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The Hausdorff dimension of horseshoes of diffeomorphisms of surfaces

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Abstract. Let f be a C^r diffeomorphism, $r \geq 2$, of a two dimensional manifold M^2, and let Λ be a horseshoe of f (i.e. a transitive and isolated hyperbolic set with topological dimension zero). We prove that there exist a C^r neighborhood $\mathcal{U}$ of f and a neighbourhood U of Λ such that for $g \in \mathcal{U}$, the Hausdorff dimension of $\bigcap_n g^n(U)$ is a C^{r-1} function of g.

1. Introduction

Let M be a closed manifold and let $\mathrm{Diff}^r(M)$ be the space of C^r diffeomorphisms of M endowed with the C^r topology. We say that $\Lambda \subset M$ is a *basic set* of $f \in \mathrm{Diff}^r(M)$ if it is hyperbolic, isolated (i.e. there exists a compact neighborhood U of Λ such that $\Lambda = \bigcap_n f^n(U)$) and f/Λ is transitive. If moreover Λ is totally disconnected (i.e. the connected component of every $p \in \Lambda$ is $\{p\}$) we say that Λ is a *horseshoe*.

The objective of this paper is to prove the following result:

Theorem A. *Let Λ be a horseshoe of $f \in \mathrm{Diff}^r(M)$, $\dim M = 2$, $r \geq 2$, and let U be a compact neighborhood of Λ such that $\bigcap_n f^n(U) = \Lambda$. Then there exists a C^r neighborhood $\mathcal{U}$ of f such that the Hausdorff dimension of $\bigcap_n g^n(U)$ is a C^{r-1} function of $g \in \mathcal{U}$.*

When $r = 1$ this result was proved by Manning and McCluskey ([4]). A different proof was given by Palis and Viana ([5]). Actually Manning and McCluskey work with the dimension of $\Lambda \cap W^s(p)$, where $W^s(p)$ is the stable manifold of a periodic point. Our proof of Theorem A does not cover the C^1 case.

Our proof relies in a method, introduced by Bowen in [2], that makes possible

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to read, through thermodynamic formalism, the Hausdorff dimension of hyperbolic conformal invariant sets. Since $\dim M = 2$, our horseshoe is roughly speaking, a Cartesian product of two such objects. However, in our case, a technical obstruction appears in the application of Bowen's method forcing us to deviate it along a cumbersome roundabout.

In order to explain this technical obstruction and the result through which we shall circumvent it, we shall first recall some definitions. if X and Y are metric spaces we say that a function $f: X \to Y$ is Hölder γ-continuous, $0 < \gamma \leq 1$, if

$$\sup_{x \neq y} \frac{d(f(x), f(y))}{d(x,y)^\gamma} < \infty.$$

Denote $C^\gamma(X,Y)$ the set of Hölder γ-continuous maps from X into Y. When Y is a normed space and X is compact we shall consider $C^\gamma(X,Y)$ endowed with the norm $\| \cdot \|_\gamma$ given by

$$\|\varphi\|_\gamma = \sup_x \|\varphi(x)\| + \sup_{x \neq y} \frac{\|\varphi(x) - \varphi(y)\|}{d(x,y)^\gamma}.$$

When X is compact and Y is an n-dimensional manifold, $C^\gamma(X,Y)$ is a Banach manifold modelled on $C^r(X, \mathbb{R}^n)$. Given an $m \times m$ matrix A whose entries a_{ij} are 0 or 1, define $B^+(A)$ as the space of sequences $\theta: \mathbb{Z}^+ \to \{1, \dots, m\}$ such that $a_{\theta(n)\theta(n+1)} = 1$ for all $n \geq 0$. Endow $B^+(A)$ with the metric $d(\alpha, \beta) = \sum_{n \geq 0} 2^{-n} |\alpha(j) - \beta(j)|$. The shift $\sigma: B^+(A) \hookleftarrow$ is defined by $\sigma(\theta)(n) = \theta(n+1)$. Define $B(A)$ as the space of sequences $\theta: \mathbb{Z} \to \{1, \dots, m\}$ that satisfy $a_{\theta(n)\theta(n+1)} = 1$ for all n endowed with the metric

$$d(\alpha, \beta) = \sum 2^{-|n|} |\alpha(n) - \beta(n)|.$$

The shift $\sigma: B(A) \hookleftarrow$ is defined as before. Given $\psi \in C^0(B^+(A), \mathbb{R})$, the Perron-Froebenius operator $\mathcal{L}_\psi: C^0(B^+(A), \mathbb{R}) \hookleftarrow$ is defined by

$$(\mathcal{L}_\psi \varphi)(x) = \sum_{\sigma(y) = x} \varphi(y) \exp \psi(x).$$

Then

$$(\mathcal{L}_\psi^n \varphi)(x) = \sum_{\sigma^n(y) = x} \varphi(y) \exp S_n\psi(x)$$

where

$$S_n \psi = \sum_{j=0}^{n-1} \psi \circ \sigma^j.$$

It is easy to prove that for all $x \in B^+(A)$ the limit

$$\lim_{n \to +\infty} \frac{1}{n} \log(\mathcal{L}_\psi^n 1)(x)$$

exists and is independent of x (see section I). Define $P(\psi)$ as this limit. Clearly $\exp P(\psi)$ is the spectral radius of $\mathcal{L}_\psi$ and it follows from Ruelle's theorem (whose statement will be recalled in the next section) that when $\psi \in C^\gamma(B^+(A), \mathbb{R})$ then $\exp P(\psi)$ is a simple eigenvalue of $\mathcal{L}_\psi : C^\gamma(B^+(A), \mathbb{R}) \hookleftarrow$ and the rest of the spectrum of $\mathcal{L}_\psi : C^\gamma(B^+(A), \mathbb{R}) \hookleftarrow$ is contained in the disk $|z| < \exp P(\psi)$. Moreover it is well known that $P(\psi)$ is the topological pressure of ψ, but we shall not use that concept.

The question of the smoothness of the Hausdorff dimension of $\bigcap_n g^n(U)$ as a function of $g \in \mathcal{U}$ is reduced, through Bowen's method, to the smoothness of the composition of certain function $\mathcal{U} \ni g \to \psi_g \in C^\gamma(B^+(A), \mathbb{R})$ with $P : C^\gamma(B^+(A), \mathbb{R}) \to \mathbb{R}$. The first function is C^{r-2} and the second, as we shall see below, is real analytic. Then the composition turns out to be C^{r-2} that is below what we want. To improve this method we shall show that $\mathcal{U} \ni g \to \psi_g \in C^0(B^+(A), \mathbb{R})$ is C^{r-1}. But now the problem is that P, as a map $P : C^0(B^+(A), \mathbb{R}) \to \mathbb{R}$ is only Lipschitz ([3]). To obtain our result we have to use both properties simultaneously and the following theorem.

Theorem B. *Let N be a Banach manifold and let $\Phi : N \to C^\gamma(B^+(A), \mathbb{R})$, $0 < \gamma \leq 1$, be a C^k function, $k \geq 1$, such that $\Phi : N \to C^0(B^+(A), \mathbb{R})$ is C^{k+1}. Then $P \circ \Phi : N \to \mathbb{R}$ is C^{k+1}.*

To explain the role of this theorem in the proof of Theorem A we shall give a short outline of its proof.

