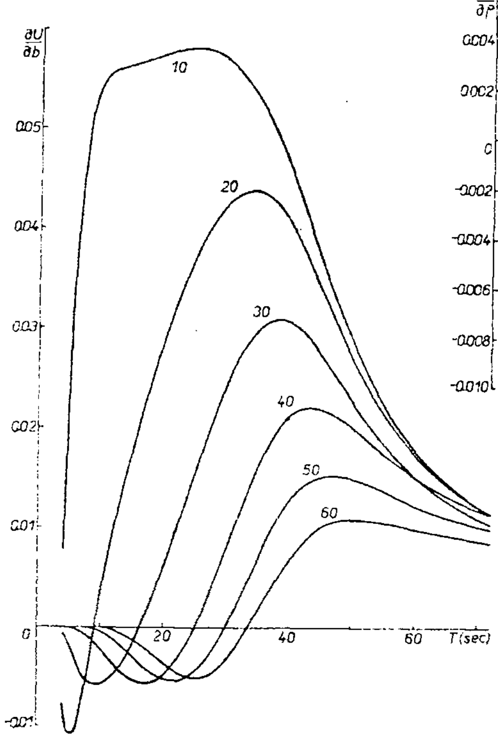


Fig. 7. Partial derivatives of the phase velocity with respect to the velocity of transverse waves for various depths of the layers. The numbers with the curves denote the depth (in km) of the centre of the layer, which is 1 km thick.

In order to be able to evaluate the effect of the depth of a layer on the partial derivatives of the dispersion curves, layers 1 km in thickness were chosen in the CANSD model, the centres of which were at depths of 10, 20, ..., 60 km. The partial derivatives of dispersion curves with respect to the parameters of these layers are shown in Figs. 7 to 10. The numbers with the curves indicate the depth of the centre of the layer. For example, curve 10 in Fig. 7 illustrates the partial derivative of the phase velocity with respect to the velocity of the transverse waves in a layer which is between 9·5 and 10·5 km deep. The properties of the partial derivatives of dispersion curves, described above, can also be observed in these figures. It also follows from Figs. 7 and 9 that the changes in the velocity of the transverse waves in the surface layers have considerable effect on the dispersion curve over a wide range of periods.



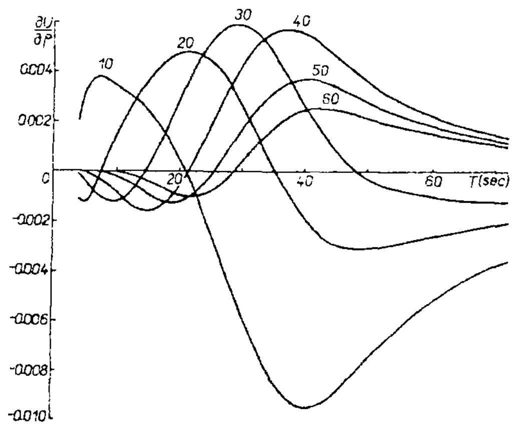


Fig. 10. Partial derivatives of the group velocity with respect to the density for various depths of the layers.

Fig. 9. Partial derivatives of the group velocity with respect to the velocity of the transverse waves for various depths of the layers.

For long periods this effect is roughly the same as the effect of layers of the same thickness at greater depth. This property must be taken into account in solving the inverse problem and in evaluating the accuracy of the velocity section, obtained by interpreting the dispersion data. The magnitude of the maxima in Fig. 7 and the corresponding periods, are given in Tab. 3. In the first column of the latter table the depths of the centres of the 1-km layers, in the second column the periods of the maxima and in the third the maximum values of the partial derivative of the phase velocity with respect to the velocities of the transverse waves are given. Curves 40, 50 and 60 in Fig. 8 and 10 have a slightly different shape than the first three curves, because at a depth of 35·2 km a high-velocity layer begins (Tab. 1). The partial derivatives of the phase velocity with respect to the thickness of the layer has not been plotted, because for the layer at the depth of 10 km they are the same as curve 2 in Fig. 3, for layers at 20 and 30 km the same as curve 3, and for the other three layers the same as curve 4. Similarly, the partial derivatives of the group velocity with respect to the thickness of the layer agree with the corresponding curves in Fig. 6.

If the curves in Fig. 7 are computed for a larger number of various depths, Figs. 11 and 12 can be plotted. In the latter figures the horizontal axis is the depth of the centre of the 1-km layer, the vertical is the period. The curves in Fig. 11 join the points,

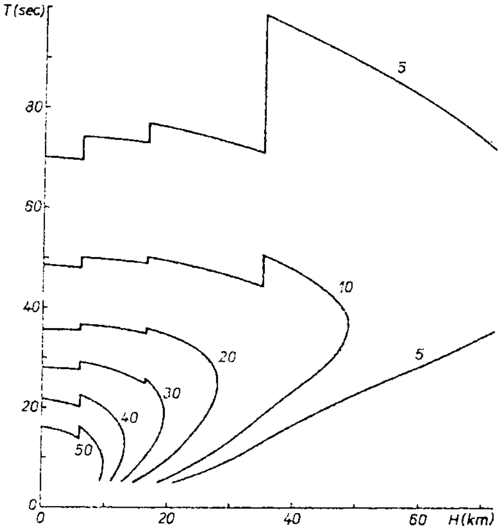


Fig. 11. Isolines of the partial derivative of the phase velocity with respect to the velocity of the transverse waves in a 1-km layer. The numbers with the curves denote the magnitude of the partial derivative multiplied by 1000.

