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Existence of periodic solutions of Hamiltonian systems on almost every energy surface

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Abstract. We estab lish the existence of periodic solutions of Hamiltonian systems on almost every smooth, compact energy surface in the sense of Lebesgue measure.

1. Introduction

In a celebrated paper Viterbo [6] showed that for any C^2-Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ and any regular value β_0 of H, if $S = H^{-1}(\beta_0)$ is compact, connected, and of contact type in the sense of Weinstein [7], there is a periodic solution x of the Hamiltonian system

$$\dot{x} = J\nabla H(x) \tag{1.1}$$

on the energy surface S. Here, $J = \begin{pmatrix} 0 & \text{-id} \\ \text{id} & 0 \end{pmatrix}$ is the standard symplectic matrix on $\mathbb{R}^{2n} = \mathbb{R}^n \times \mathbb{R}^n$. More generally, Hofer and Zehnder [3] were able to show that for any Hamiltonian $H \in C^2(\mathbb{R}^{2n}; \mathbb{R})$ and any regular value β_0 such that $S = H^{-1}(\beta_0)$ is compact and connected but not necessarily of contact type, then in any neighbourhood of S there is a periodic solution x of (1.1). Subsequently, Rabinowitz [4] observed that their argument can be adjusted to yield in fact uncountably many periodic solutions of (1.1) in any neighborhood of such S.

Here, by a slight modification of the proof used by Hofer and Zehnder we will even establish the existence of periodic solutions of (1.1) on any energy surface $H^{-1}(\beta)$ for almost every β sufficiently near β_0 in the sense of Lebesgue measure.

Theorem 1.1. *Suppose β_0 is a regular value of $H \in C^2(\mathbb{R}^{2n})$ and $H^{-1}(\beta_0)$ is*

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compact and connected. Then for almost every β near β_0 there is a periodic solution of (1.1) *on the energy surface $H^{-1}(\beta)$.*

However, although we also obtain uniform bounds on the action of solutions, we cannot obtain existence of periodic solutions on a *given* energy surface, in general; see [1] together with [3] for some recent results in this regard.

2. We will basically use the variational approach of Hofer and Zehnder [3]. However, the presentation in Struwe [5] is sometimes suited better for our purpose. Note that periodic solutions of (1.1) uniquely correspond to critical points of the energy functional

$$E(x) = \frac{1}{2}\int_0^1 < \dot{x}, Jx > dt - \int_0^1 H(x(t))dt,$$

on the space of 1-periodic loops $x \in C^1\left([0,1];\mathbb{R}^{2n}\right)$. Suppose $\beta_0 = 1$ is regular and $X = H^{-1}(1)$ is compact and connected. Then there exists $\delta > 0$ such that all numbers $\beta \in]1-2\delta, 1+2\delta[$ are regular values of H and $S_\beta = H^{-1}(\beta)$ is a compact and connected energy hypersurface diffeomorphic to $S = S_1$ for all such β.

For $m \in \mathbb{N}, |\alpha| < \delta$ let $U_{m,\alpha} = \bigcup_{|\beta-(1+\alpha)|<\frac{\delta}{m}} S_\beta$, and let $A_{m,\alpha}$ be the unbounded, $B_{m,\alpha}$ be the bounded component of the complement of $U_{m,\alpha}$. We may assume that 0 lies in the interior of $\bigcap_{m,\alpha} B_{m,\alpha}$. Also let $\gamma = \sup_{m,\alpha}(\text{diam } U_{m,\alpha})$. Choose smooth functions f, g and constants b, r as in Hofer-Zehnder [3], satisfying

$$\gamma < r < 2\gamma, \ \frac{3}{2}\pi r^2 < b < 2\pi r^2,$$

$$f(s) = 0 \text{ for } s \le -\delta, f(s) = b \text{ for } s \ge \delta, f'(s) > 0 \text{ for } |s| < \delta,$$

$$g(s) = b \text{ for } s \le r, \ g(s) \ge \frac{3}{2}\pi s^2 \text{ for } s > r, \ g(s) = \frac{3}{2}\pi s^2 \text{ for large } s,$$