Let M be a two dimensional manifold and Λ a horseshoe of $f \in \mathrm{Diff}^r(M)$, $r \geq 2$. Let U be a neighborhood of Λ such that $\bigcap_n f^n(U) = \Lambda$. Take a neighborhood $\mathcal{U}$ of f such that $\Lambda_g = \bigcap_n g^n(U)$ is a horseshoe of g for all $g \in \mathcal{U}$ and there exists a C^r map $\mathcal{U} \ni g \to h_g \in C^0(\Lambda, M)$ such that h_g is a topological equivalence between $f|\Lambda$ and $g|\Lambda_g$. Define $\delta^s(g)$ and $\delta^u(g)$ as the Hausdorff dimensions of $W_g^s(x) \cap \Lambda_g$ and $W_g^u(x) \cap \Lambda_g$, $x \in \Lambda_g$. These numbers are independent of the point x. There are several ways to prove this, our proof will implicitly contain one. We shall also see that the Hausdorff dimension of Λ_g is $\delta^s(g) + \delta^u(g)$. Therefore we have only to prove that $\delta^s(g)$ and $\delta^u(g)$ are C^{r-1} functions of g. Take a shift $\sigma : B(A) \hookleftarrow$ topologically equivalent to $f|\Lambda$.

Let $h: B(A) \to \Lambda$ be a homeomorphism realizing this equivalence. Given $g \in \mathcal{U}$ define $\psi_g: B(A) \to \mathbb{R}$ by

$$\psi_g(\theta) = -\log \left| g'(h_g h(\theta))/E^u_{h_g h(\theta)} \right|.$$

There exists $0 < \gamma \leq 1$ such that $\psi_g \in C^\gamma(B(A), \mathbb{R})$. Moreover there exists a continuous linear map $T: C^\gamma(B(A), \mathbb{R}) \hookleftarrow$ such that $(T\psi)(\theta)$ is homologous to ψ and independent of the values $\theta(n)$ for $n < 0$. ([1]). This means that $T\psi$ can be regarded as an element of $C^\gamma(B^+(A), \mathbb{R})$. Define a function $B: \mathbb{R} \times \mathcal{U} \to \mathbb{R}$ by $B(\delta, g) = P(\delta T\psi_\delta)$. Essentially following Bowen ([2]) one proves that $\delta^u(g)$ satisfies $B(\delta^u(g), g) = 0$. Now suppose that we were able to prove that the map $\mathcal{U} \ni g \to \psi_g \in C^\gamma(B^+(A), \mathbb{R})$ is, say, $C^k, k \geq 1$. Then, since T is linear and $P: C^\gamma(B^+(A), \mathbb{R}) \to \mathbb{R}$ is real analytic, it would follow that B is C^k.

Moreover, as we shall see, it is easy to prove that for each $g \in \mathcal{U}$ there exists $C(g) > 0$ such that $(\partial B/\partial \delta)(\delta, g) \leq -C(g)$ for all δ. Then for each $g \in \mathcal{U}$ there exists a unique $\delta(g)$ satisfying $B(\delta(g), g) = 0$. Hence $\delta(g) = \delta^u(g)$ and by the implicit function theorem, the function $\delta^u: \mathcal{U} \to \mathbb{R}$ is C^k. Therefore this approach would work if we could prove that, for some $0 < \gamma \leq 1$, the function $\mathcal{U} \ni g \to \psi_g \in C^\gamma(B(A), \mathbb{R})$ is C^{r-1}. However we can only prove that it is C^{r-2}. But we can also prove that $\mathcal{U} \ni g \to \psi_g \in C^o(B(A), \mathbb{R})$ is C^{r-1}. Hence we can apply Theorem B to $N = \mathcal{U}$ and Φ being the map $g \to \psi_g$ and we obtain that B is C^{r-1} and then that $\delta^u: \mathcal{U} \to \mathbb{R}$ is also C^{r-1}.

1. Proof of Theorem B.

Let $\sigma: B^+(A) \hookleftarrow$ be a subshift of finite type. To simplify the notation we shall denote $K = B^+(A)$. Given $\psi \in C^\gamma(K, \mathbb{R}), 0 \leq \gamma \leq 1$, the Perron-Froebenius operator $\mathcal{L}_\psi: C^o(K, \mathbb{R}) \hookleftarrow$ is defined by

$$(\mathcal{L}_\psi \varphi)(x) = \sum_{y \in \sigma^{-1}(x)} \varphi(y) \exp \psi(y).$$

Theorem 1.1. (Ruelle [1]) *If $\psi \in C^\gamma(K, \mathbb{R}), 0 < \gamma \leq 1$, the spectrum of $\mathcal{L}_\psi: C^\gamma(K, \mathbb{R}) \hookleftarrow$ consists in a simple eigenvalue $\lambda(\psi) > 0$ and a set contained in the disk $\{z \in \mathbb{C}/|z| < \lambda(\psi)\}$. Moreover there exist a strictly positive function $h_\psi \in C^\gamma(K, \mathbb{R})$ and a probability ν_ψ on the Borel σ-algebra of K satisfying*

a) $\mathcal{L}_\psi h_\psi = \lambda(\psi) h_\psi$

b) $\int h_\psi \, \mathrm{d}\, \nu_\psi = 1$

c) $\mathcal{L}_\psi^* \nu_\psi = \lambda(\psi) \nu_\psi$

d) *For all* $\varphi \in C^\beta(K, \mathbb{R})$, $0 \leq \beta \leq \gamma$:

$$\lim_{n \to +\infty} \left\| \lambda(\psi)^{-n} \mathcal{L}^n \varphi - h_\psi \int \varphi \, \mathrm{d}\, \nu_\psi \right\|_\beta = 0$$

and for $0 < \beta \leq \gamma$ *the convergence is uniform in the unit* C^β *ball*

e) *There exists* $C_1 > 0$ *such that if for* $\theta \in B^+(A)$ *and* $n > 0$ *we define* $B(\theta, n) = \{\alpha | \alpha(j) = \theta(j)$ *for* $0 \leq j \leq n\}$, *then*

$$C_1^{-1} \lambda(\psi)^{-n} \exp(S_n \psi)(\theta) \leq \nu_\psi(B(\theta, n)) \leq C_1 \lambda(\psi)^{-n} \exp(S_n \psi)(\theta).$$

Corollary 1.2. *If* $\psi \in C^\gamma(K, \mathbb{R}), 0 < \gamma \leq 1$, *then*

$$\lim_{n \to +\infty} \frac{1}{n} \log \sum_{y \in \sigma^{-n}(x)} \exp(S_n \psi)(y) = \log \lambda(\psi)$$

uniformly on $x \in K$.

Proof. From part (d) of 1.1 follows that

$$\lim_{n \to +\infty} \frac{1}{n} \log \sum_{y \in \sigma^{-n}(x)} \exp(S_n \psi)(y) = \lim_{n \to +\infty} \frac{1}{n} \log(\mathcal{L}^n 1)(x) =$$

$$= \log \lambda(\psi) + \lim_{n \to +\infty} \frac{1}{n} \log \lambda(\psi)^{-n} (\mathcal{L}_\psi^n 1)(x) = \log \lambda(\psi)$$

uniformly on $x \in K$.