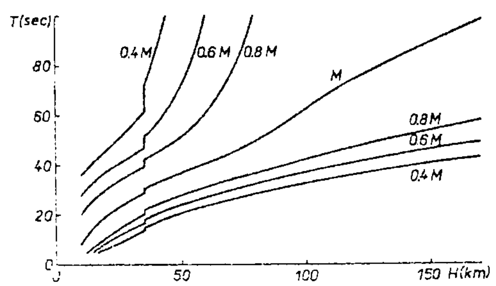


Fig. 12. Periods in which the partial derivative of the phase velocity with respect to the velocity of the transverse waves achieves maximum and multiples of the maximum.

in which the partial derivatives of the phase velocity with respect to the velocity of the transverse waves have the same value. The numbers with the curves give the magnitude of the partial derivative multiplied by 1000. For example on curve 20 the partial derivatives have a value of 0·02. Figure 11 thus illustrates the partial derivative of the phase velocity with respect to the velocity of transverse waves as a function of two variables, the depth of the layer H and the period T. Figure 12 shows the periods, at which the partial derivative of the phase velocity with respect to the velocities of the transverse waves achieves, for a given depth H, maximum (curve M), eight tenths six tenths and four tenths of the maximum value (curves 0·8 M, 0·6 M, 0·4 M). For example, for the layer at a depth of 50 km the said partial derivative achieves its maximum value, equal to 0·00968, at a period of 36·9 s (Fig. 7 and Tab. 3), and values of 0·8 × 0·00968 = 0·00774 at periods of 28 and 52 s.

Finally, an attempt will be made at clarifying the occurrence of negative values on the curves in Fig. 4. Let us transcribe Eq. (7) to

$$U = c : [1 + (T/c)(\partial c/\partial T)]. \tag{26}$$

Table 3. Maximum values of partial derivatives in Fig. 7.

H (km)	T_m (s)	$\partial c/\partial b$
10	7·8	0·05028
20	19·7	0·02898
30	26·2	0·01842
40	33·0	0·01331
50	36·9	0·00968
60	40·8	0·00749

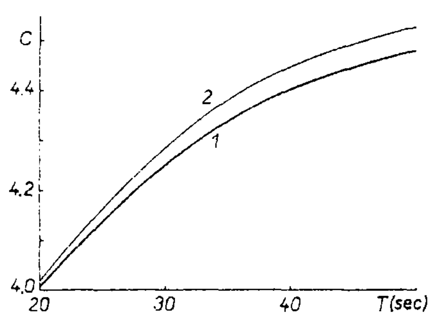


Fig. 13a. Curves of the phase velocity. Curve 1 is for model CANSD, curve 2 for the case when b_4 has been increased by 0.1 km/s.

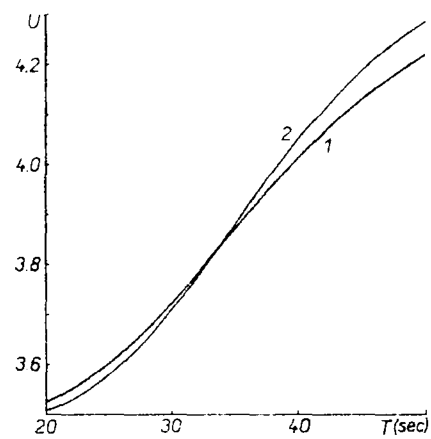


Fig. 13b. Curves of the group velocity.

From Eq. (26) follows the well-known fact that small values of the group velocity correspond to "steep" sections of the phase-velocity curve, i.e. to large partial derivatives of c with respect to T. Curve 1 in Fig. 13a illustrates the phase velocity for model CANSD. If the velocity b_4 in the CANSD model is increased by 0.1 km/s and the other parameters are left unchanged, we obtain curve 2. The appropriate group velocities are in Fig. 13b. In increasing velocity b_4, the phase velocity was increased, but for short periods also the slope of the phase-velocity curve increased. It appears that for short periods this increase in slope in (26) overrides the increase in phase velocity, so that the group velocity is then decreased (Fig. 13b). The above yields one more conclusion. Let us denote by T_j the period in which the j-th curve in Fig. 1 achieves maximum. If the velocity of the transverse waves is slightly increased in the j-th layer, the phase velocity will increase for period T_j, but the slope of the curve will remain unchanged. According to (26) the group velocity for period T_j is increased, i.e. the j-th curve in Fig. 4 must achieve positive values for period T_j. The period in which the j-th curve in Fig. 4 achieves zero value must always, therefore, be smaller than T_j.

5. CONCLUSION

In this paper formulas have been derived, which make it possible to compute the group velocity and the partial derivatives of the phase and group velocities for Love waves in a layered medium without numerical differentiation. The computations on the basis of these formulas are multiply quicker than numerical differentiation, as well as more accurate, especially in computing the partial derivatives of the group velocity. The partial derivatives of dispersion curves of Love waves for the model of the Canadian shield CANSD have been studied in greater detail, and this has also yielded certain more general properties of the partial derivatives of dispersion curves.

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Zusammenfassung

DIE PARTIELLE ABLEITUNGEN DER DISPERSIONSKURVEN DER LOVE-WELLEN IM GESCHICHTETEN MEDIUM

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Es werden Formeln abgeleitet, die die Gruppengeschwindigkeit und partielle Ableitungen der Phasen- und Gruppengeschwindigkeit der Love-Wellen im geschichteten Medium zu berechnen ermöglichen, und zwar ohne Verwendung der numerischen Ableitungen. Die Berechnungen mit Hilfe dieser Formeln sind vielfach schneller als bei Verwendung der numerischen Methoden; sie sind sogar genauer, besonders bei der Berechnung der partiellen Ableitungen der Gruppengeschwindigkeit. Ausführlich werden die partiellen Ableitungen der Dispersionskurven der Love-Wellen für das Model des Kanadischen-Schilds studiert.

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