$$0 < g'(s) \le 3\pi s \text{ for } s > r.$$

We modify H as follows: For $m \in \mathbb{N}, |\alpha| < \delta$ let

$$H_{m,\alpha}(x) = \begin{cases} 0, & \text{if } x \in B_{m,\alpha} \\ f(ms), & \text{if } H(x) = 1 + \alpha + s \text{ for some } |s| \le \frac{\delta}{m} \\ b, & \text{if } x \in A_{m,\alpha}, |x| \le r \\ g(|x|), & \text{else} \end{cases}$$

and denote

$$E_{m,\alpha}(x) = \frac{1}{2}\int_0^1 < \dot{x}, Jx > dt - \int_0^1 H_{m,\alpha}(x)dt.$$

As our space of admissible loops we choose the Hilbert space

$$V = H^{1/2,2}\left(S^1; \mathbb{R}^{2n}\right) =$$

$$= \left\{ x = \sum_{k\in\mathbb{Z}} x_k \exp(2\pi Jkt); x_k \in \mathbb{R}^{2n}, ||x||^2 = 2\pi \sum_k |k|\,|x_k|^2 + |x_0|^2 < \infty \right\}$$

with scalar product

$$(x, y) =< x_0, y_0 > +2\pi \sum_{k\in\mathbb{Z}} |k| < x_k, y_k >$$

inducing the norm $||\cdot||$ in V. Note that $E_{m,\alpha}$ is well-defined of class C^2 on this space. Moreover, the operator $L : x \mapsto -J\dot{x}$ gives rise to a decomposition of V into eigenspaces

$$V_k = \left\{ x_k \exp(2\pi Jkt); x_k \in \mathbb{R}^{2n} \right\}$$

corresponding to eigenvalues $2\pi k$, $k \in \mathbb{Z}$. In particular, L induces a splitting $V = V^- \oplus V^0 \oplus V^+$, with orthogonal projections P^-, P^0, P^+, respectively, where $V^- = \bigoplus_{k<0} V_k, V^0 = V_0, V^+ = \bigoplus_{k>0} V_k$ denote the orthogonal subspaces of V on which the action

$$A(x) = \frac{1}{2}\int_0^1 < \dot{x}, Jx > dt$$

is negative, null, or positive.

Finally, if we denote $\nabla E_{m,\alpha}(x) \in V$ the gradient of $E_{m,\alpha}$, we have

$$\nabla E_{m,\alpha}(x) = -x^- + x^+ - \nabla G_{m,\alpha}(x)$$

where $x^\pm = P^\pm x \in V^\pm$ and $G_{m,\alpha}(x) = \int_0^1 H_{m,\alpha}(x(t))dt$. Note that $\nabla G_{m,\alpha}$ is represented by

$$(\varphi, \nabla G_{m,\alpha}(x)) = \int_0^1 \nabla H_{m,\alpha}(x)\varphi dt$$

and hence by Rellich's theorem is compact. Moreover, $E_{m,\alpha}$ satisfies the Palais-Smale condition:

Any sequence $(x_k) \subset V$ *such that* $|E_{m,\alpha}(x_k)| \leq C$ *and* $\nabla E_{m,\alpha}(x_k) \to 0$ *as* $k \to \infty$ *has a strongly convergent subsequence.* (P.-S.)

Similar to Hofer-Zehnder, [3]; Lemma 1, we have:

Lemma 2.1. *Let* $x \in V$ *be a critical point of* $E_{m,\alpha}$ *with* $E_{m,\alpha}(x) > 0$, *then*

$$T(x) = mf'\Big(m(H(x) - 1 - \alpha)\Big) > 0,$$

and the function $y(t) = x\left(\frac{t}{T(x)}\right)$ *is a* $T(x)$*-periodic solution of* (1.1).