Corollary 1.3. *If* $\psi \in C^0(K, \mathbb{R})$ *the limit*

$$\lim_{n \to +\infty} \frac{1}{n} \log \sum_{\sigma^n(y) = x} \exp(S_n \psi)(y)$$

exists for all $x \in K$ *and is independent of* x.

Proof. Define $\Phi_n : C^0(K) \hookleftarrow$ by

$$\Phi_n(\psi)(x) = \frac{1}{n} \log \sum_{\sigma^n(y) = x} \exp(S_n \psi)(y).$$

Then

$$(\Phi_n'(\psi)\varphi)(x) = \frac{\sum_{\sigma^n(y)=x} \frac{1}{n}(S_n \varphi)(y) \exp(S_n \psi)(y)}{\sum_{\sigma^n(y)=x} \exp(S_n \psi)(y)}$$

Hence

$$|(\Phi_n'(\psi)\varphi)(x)| \leq \|\varphi\|_0$$

for all $x \in K$. Then

$$\|\Phi_n'(\psi)\|_0 \leq 1$$

for all n. Then the sequence of maps $\Phi_n: C^0(K,\mathbb{R}) \hookleftarrow$ is uniformly Lipschitz and is pointwise convergent (by 1.2), in the dense subset $C^\gamma(K,\mathbb{R}) \subset C^0(K,\mathbb{R})$. Therefore the sequence Φ_n converges uniformly on compact subsets of $C^0(K,\mathbb{R})$ to a continuous function $\Phi: C^0(K,\mathbb{R}) \hookleftarrow$. Since $\Phi(\psi) \in C^0(K,\mathbb{R})$ is (by 1.2) a constant function when $\psi \in C^\gamma(K,\mathbb{R}), 0 < \gamma \leq 1$, then, by the density of $C^\gamma(K,\mathbb{R})$ in $C^0(K,\mathbb{R})$ and the continuity of Φ in $C^0(K,\mathbb{R})$, it follows that $\Phi(\psi)$ is a constant function for each $\psi \in C^0(K,\mathbb{R})$.

Using 1.3, given $\psi \in C^0(K,\mathbb{R})$ define

$$P(\psi) = \lim_{n\to+\infty} \frac{1}{n} \log \sum_{y\in\sigma^{-n}(x)} \exp(S_n\psi)(y).$$

Corollary 1.4. *For all* $0 < \gamma \leq 1$ *the functions*

$$P: C^\gamma(K,\mathbb{R}) \to \mathbb{R}$$
$$C^\gamma(K,\mathbb{R}) \ni \psi \to \nu_\psi \in C^\gamma(K,\mathbb{R})'$$
$$C^\gamma(K,\mathbb{R}) \ni \psi \to h_\psi \in C^\gamma(K,\mathbb{R})$$

are real analytic.

Proof. Let D be a closed disk centered at $\lambda(\psi)$ such that

$$D \cap \mathrm{sp}(\mathcal{L}_\psi) = \{\lambda(\psi)\}.$$

Let F be the space of linear maps of $C^\gamma(K,\mathbb{R})$ into itself endowed with the topology of the norm. Let U be a neighborhood of $\mathcal{L}_\psi$ in F such that $\partial D \cap \mathrm{sp}(L) = \phi$ for all $L \in U$. Then, if $L \in U$, we can define the projection of $C^\gamma(K,\mathbb{R})$ given by:

$$\pi_L = \frac{1}{2\pi i}\int_{\partial D} (\lambda I - L)^{-1} d\lambda. \tag{1}$$

When $L = \mathcal{L}_\psi$ the image of this projection is the eigenspace associated to the eigenvalue $\lambda(\psi)$. By part (d) of Theorem 1.1 this space is spanned by h_ψ, hence it is one dimensional. Therefore if U is small enough the image of π_L is also one dimensional and invariant under L. Hence it is an eigenspace associated to an eigenvalue $\mu(L)$ that is near to $\lambda(\psi)$ because the image of π_L is near to the image of $\pi_{\mathcal{L}_\psi}$. Then $\mu(L) > 0$. Hence we can calculate it as

$$\mu(L) = \frac{\langle w, \pi_L L_v \rangle}{\langle w, \pi_L v \rangle}. \tag{2}$$

where $v \in C^\gamma(K, \mathbb{R})$ and $w \in C^\gamma(B^+(A), \mathbb{R})'$ are vectors such that the inner product in the denominator is $\neq 0$. Choose v and w such that $\langle w, \pi_{\mathcal{L}_\psi} v \rangle \neq 0$. Then, the previous requirement, is satisfied for all $L \in U$ if U is small enough. Then (1) and (2) show that $\mu(L)$ is a real analytic function of L. Moreover, by elementary arguments, the spectrum of L consists of $\mu(L)$ and a set contained in a disk $\{z/|z| < r\}$ with $r < \mu(L)$ (recall that by Theorem 1.1 this property holds for $L = \mathcal{L}_\psi$). Hence, if φ is so near to ψ that $\mathcal{L}_\varphi \in U$, then

$$\mu(\mathcal{L}_\varphi) = \lambda(\varphi). \tag{3}$$

Take a neighborhood V of ψ in $C^\gamma(K, \mathbb{R})$ such that $\mathcal{L}_\varphi \in U$ when $\varphi \in V$. Clearly the map $V \ni \varphi \to \mathcal{L}_\varphi \in F$ is real analytic. Hence the map $V \ni \varphi \to \lambda(\varphi) \in \mathbb{R}$ is real analytic because it is the composition of the maps $V \ni \varphi \to \mathcal{L}_\varphi \in U$ and $U \ni L \to \mu(L) \in \mathbb{R}$. Therefore the map $V \ni \varphi \to P(\varphi) = \log \lambda(\varphi) \in \mathbb{R}$ is real analytic. Since the spectrum of $\mathcal{L}_\psi^* : C^\gamma(K, \mathbb{R})' \hookleftarrow$, is the same as the spectrum of $\mathcal{L}_\psi : C^\gamma(K, \mathbb{R}) \hookleftarrow$, a similar argument shows that there exists a neighborhood U of $\mathcal{L}_\psi^*$ in the space of linear continuous maps of $C^\gamma(K, \mathbb{R})'$ endowed with the topology of the norm, such that for each $L \in U$ there exists a projection $\hat{\pi}_L : C^\gamma(K, \mathbb{R})' \hookleftarrow$, depending analytically on L, and whose image is the one dimensional eigenspace associated to the eigenvalue $\lambda(\psi)$. We leave to the reader to check that given a neighborhood W of ψ such that $\mathcal{L}_\varphi^* \in U$ when $\varphi \in W$, then ν_φ and h_φ are given by

$$\nu_\varphi = \langle \hat{\pi}_{\mathcal{L}_\varphi^*} \nu_\psi, 1 \rangle^{-1} \hat{\pi}_{\mathcal{L}_\varphi^*} \nu_\psi$$

$$h_\varphi = \langle \pi_{\mathcal{L}_\varphi} h_\psi, \nu_\varphi \rangle^{-1} \pi_{\mathcal{L}_\nu} h_\psi.$$

Then they are real analytic functions of $\psi \in W$.

Corollary 1.5. *For all* $0 < \gamma \leq 1, \nu_\psi$ *is a weakly continuous function of* $\psi \in C^\gamma(K, \mathbb{R})$, *i.e.*

$$\lim_{n \to +\infty} \int \varphi d\nu_{\psi_n} = \int \varphi d\nu_\psi$$

for every convergent sequence $\psi_n \to \psi$ *in* $C^\gamma(K, \mathbb{R})$ *and all* $\varphi \in C^0(K, \mathbb{R})$.

Proof. Let $\psi_n \to \psi$ be a convergent sequence in $C^\gamma(K,\mathbb{R})$ and suppose that ν_{ψ_n} does not converge weakly to ν_ψ. Then we can assume that ν_{ψ_n} converges weakly to a probability $\nu \neq \nu_\psi$. Then

$$\mathcal{L}_\psi^* \nu = \lim_{n\to+\infty} \mathcal{L}_{\psi_n}^* \nu_{\psi_n} = \lim_{n\to+\infty} \lambda(\psi_n)\nu_{\psi_n} = \lambda(\psi)\nu.$$

Hence, $\nu \in C^0(K,\mathbb{R})' \subset C^\gamma(K,\mathbb{R})'$ is an eigenvector of $\mathcal{L}_\psi^*: C^\gamma(K,\mathbb{R})' \hookleftarrow$ associated to the eigenvalue $\lambda(\psi)$. But $\lambda(\psi)$ is a simple eigenvalue of $\mathcal{L}_\psi^*: C^\gamma(K,\mathbb{R})' \hookleftarrow$ because $\lambda(\psi)$ is a simple eigenvalue of $\mathcal{L}_\psi: C^\gamma(K,\mathbb{R}) \hookleftarrow$. Since ν_ψ is an eigenvector of $\mathcal{L}_\psi^*: C^\gamma(K,\mathbb{R})' \hookleftarrow$ associated to $\lambda(\psi)$, it follows that ν is a scalar multiple of ν_ψ. But since both are probabilities it follows that $\nu = \nu_\psi$.

Corollary 1.6. *If $\psi \in C^\gamma(K,\mathbb{R}), 0 < \gamma \leq 1$, then*

$$\lim_{n\to+\infty} \left\| \frac{1}{n} \frac{\mathcal{L}_\psi^n(S_n\varphi)}{\mathcal{L}_\psi^n 1} - \int \varphi h_\psi d\nu_\psi \right\|_\beta = 0$$

for all $\varphi \in C^\beta(K,\mathbb{R}), 0 \leq \beta \leq \gamma$. Moreover, when $0 < \beta \leq \gamma$, the convergence is uniform in the unit ball of C^β

Proof. It is easy to check that

$$\mathcal{L}_\psi^n(S_n\varphi) = \sum_{j=0}^{n-1} \mathcal{L}_\psi^{n-j}(\varphi \mathcal{L}_\psi^j 1)$$

Then

$$\lambda^{-n}(\psi)\mathcal{L}_\psi^n(S_n\varphi) = \sum_{j=0}^{n-1} \lambda(\psi)^{-(n-j)} \mathcal{L}_\psi^{n-j}(\varphi\lambda(\psi)^{-j}\mathcal{L}_\psi^j 1).$$

But by Ruelle's theorem

$$\lim_{j\to+\infty} \left\| \varphi\lambda(\psi)^{-j}\mathcal{L}_\psi^j 1 - \varphi h_\psi \right\|_\beta = 0$$

and

$$\sup_m \left\| \lambda(\psi)^{-m}\mathcal{L}_\psi^m \right\|_\beta < \infty.$$

Hence

$$\lim_{n\to++\infty} \left\| \frac{1}{n}\lambda^{-n}(\psi)\mathcal{L}_\psi^n(S_n\varphi) - \sum_{j=0}^{n-1} \lambda(\psi)^{-(n-j)}\mathcal{L}_\psi^{n-j}\varphi h_\psi \right\|_\beta = 0. \tag{1}$$

But, by Ruelle's Theorem,

$$\lim_{m\to+\infty} \left\| \lambda(\psi)^{-m}\mathcal{L}_\psi^n \varphi h_\psi - h_\psi \int \varphi h_\psi d\nu_\psi \right\|_\beta = 0. \tag{2}$$

From (2) it follows that

$$\lim_{n\to+\infty}\left\|\frac{1}{n}\sum_{j=0}^{n-1}\lambda(\psi)^{-(n-j)}\mathcal{L}_\psi^{n-j}\varphi h_\psi - h_\psi\int\varphi h_\psi d\nu_\psi\right\| = 0. \tag{3}$$

Also from Ruelle's theorem it follows that

$$\lim_{n\to+\infty}\left\|\lambda^{-n}(\psi)\mathcal{L}_\psi^n 1 - h_\psi\right\|_0 = 0 \tag{4}$$

Then, by (1), (3) and (4):

$$\lim_{n\to+\infty}\left\|\frac{1}{n}\frac{\mathcal{L}_\psi^n(S_n\varphi)}{\mathcal{L}_\psi^n 1} - \int\varphi h_\psi d\nu_\psi\right\|_0 =$$

$$= \lim_{n\to+\infty}\left\|\frac{\lambda^n(\psi)}{\mathcal{L}_\psi^n 1}\ \frac{1}{n}\lambda^{-n}(\psi)\mathcal{L}_\psi^n(S_n\varphi) - \int\varphi h_\psi d\nu_\psi\right\|_\beta \le$$

$$\le \lim_{n\to+\infty}\left\|\frac{\lambda^n(\psi)}{\mathcal{L}_\psi^n 1}\right\|_\beta \lim_{n\to+\infty}\left\|\frac{1}{n}\lambda^{-n}(\psi)\mathcal{L}_\psi^n(S_n\varphi) - (\lambda(\psi)^{-n}\mathcal{L}_\psi^n 1)\int\varphi h_\psi d\nu_\psi\right\|_\beta =$$

$$= \lim_{n\to+\infty}\left\|\frac{1}{n}\lambda^{-n}(\psi)\mathcal{L}_p^n si\varphi) - h_\psi\int\varphi h_\psi d\nu_\psi\right\|_\beta = 0.$$

The uniform convergence in the unit C^β ball follows from the fact that, by part (d) of Ruelle's theorem, all the convergences involved in this argument are uniform in the unit ball of C^β for $0 < \beta \le \gamma$.

Corollary 1.7. *If* $0 < \gamma \le 1$, *and* $\psi \in C^\gamma(K,\mathbb{R})$, *then the derivative*

$$P'(\psi): C^\gamma(K,\mathbb{R}) \to \mathbb{R}$$

is given by

$$P'(\psi)\varphi = \int\varphi h_\psi d\nu_\psi.$$

Proof. Fix $p \in K$ and define $P_n: C^\gamma(K,\mathbb{R}) \longrightarrow \mathbb{R}$ by

$$P_n(\psi) = \frac{1}{n}\log(\mathcal{L}_\psi^n 1)(p).$$

Then an easy calculation shows that

$$P_n'(\psi)\varphi = \frac{1}{n}\frac{\mathcal{L}_\psi^n(S_n\varphi)}{\mathcal{L}_\psi^n 1}.$$

Hence, 1.6 implies that

$$P'(\psi)\varphi = \lim_{n\to+\infty} P_n(\psi)\varphi = \int\varphi h_\psi d\nu_\psi.$$

Now we shall prove Theorem B. This proof requires two Lemmas.