Proof. If x is constant, $E_{m,\alpha}(x) = -H_{m,\alpha}(x) \leq 0$. If x is non-constant, and $|x| \geq r$ somewhere, then from

$$\dot{x} = J\nabla H_{m,\alpha}(x) = J\frac{g'(|x|)}{|x|}x$$

we obtain that $|x(t)| = \text{const.} = s_0$, whence

$$E_{m,\alpha}(x) = \frac{1}{2}g'(s_0)s_0 - H_{m,\alpha}(x) \leq \frac{3}{2}\pi s_0^2 - \frac{3}{2}\pi s_0^2 = 0.$$

It remains the case that $H(x) = 1 + \alpha + s$ for some $|s| \leq \frac{\delta}{m}$, whence

$$\dot{x} = J\nabla H_{m,\alpha}(x) = mf'\Big(m(H(x) - 1 - \alpha)\Big)J\nabla H(x),$$

as desired. □

Moreover, a trivial computation yields:

Lemma 2.2. *For fixed* $x \in V$, $m \in \mathbb{N}$ *the functional* $E_{m,\alpha}(x)$ *is monotone non-decreasing in* $\alpha \in]-\delta, \delta[$, *and there holds*

$$\frac{\partial}{\partial\alpha}E_{m,\alpha}(x) = m\int_0^1 f'\Big(m\big(H(x(t)) - 1 - \alpha\big)\Big)dt.$$

In particular, for a critical point x of $E_{m,\alpha}$ with $E_{m,\alpha}(x) \geq 0$ corresponding to a $T(x)$-periodic solution of (1.1), by Lemma 2.1 we have

$$T(x) = \frac{\partial}{\partial \alpha} E_{m,\alpha}(x).$$

The remainder of the paper will be concerned with obtaining some uniform (in m) control of the α-derivative of suitable critical points of $E_{m,\alpha}$ for almost every $|\alpha| < \delta$.

To obtain suitable critical points $x_{m,\alpha}$, for fixed $m \in \mathbb{N}$, $|\alpha| < \delta$ we set up a mini-max scheme as in Hofer-Zehnder [3].

Lemma 2.3. *There exist numbers $\mu_0 > 0, \rho > 0$ such that $E_{m,\alpha}(x) \geq \mu_0$ for all $x \in V^+, ||x|| = \rho$, uniformly in $m \in \mathbb{N}, |\alpha| < \delta$.*

Proof. Let $r_0 = \sup\{R > 0; B_R(0) \subset B_{m,\alpha} \text{ for all } m, \alpha\} > 0$. Let C_0 be a constant independent of m and α such that

$$H_{m,\alpha}(x) \leq C_0 |x|^2 \text{ for all } x.$$

Note that $V \hookrightarrow L^p([0,1]; \mathbb{R}^{2n})$ continuously for any $p < \infty$. In particular, there exists $C > 0$ such that

$$\text{meas}\{t; |x(t)| > r_0\} \leq \frac{1}{r_0^2} \int_0^1 |x(t)|^2 dt \leq C||x||^2.$$

Hence by Hölder's inequality, for $x \in V$ we have

$$\int_0^1 H_{m,\alpha}(x(t))dt \leq C_0 \int_{\{t; |x(t)| \geq r_0\}} |x(t)|^2 dt$$

$$\leq C_0 \Big(\text{meas}\{t; |x(t)| \geq r_0\}\Big)^{\frac{1}{2}} \left(\int_0^1 |x(t)|^4 dt\right)^{\frac{1}{2}} \leq C||x||^3.$$

Since on the other hand, for $x \in V^+$ we have

$$A(x) = ||x||^2,$$

the claim follows. □

Fix $e = \frac{1}{\sqrt{2\pi}} \exp(2\pi Jt)a \in V^+$, $|a| = 1$, and define

$$Q = \{x = x^- + x^0 + se \in V^- \oplus V^0 \oplus \mathbb{R} \cdot e \subset V; ||x^- + x^0|| \leq R, 0 \leq s \leq R\}$$

with $R > 0$ to be determined. Denote ∂Q the relative boundary of Q,

$$\partial Q = \{x = x^- + x^0 + se; ||x^- + x^0|| = R, \ \text{or } s \in \{0, R\}\}.$$