We say that a map $f: K_1 \to K_2$, where K_1, K_2 are metric spaces, is *compact* if it maps bounded sets onto relatively compact sets. It is easy to see that if a sequence of compact maps $f_n: K_1 \to K_2$ converges to a map $f: K_1 \to K_2$ uniformly on bounded sets, then f is compact.

Lemma 1.9. *Let E_1, E_2 be Banach spaces and $U \subset E_1$ an open set. If $f: U \to E_2$ is a C^k compact map, then for all $x \in U$, the derivatives $f^{(j)}(x): \overbrace{E_1 \times \cdots \times E_1}^{j} \to E_2$ are compact for all $1 \le j \le k$.*

Proof. Given $x \in U$ let B be the unit ball centered at 0 and define maps $f_n: B \to E_2$ by

$$f_n(v) = n(f(x + \frac{1}{n}v) - f(x)).$$

Then the sequence f_n converges uniformly to $f'(x)/B$. Since clearly each map f_n is compact, it follows that $f'(x)$ is compact. Now suppose that we have proved that $f^{(j)}(x)$ is compact for $1 \le j < m$. Define maps $f_n: B \to E_2$ by

$$f_n(v) = n^m(f(x + \frac{1}{n}v) - f(x) - \sum_{j=1}^{m-1} \frac{1}{j!} f^{(j)}(x) \overbrace{\left(\frac{1}{n}v, \cdots, \frac{1}{n}v\right)}^{j}).$$

Then the sequence f_n converges uniformly to the map

$$B \ni v \to f^{(m)}(x)(v, \ldots, v) \in E_2.$$

Hence this map is compact. Since using the symmetry of the m-linear map $f^{(m)}(x)$ it is possible to write $f^{(m)}(x)(v_1, \ldots, v_m)$ as a linear combination of the vectors $f^{(m)}(x)(v_i, \ldots, v_i), 1 \le i \le m$, it follows that $f^{(m)}$ is compact.

Lemma 1.10. *Let E_0, E_1, E_2 be Banach spaces, $U \subset E_0$ an open set and suppose that $f: U \to E_1, L: E_1 \to E_2$ and $P: E_2 \to \mathbb{R}$ are maps satisfying*

a) *L is linear and compact*

b) *f is $C^k, k \ge 0$*

c) *$L \circ f$ is C^{k+1}*

d) *$P \circ L$ is C^{k+1}*

e) *There exists a function T that associates to each $x \in E_1$ a continuous linear map $T(x): E_2 \to \mathbb{R}$ satisfying*

$$(P \circ L)' = T(x)L \tag{1}$$

for all $x \in E_1$, and

$$\lim_{n \to +\infty} T(x_n)v = T(\lim x_n)v \tag{2}$$

for every convergent sequence $\{x_n\} \subset E_1$ and all $v \in E_2$.

Then $P \circ L \circ f$ is C^{k+1}

Proof. Obviously $P \circ L \circ f$ is C^k because $P \circ L$ is C^k and f is C^k. Suppose that $k \geq 1$. The derivative $(P \circ L \circ f)^{(k)}(x)$ can be written as the sum of

$$(P \circ L)'(f(x))f^{(k)}(x) \tag{3}$$

and a linear combination of compositions of derivatives $(P \circ L)^{(i)}$ and $f^{(j)}$ with $1 < i \leq k$ and $1 \leq j < k$. Hence all these terms are C^1, because f is C^k and $P \circ L$ is C^k. This means that to prove that f is C^{k+1} we have only to prove that (3) is C^1. Observe that given Banach spaces E and F, an open set $U \subset E$ and a map $\Phi: U \to F$, then, to prove that Φ is C^1 it suffices to show that for every $x \in U$ there exists a continuous linear map $A(x): E \to F$, depending continuously on x, such that

$$\lim_{t \to 0} \frac{1}{t}(\Phi(x + tv) - \Phi(x)) = A(x)v$$

for all $x \in U$ and $v \in E$. This is proved by writing

$$\Phi(x + v) - \Phi(x) = \int_0^1 \frac{d}{dt}\Phi(x + tv)dt =$$

$$= \int_0^1 A(x + tv)vdt = A(x)v + \int_0^1 (A(x + tv) - A(x))vdt.$$

Hence

$$\|\Phi(x + v) - \Phi(x) - A(x)v\| \leq \|v\| \sup_{\|y - x\| \leq \|v\|} \|A(y) - A(x)\|$$

thus proving that $\Phi'(x) = A(x)$. We shall use this criteria to prove that (3) is C^1. Observe that

$$(P \circ L)'(f(x + tw))f^{(k)}(x + tw) - (P \circ L)'(f(x))f^{(k)}(x) =$$

$$= ((P \circ L)'(f(x + tw)) - (P \circ L)'(f(x)))f^{(k)}(x + tw) +$$

$$+(P\circ L)'(f(x))(f^{(k)}(x+tw)-f^{(k)}(x)).$$

Since $P\circ L$ is C^2 by (d) and the assumption $k\geq 1$, it follows that

$$\lim_{t\to 0}\frac{1}{t}((P\circ L)'(f(x+tw))-(P\circ L)'(f(x)))f^{(k)}(x+tw)=$$
$$=(P\circ L)''(f(x))f'(x)wf^{(k)}(x).$$

Moreover, by (1):

$$(P\circ L)'(f(x))(f^{(k)}(x+tw)-f^{(k)}(x))=$$
$$=T(f(x))((Lf)^{(k)}(x+tw)-(Lf)^{(k)}(x)).$$

Hence, since Lf is C^{k+1},

$$\lim_{t\to 0}\frac{1}{t}T(f(x))((Lf)^{(k)}(x+tw)-(Lf)^{(k)}(x))=$$
$$=T(f(x))(Lf)^{(k+1)}(x)w.$$

Hence

$$\lim_{t\to 0}\frac{1}{t}((P\circ L)'(f(x+tw))f^{(k)}(x+tw)-(P\circ L)'(f(x))f^{(k)}(x))=$$
$$=(P\circ L)''(f(x))(f'(x)w)f^{(k)}(x)+T(f(x))(Lf)^{(k+1)}(x)w.$$

Therefore, if we prove that the $(k+1)$-linear map $T(f(x))(Lf)^{(k+1)}(x)$ is a continuous function of x, it will follow that (2) is C^1 and then that $P\circ L\circ f$ is C^{k+1}. Since $P\circ L$ is C^2 and f is C^k it follows that the first term of the sum depends continuously on x. To prove the continuity of $T(f(x))(Lf)^{(k+1)}(x)$ first observe that if $x_n\to x$ and $S\subset E_2$ is a relatively compact set then, by (2), $T(x_n)/S$ converges uniformly to $T(x)/S$. Moreover, Lf is compact because L is compact, and then, by the previous Lemma, $(Lf)^{(k+1)}(y)$ is compact for all $y\in E_0$. Let B be the unit ball of $\overbrace{E_0\times\cdots\times E_0}^{k+1}$. Define

$$S=(Lf)^{(k+1)}(x)B\cup(\bigcup_{n\geq 1}(Lf)^{(k+1)}(x_n)B).$$

This set is relatively compact because every sequence $\{u_n\}\subset S$ either has a subsequence contained in some $(Lf)^{(k+1)}(p)B$, $p\in\{x,x_1,\dots\}$ (and then, since this set is relatively compact, has a convergent subsequence) or has a subsequence that can be written as

$$u_{n_j}=(Lf)^{(k+1)}(x_{m_j})\theta_{m_j}$$

with $\theta_{m_j} \in B$ and $m_j \to +\infty$. But now, by the compacity of $(Lf)^{(k+1)}$, we can assume that the sequence $(Lf)^{(k+1)}(x)\theta_{m_j}, j \geq 1$, converges to a point $y \in E_2$ and then it is easy to prove that $(Lf)^{(k+1)}(x_{m_j})\theta_{m_j}$ converges to y. This concludes the proof of the relative compacity of S and then $T(x_n)/S$ converges uniformly to $T(x)/S$. Since $(Lf)^{(k+1)}(x)B \subset S$ for all $p \in \{x, x_1, \ldots\}$, it follows that $T(x_n)(Lf)^{(k+1)}(x_n)/B$ converges to $T(x)(Lf)^{(k+1)}(x)/B$ uniformly. This completes the proof of the Lemma when $k \geq 1$. The case $k = 0$ is handled by similar methods.

To prove Theorem B we shall apply Lemma 1.10 to an open set $U \in N$, the Banach spaces $C^\gamma(K, \mathbb{R})$ and $C^0(K, \mathbb{R})$, the C^k map $\Phi: U \to C^\gamma(K, \mathbb{R})$, the compact linear map $i: C^\gamma(K, \mathbb{R}) \to C^0(K, \mathbb{R})$ given by the inclusion and the function $P: C^0(K, \mathbb{R}) \to \mathbb{R}$. Hypothesis (a), (b) and (c) of 1.10 are obviously satisfied. Hypothesis (d) holds because we proved (Corollary 1.4) that $P: C^\gamma(K, \mathbb{R}) \to \mathbb{R}$ is real analytic. To check (e) associate, to each $\psi \in C^\gamma(K, \mathbb{R})$, the functional $T(\psi) \in C^0(K, \mathbb{R})'$ given by

$$T(\psi)\varphi = \int \varphi h_\psi d\nu_\psi.$$

Then, by 1.7,

$$(P \circ i)'(\psi)\varphi = T(\psi)\varphi$$

thus proving property (1) of hypothesis (e). Property (2) follows from the fact that ν_ψ is, by 1.5, a weakly continuous function of $\psi \in C^\gamma(K, \mathbb{R})$ and h_ψ is a continuous (in fact real analytic) function of ψ by 1.4. Then, 1.11 can be applied and proves that $P \circ i \circ \Phi$ is C^{k+1}.

2. Proof of Theorem A

The proof of Theorem A requires the following properties:

Proposition 2.1. ([1]) *For all* $0 < \gamma \leq 1$ *there exists a continuous linear map* $T: C^\gamma(B(A), \mathbb{R}) \to G^\gamma(B^+, (A), \mathbb{R})$ *such that, denoting* $\pi: B(A) \to B^+(A)$ *the canonical projection (i.e.* $\pi(\theta) = \theta/\mathbb{Z}^+$*) then, for all* $\psi \in C^\gamma(B^+(A), \mathbb{R}), \psi$ *is homologous to* $(T\psi) \circ \pi$*, i.e. there exists* $u \in C^0(B(A), \mathbb{R})$ *such that*

$$u \circ \sigma - u = \psi - (T\psi) \circ \pi.$$

Lemma 2.2. *If Λ is a horseshoe of $f \in \text{Diff}^r(M), r \geq 1$, and $h: B(A) \to \Lambda$ is a topological equivalence between $\sigma: B(A) \hookleftarrow$ and $f|\Lambda$, then h is Hölder continuous*

Lemma 2.3. *Let Λ be a horseshoe of $f \in \text{Diff}^r(M), r \geq 2, \dim M = 2$. Let $h: B(A) \to \Lambda$ be a topological equivalence between $\sigma: B(A) \hookleftarrow$ and $f|\Lambda$. Define $\psi: B(A) \to \mathbb{R}$ by*

$$\psi(\theta) = -\log\left|f'(h(\theta))/E^u_{h(\theta)}\right|.$$

Then the following properties hold:

a) The function $\mathbb{R} \ni \delta \to P(\delta T\psi) \in \mathbb{R}$ is real analytic (where T is given by 2.1).

b) There exists $c > 0$ such that

$$\frac{\partial}{\partial\delta}P(\delta T\psi) \leq -c$$

for all δ.

c) There exists a unique $\delta(f) > 0$ such that $P(\delta(f)T\psi) = 0$. Moreover every $x \in \Lambda$ is contained in an open interval $J \subset W^s(x)$ such that there exists a probability μ on the Borel σ-algebra of J and a constant $C > 0$ satisfying

$$C^{-1}r^\delta \leq \mu(B_r(x)) \leq Cr^\delta$$

for all $r \geq 0$.

Lemma 2.4. *If Λ is a basic set of $f \in \text{Diff}^r(M), r \geq 2$, there exist neighborhoods U and $\mathcal{U}$ of Λ and f respectively such that, defining $\Lambda_g = \bigcap_n g^n(U)$, there exist $0 < \gamma < 1$ and a C^{r-1} function $\mathcal{U} \ni g \to h_g \in C^\gamma(\Lambda, M)$ satisfying the following properties:*

a) *$h_g(\Lambda) = \Lambda_g$ and h_g is a topological equivalence between $f|\Lambda$ and $g|\Lambda_g$;*

b) *The function $\mathcal{U} \ni g \to |(\det(g'/E^u)) \circ h_g| \in C^\gamma(\Lambda, \mathbb{R})$ is C^{r-2};*

c) *The function $\mathcal{U} \ni g \to |(\det(g'/E^u)) \circ h_g| \in C^0(\Lambda, \mathbb{R})$ is C^{r-1}.*

Now let us prove Theorem A. Let Λ be a horseshoe of $f \in \text{Diff}^r(M), r \geq 2$ and suppose that $\dim M = 2$. Let $\mathcal{U}$ and U be the neighborhoods given by Lemma 2.4. It is known (Bowen [1]) that there exists a topological equivalence $h: B(A) \to \Lambda$ between $\sigma: B(A) \hookleftarrow$ and $f|\Lambda$. Then, if $h_g: \Lambda \to \Lambda_g$ is given by Lemma 2.4, the map $h_g h\colon B(A) \to \Lambda_g$ is a topological equivalence between $\sigma\colon B(A) \hookleftarrow$ and g/Λ_g. Moreover by Lemmas 2.2 and 2.4 there exists $0 < \gamma < 1$ such that $h_g h \in C^\gamma(B^+(A), M)$ and the function $\mathcal{U} \ni g \to h_g h \in C^\gamma(B(A), M)$ is C^{r-1}. Define $\psi_g \in C^\gamma(B^+(A), \mathbf{R})$ by