Lemma 2.4. *If R is sufficiently large independent of $m \in \mathbb{N}, |\alpha| < \delta$, then $E_{m,\alpha}|_{\partial Q} \leq 0$.*

Proof. See Hofer-Zehnder [3], Lemma 3. □

Fix $0 < \rho < R$ according to Lemmas 2.3, 2.4. Define a class Γ of maps $V \to V$ as follows:

$h \in C^0(V; V)$ belongs to Γ if h is homotopic to the identity through a family of maps $h_t = L_t + K_t$, $0 \leq t \leq T$, where $L_0 = \text{id}, K_0 = 0$ and where for each $t \in [0, T]$ the map K_t is compact while

$$L_t = (L_t^-, L_t^0, L_t^+) : V^- \oplus V^0 \oplus V^+ \to V^- \oplus V^0 \oplus V^+$$

is a linear isomorphism preserving the sub-spaces V^-, V^0, and V^+.

Finally, for $m \in \mathbb{N}, |\alpha| < \delta$ define $\Gamma_{m,\alpha}$ to be the class of maps $h \in \Gamma$ such that

$$E_{m,\alpha}|_{h_t(\partial Q)} \leq 0 \quad \text{for all } t \in [0, T].$$

Note that by Lemma 2.2 we have $\Gamma_{m,\alpha'} \subset \Gamma_{m,\alpha}$ if $\alpha \leq \alpha'$.

Following Benci-Rabinowitz [2], as an application of Leray-Schauder degree theory one can show that ∂Q and $S_\rho^+ = \{x \in V^+; ||x|| = \rho\}$ link with respect to $\Gamma_{m,\alpha}$; that is, we have:

Lemma 2.5. *For any $m \in \mathbb{N}, |\alpha| < \delta$, $h \in \Gamma_{m,\alpha}$ there holds $h(Q) \cap S_\rho^+ \neq \emptyset$.*

Proof. See Benci-Rabinowitz [2], or Struwe [5], Lemma II. 8.12. □

For $m \in \mathbb{N}, |\alpha| < \delta$ define

$$\mu_m(\alpha) = \inf_{h \in \Gamma_{m,\alpha}} \sup_{x \in Q} E_{m,\alpha}(h(x)).$$

Note that by Lemmas 2.3, 2.5 we have $\mu_m(\alpha) \geq \mu_0 > 0$. Moreover, choosing $h = \mathrm{id}$ as comparison function we also have $\mu_m(\alpha) \leq \mu_\infty < \infty$ for some real number μ_∞ independent of m and α.

By Lemma 2.2 and since $\Gamma_{m,\alpha}$ is non-increasing, clearly we have

Proposition 2.6. *For fixed $m \in \mathbb{N}$ the map $\alpha \longmapsto \mu_m(\alpha)$ is monotone non-decreasing.*

Hence for any $m \in \mathbb{N}$ the function μ_m is almost everywhere differentiable with $0 \leq \frac{\partial}{\partial\alpha}\mu_m \in L^1([-\delta,\delta])$, and there holds

$$\int_{-\delta}^{\delta} \frac{\partial}{\partial\alpha}\mu_m d\alpha \leq \mu_\infty - \mu_0 < \infty,$$

independent of m. But then also $\liminf_{m\to\infty}\left(\frac{\partial}{\partial\alpha}\mu_m\right) \in L^1([-\delta,\delta])$, and by Fatou's lemma

$$\int_{-\delta}^{\delta} \liminf_{m\to\infty} \frac{\partial}{\partial\alpha}\mu_m d_\alpha \leq \liminf_{m\to\infty} \int_{-\delta}^{\delta} \frac{\partial}{\partial\alpha}\mu_m d\alpha \leq \mu_\infty - \mu_0.$$

In particular, $\liminf_{m\to\infty}\left(\frac{\partial}{\partial\alpha}\mu_m(\alpha_0)\right) < \infty$ for almost every $\alpha_0 \in]-\delta,\delta[$. Fix such an α_0 and let $\Lambda \subset \mathbb{N}$ be a sequence such that

$$\frac{\partial}{\partial\alpha}\mu_m(\alpha_0) \to \liminf_{m\to\infty}\left(\frac{\partial}{\partial\alpha}\mu_m(\alpha_0)\right) =: C_0$$

as $m \to \infty, m \in \Lambda$.

Lemma 2.7. *For any $m \in \Lambda$ there exists a critical point x_m of E_{m,α_0} such that $E_{m,\alpha_0}(x_m) = \mu_m(\alpha_0)$ and $T(x_m) = \frac{\partial}{\partial\alpha}E_{m,\alpha}(x_m)_{|\alpha=\alpha_0} \leq C_0 + 4$.*

Proof. (We omit the index m for brevity.) Choose a sequence $a_k \searrow \alpha_0$. We claim there exists a sequence (x^k) such that $\nabla E_{\alpha_0}(x^k) \to 0$ and

$$\begin{aligned}\mu(\alpha_0) - 2(\alpha_k - \alpha_0) &\leq E_{\alpha_0}(x^k) \\ &\leq E_{\alpha_k}(x^k) \\ &\leq \mu(\alpha_k) + (\alpha_k - \alpha_0) \\ &\leq \mu(\alpha_0) + (C_0 + 2)(\alpha_k - \alpha_0)\end{aligned} \tag{2.1}$$

for large k. This will imply the assertion of the lemma: By (P.-S.), (x^k) will accumulate at a critical point x of E_{α_0}, satisfying $E_{\alpha_0}(x) = \mu(\alpha_0)$ and

$$
\begin{aligned}
C_0 + 4 &\geq \liminf_{k\to\infty} \frac{E_{\alpha_k}(x^k) - E_{\alpha_0}(x^k)}{\alpha_k - \alpha_0} \\
&= \liminf_{k\to\infty} \frac{\int_0^1 \int_{\alpha_0}^{\alpha_k} m\, f'\Big(m\big(H(x^k(t)) - 1 - \alpha\big)\Big) d\alpha dt}{\alpha_k - \alpha_0} = \\
&= \int_0^1 m f'\Big(m\big(H(x(t)) - 1 - \alpha\big)\Big) dt = T(x).
\end{aligned}
$$

Negating the above claim, there exists a number $\varepsilon > 0$ such that for all $x \in V$ satisfying (2.1) there holds

$$||\nabla E_{\alpha_0}(x)||^2 \geq \varepsilon. \tag{2.2}$$

Let $0 \leq \varphi \leq 1$ be a Lipschitz continuous function such that $\varphi(s) = 0$ for $s \leq 0, \varphi(s) = 1$ for $s \geq 1$ and define a family of vector fields by letting

$$
\begin{aligned}
e_k(x) &= -\Big((1 - \varphi_k(x))\nabla E_{\alpha_k}(x) + \varphi_k(x)\nabla E_{\alpha_0}(x)\Big) \\
&= x^- - x^+ + \Big((1 - \varphi_k(x))\nabla G_{\alpha_k}(x) + \varphi_k(x)\nabla G_{\alpha_0}(x)\Big),
\end{aligned} \tag{2.3}
$$

where

$$\varphi_k(x) = \varphi\left(\frac{E_{\alpha_0}(x) - (\mu(\alpha_0) - 2(\alpha_k - \alpha_0))}{\alpha_k - \alpha_0}\right).$$

Let $\Phi_k : V \times [0, \infty[\to V$ be the corresponding flows

$$
\begin{aligned}
\frac{\partial}{\partial t}\Phi_k(x,t) &= e_k(\Phi_k(x,t)), \\
\Phi_k(x,0) &= x.
\end{aligned}
$$

By (2.3), as in Hofer-Zehnder [3] or Struwe [5], Lemma 8.13, $\Phi_k(\cdot, t) \in \Gamma$ for all $t \geq 0$. We may assume that $\mu(\alpha_0) \geq 2(\alpha_k - \alpha_0)$ for all k, whence if $E_{\alpha_k}(x) \leq 0$, then $e_k(x) = -\nabla E_{\alpha_k}(x)$ and hence $E_{\alpha_k}(\Phi_k(x,t)) \leq 0$ for all $t \geq 0$. That is,

$$\Phi_k(\cdot, t) \circ h \in \Gamma_{\alpha_k} \text{ for all } h \in \Gamma_{\alpha_k}, \text{ all } t \geq 0.$$