$$\psi_g(\theta) = -\log\left|g'(h_g h(\theta))/E^u_{h_g h(\theta)}\right|$$

and $B\colon \mathcal{U} \times \mathbf{R} \to \mathbf{R}$ by

$$B(g,\delta) = P(\delta T \psi_g).$$

Then B is C^{r-1}. This follows from applying Theorem B to the Banach manifold $\mathcal{U}$ and the C^{r-2} map $\mathcal{U} \ni g \to \psi_g \in C^\delta(B^+(A), \mathbf{R})$. Moreover, by Lemma 2.3, for each $g \in \mathcal{U}$ there exists a unique $\delta^u(g)$ satisfying

$$P(g, \delta^u(g)) = 0$$

$$\frac{\partial P}{\partial t}(g, \delta^u(g)) < 0.$$

Then, by the implicit function theorem the function $\mathcal{U} \ni g \to \delta^u(g)$ is C^{r-1}. Take a point $x \in \Lambda_g$ and let J^u be an interval contained in $W^u(x)$ and containing x such that, according to Lemma 2.3 there exists a finite measure μ_u on the Borel σ-algebra of J^u and a constant $C_u > 0$ such that

$$C^{-1} r^{\delta^u(g)} \le \mu_u(B_r(p)) \le C_u r^{\delta^u(g)} \tag{1}$$

for all $p \in J$ and $r > 0$. In a similar way (replacing g by g^{-1}) there exists a C^{r-1} function $\mathcal{U} \ni g \to \delta^s(g) \in \mathbf{R}$ such that there exists an interval $J^s \subset W^s(x)$ containing x and a finite measure μ_u on the Borel σ-algebra of J^s such that there exists $C_s > 0$ satisfying

$$C^{-1} r^{\delta^s(g)} \le \mu_s(B_r(p)) \le C_s r^{\delta^s(g)} \tag{2}$$

for all $r > 0$ and $p \in J^s$. By standard properties of hyperbolic sets, there exists $\epsilon > 0$ such that if J^s and J^u are sufficiently small then $W^s_\epsilon(a) \cap W^u_\epsilon(b)$ contains exactly one point for all $a \in J^u$ and $b \in J^s$. Given $A \subset J^u$ and $B \subset J^s$ define

$$A \times B = \{W^s_\epsilon(a) \cap W^u_\epsilon(b) | a \in A, b \in B\}.$$

Take a measure μ on the Borel σ-algebra of $J^u \times J^s$ such that

$$\mu(A \times B) = \mu_u(A)\mu_s(B) \tag{3}$$

for every pair of Borel sets $A \subset J^u, B \subset J^s$. Since g is at least C^2, the stable and unstable foliations extend to C^1 foliations of a neighborhood of Λ_g. Then there exists $k > 1$ such that

$$(B_{r/k}(p) \cap J^u) \times (B_{r/k}(p) \cap J^s) \subset B_r(p) \subset (B_{kr}(p) \cap J^u) \times (B_{kr}(p) \cap J^s)$$

for all $p \in J^u \times J^s$. Then, from (1), (2) and (3), there exists $C > 0$ such that

$$C^{-1} r^{\delta^s(g)+\delta^u(g)} \leq \mu(B_r(p)) \leq C r^{\delta^s(g)+\delta^u(g)}$$

for all $p \in J^u \times J^s$ and $r > 0$. Then the Hausdorff dimension of $J^u \times J^s$ is $\delta^s(g) + \delta^u(g)$. This follows from the following easy Lemma:

Lemma. *Let K be a compact metric space and μ a probability on the Borel σ-algebra of K such that there exist $0 \leq \delta_1 < \delta_2$ and $C > 0$ satisfying:*

$$C^{-1} r^{\delta_2} \leq \mu(B_r(x)) \leq C r^{\delta_1}$$

for all $x \in K$ and $r > 0$. Then, if $HD(K), c^-(K), c^+(K)$ denote respectively the Hausdorff dimension and the lower and upper capacities of K

$$\delta_1 \leq HD(K) \leq c^-(K) \leq c^+(K) \leq \delta_2$$

Proof. $c^+(K)$ can be defined as

$$c^+(K) = \limsup_{r \to 0} \frac{\log S(r)}{\log(1/r)}$$

where $S(r)$ is the maximum number m such that there exists points $x_1, \dots, x_m$ such that $d(x_i, x_j) \geq r$ for all $1 \leq i < j \leq m$. Then the balls $B_{r/2}(x_i), i = 1, \dots, m$ are disjoint. Hence

$$1 \geq \mu\left(\bigcup_{i=1}^{m} B_{r/2}(x_i)\right) \geq \sum_{i=1}^{m} \mu(B_{r/2}(x_i)) \geq mC^{-1}\left(\frac{r}{2}\right)^{\delta_2}.$$

Hence

$$m \leq C\left(\frac{2}{r}\right)^{\delta_2}.$$

This implies easily $c^+(K) \leq \delta_2$. In a similar way one proves $HD(K) \geq \delta_1$ completing the proof of the Lemma.

Then $HD(J^u \times J^s) = \delta^u(g) + \delta^s(g)$. Since $J^u \times J^s$ is a neighborhood of x and x is arbitrary, it follows that

$$HD(\Lambda_g) = \delta^u(g) + \delta^s(g).$$

Since δ^s and δ^u are C^{r-1} functions of g this completes the proof of the Theorem.

Proof of Lemma 2.2. Due to the hyperbolicity of Λ there exists $\delta > 0, C > 0$ and $0 < \lambda < 1$ such that if $x \in \Lambda$ and $y \in M$ satisfy $\mathrm{d}(f^n(x), f^n(y)) < \delta$ for all $-N \leq n \leq N$ then

$$\mathrm{d}(x,y) \leq C\lambda^N.$$

Moreover, recalling that we endowed $B(A)$ with the metric

$$\mathrm{d}(\alpha,\beta) = \sum_{-\infty}^{\infty} 2^{-|n|}|\alpha(n) - \beta(n)|,$$

it is easy to see that $\alpha(n) = \beta(n)$ for all n satisfying

$$|n| < -(\log 2)^{-1} \log \mathrm{d}(\alpha,\beta).$$

Since h is continuous there exists $k > 0$ such that $\alpha(n) = \beta(n)$ for $-k \leq n \leq k$ implies $\mathrm{d}(h(\alpha), h(\beta)) \leq \delta$. Then, given $\alpha, \beta \in B(A)$ define

$$N = -(\log 2)^{-1} \log \mathrm{d}(\alpha,\beta) - 1. \tag{1}$$

Then $\alpha(n) = \beta(n)$ for all $-N \leq n \leq N$. Therefore $(\sigma^j\alpha)(n) = (\sigma^j\beta)(n)$ for all $-k < n < k$ if $-(N-k) \leq j \leq N-k$. Hence $\mathrm{d}(h(\sigma^j\alpha), h(\sigma^j\beta)) \leq \delta$ for $-(N-k) \leq j \leq N-k$. Since $h\sigma^j = f^j h$, this implies

$$\mathrm{d}(f^j(h(\alpha)), f^j(h(\beta))) \leq \delta$$

for $-(N-k) \leq j \leq N-k$. Then

$$\mathrm{d}(h(\alpha), h(\beta)) \leq C\lambda^{N-k}.$$

Replacing (1) in this inequality, we obtain

$$\mathrm{d}(h(\alpha), h(\beta)) \leq C_0\, \mathrm{d}(\alpha,\beta)^\gamma$$

with $C_0 = C/\lambda^{k+1}$ and $\gamma = (\log 2)^{-1} \log \lambda$.