For any k, any $x \in V$, moreover,

$$\Big(e_k(x), \nabla E_{\alpha_k}(x)\Big) \leq -\varphi_k(x)\Big(\nabla E_{\alpha_0}(x), \nabla E_{\alpha_k}(x)\Big).$$

But for $x \in V$ satisfying (2.1), by assumption (2.2) we have

$$\begin{aligned}\left(\nabla E_{\alpha_0}(x), \nabla E_{\alpha_k}(x)\right) &= \|\nabla E_{\alpha_0}(x)\|^2 - \left(\nabla E_{\alpha_0}(x), \nabla G_{\alpha_k}(x) - \nabla G_{\alpha_0}(x)\right) \\ &\geq \frac{1}{2}\|\nabla E_{\alpha_0}(x)\|^2 - \frac{1}{2}\|\nabla G_{\alpha_k}(x) - \nabla G_{\alpha_0}(x)\|^2 \\ &\geq \frac{\varepsilon}{2} - C \int_{\{t;|x(t)|\leq r\}} \left|\nabla H_{\alpha_k}(x) - \nabla H_{\alpha_0}(x)\right|^2 dt \\ &\geq \frac{\varepsilon}{2} - o(1)\end{aligned} \tag{2.4}$$

with error $o(1) \to 0$ as $k \to \infty$. Hence for large k the map $t \to E_{\alpha_k}(\Phi_k(x,t))$ is non-increasing for any $x \in V$ satisfying $E_{\alpha_k}(x) \leq \mu(\alpha_k) + (\alpha_k - \alpha_0)$.

Choose $h_0 \in \Gamma_{\alpha_k}$ such that

$$\sup_{x \in Q} E_{\alpha_k}(h_0(x)) \leq \mu(\alpha_k) + (\alpha_k - \alpha_0). \tag{2.5}$$

Then letting $h_t = \Phi_k(\cdot, t) \circ h_0 \in \Gamma_{\alpha_k}$ for all $t \geq 0$, (2.5) also holds for h_t.

But $\Gamma_{\alpha_k} \subset \Gamma_{\alpha_0}$ and hence by definition of $\mu(\alpha_0)$ we have

$$M(t) = \sup_{x \in Q} E_{\alpha_0}(h_t(x)) \geq \mu(\alpha_0) \tag{2.6}$$

for all t. Combining (2.5) and (2.6) we see that $M(t)$ is achieved only at points $h_t(x) \in V$ satisfying (2.1), whence by (2.2) and definition of e_k for sufficiently large k we have

$$\frac{d}{dt} M(t) \leq -\varepsilon$$

uniformly in $t \geq 0$, contradicting (2.6). This proves the lemma. □

Proof of Theorem 1.1. For any $\alpha_0 \in]-\delta, \delta[$ with

$$\liminf_{m \to \infty} \frac{\partial}{\partial \alpha} \mu_m(\alpha_0) < \infty$$

let Λ and $(x_m)_{m \in \Lambda}$ be as in Lemma 2.7. For any m the function x_m is a 1-periodic solution of

$$\dot{x}_m = T(x_m) \nabla H(x_m)$$

satisfying

$$|1 + \alpha_0 - H(x_m)| \leq \frac{\delta}{m}, \quad A(x_m) \geq E_{m,\alpha_0}(x_m) \geq \mu_0;$$

see Lemma 2.1. Since $0 \leq T(x_m) \leq C$ uniformly, by the theorem of Arzéla-Ascoli we may assume that (x_m) converges C^1-uniformly to a 1-periodic solution x of

$$\dot{x} = T\nabla H(x)$$

for some $T \in \mathbb{R}$ with $H(x) = 1 + \alpha_0$. Since $A(x) \geq \mu_0 > 0$, in particular $x \not\equiv$ const. and $T \neq 0$. The proof is complete. □

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