Proof of Lemma 2.3. To prove (a) first recall that the subbundle $E^u \subset TM|\Lambda$ is Hölder continuous because $r \geq 2$. Moreover 2.2 implies that h is Hölder continuous. Hence, $\psi \in C^\gamma(B(A), \mathbb{R})$ for some $0 < \gamma < 1$, and then $T\psi \in C^\gamma(B^r(A), \mathbb{R})$. Therefore the analiticity of the map $\mathbb{R} \ni \delta \to P(\delta T\psi) \in \mathbb{R}$ follows from the analiticity of $P: C^\gamma(B^+(A), \mathbb{R}) \to \mathbb{R}$. Before proving (b) let us show that there exists $A > 0 > B$ satisfying

$$S_n(T\psi)(\theta) \leq A + Bn \tag{1}$$

for all $\theta \in B(A)$ and $n \geq 0$. For this purpose take $\overline{\theta}$ such that $\pi(\overline{\theta}) = \theta$. Then

$$S_n(T\psi)(\theta) = S_n(T\psi)(\pi(\overline{\theta})) = \sum_{j=0}^{n-1}(T\psi)(\sigma^j(\pi(\overline{\theta})))$$
$$= \sum_{j=0}^{n-1}(T\psi)(\pi(\sigma^j\overline{\theta})).$$

Recalling that $(T\psi)\circ\pi - \psi$ is homologous to zero, there exists $u \in C^0(B(A), \mathbb{R})$ such that

$$(T\psi)\circ\pi = \psi + (u\circ\sigma - u).$$

Hence

$$S_n(T\psi(\theta)) = \sum_{j=0}^{n-1}((T\psi)\circ\pi)(\sigma^j(\theta))$$
$$= \sum_{j=0}^{n-1}\psi(\sigma^j(\overline{\theta})) + u(\sigma^n(\overline{\theta})) - u(\overline{\theta}).$$

Let K be the maximum of u. Then

$$S_n(T\psi)(\theta) \leq \sum_{j=0}^{n-1}\psi(\sigma^j(\overline{\theta})) + 2K. \tag{2}$$

But

$$\sum_{j=0}^{n-1}\psi(\sigma^j(\overline{\theta})) = -\sum_{j=0}^{n-1}\log\left|f'(\sigma^j(\overline{\theta}))|E^u_{h\sigma^j(\overline{\theta})}\right|$$
$$= \sum_{j=0}^{n-1}\log\left|f'(f^j(h(\overline{\theta})))|E^u_{f^jh(\overline{\theta})}\right|$$
$$= -\log\left|(f^n)'(h(\overline{\theta}))/E^u_{h(\overline{\theta})}\right|.$$

Then, if $C > 0$ and $0 < \lambda < 1$ are such that

$$|(f^n)'(x)/E^u_x|^{-1} \leq C\lambda^n$$

for all $x \in \Lambda$ and $n \geq 0$; it follows that

$$\sum_{j=0}^{n-1}\psi(\sigma^j(\overline{\theta})) \leq \log C + n\log\lambda.$$

Replacing this inequality in (2), we obtain (1) with $A = \log C + 2K$ and $B = \log\lambda$. Now, to complete the proof of (b), fix $\alpha \in B(A)$ and define $P_n\colon \mathbb{R} \hookleftarrow$ by

$$P_n(t) = \frac{1}{n}\log\sum_{\sigma^n\theta=\alpha}\exp S_n(t(T\psi))(\theta).$$

Then

$$\frac{dP_n}{dt}(t) = \frac{1}{n}\frac{\sum_{\sigma^n\theta=\alpha} S_n(T\psi)(\theta)\exp S_n(t(T\psi))(\theta)}{\sum_{\sigma^n\theta=\alpha}\exp S_n(tT\psi)(\theta)}$$
$$\leq \frac{1}{n}\sup_{\sigma^n\theta=\alpha} S_n(T\psi)(\theta)$$
$$\leq \frac{1}{n}(A+B_n).$$

This implies that there exists $c > 0$ such that, if n is large, $(dP_n/dt)(t) \leq -c$ for all t. In particular

$$P_n(t_1) - P_n(t_2) \leq -c(t_1 - t_2)$$

for all $T_1 \geq t_2$. By 1.2

$$\log\lambda(t(T\psi)) = \lim_{n\to+\infty} P_n(t).$$

Hence

$$\log\lambda(t_1(T\psi)) - \log\lambda(t_2(T\psi)) \leq -c(t_1 - t_2)$$

for all $t_1 \geq t_2$. This implies

$$\frac{d}{dt}\log\lambda(t(T\psi)) \leq -c$$

thus proving (b). To prove (c) take an interval $J \subset W^s(x)$ containing x and define $F\colon J \to B^+(A)$ by $F(p) = \pi h^{-1}(p)$. Clearly F is continuous. Let us prove that if $\operatorname{diam}(J)$ is small enough then F is injective. Let $K_i \subset \Lambda$ be the image under h of the set $\{\theta \in B(A) | \theta(0) = i\}$. The sets K_i are compact and disjoint. Then there exists $\delta_0 > 0$ such that $\mathrm{d}(K_i, K_j) > \delta_0$ for all $1 \leq i < j \leq m$. Since J is an interval contained in a unstable manifold, diminishing its diameter grants $\operatorname{diam}(f^{-n}(J)) \leq \delta_0$ for all $n \geq 0$. This means that if K_i is the set of the family $\{K_1, \ldots, K_m\}$ that contains $f^{-n}(x)$, then $f^{-n}(J\cap\Lambda) \subset K_i$ because $\operatorname{diam} f^{-n}(J\cap\Lambda) < \mathrm{d}(f^{-n}(x), \Lambda - K_i)$ and $f^{-n}(J\cap\Lambda) \subset \Lambda$. On the other hand, if $\theta \in B^+(A)$, the point $h(\theta)$ satisfies

$$f^n(h(\theta)) \in K_{\theta(n)}$$

for all $n \in \mathbb{Z}$. Hence, if $h(\theta_0)$ and $h(\theta_1)$ are contained in J, it follows that $\theta_1(-n) = \theta_0(-n)$ for all $n \geq 0$ because, for all $n \geq 0$

$$f^{-n}(h(\theta_0)) \in K_{\theta_0(-n)}$$

$$f^{-n}(h(\theta_1)) \in K_{\theta_1(-n)},$$

and, as we explained above, these two properties plus the fact that $h(\theta_0), h(\theta_1) \in J$, imply $\theta_0(-n) = \theta_1(n)$. Now suppose that $F(x_1) = F(x_2)$. This means that $\pi h^{-1}(x_1) = \pi h^{-1}(x_2)$ and then $h^{-1}(x_1)(n) = h^{-1}(x_2)(n)$ for all $n \geq 0$. But we also have $h^{-1}(x_1)(n) = h^{-1}(x_2)(n)$ for all $n \leq 0$ because $h(h^{-1}(x_1)) = x_1 \in J$ and $h(h^{-1}(x_2)) = x_2 \in J$ and therefore $h^{-1}(x_1)(n) = \theta_0(n) = h^{-1}(x_2)(n)$ for all $n \leq 0$. This completes the proof of the injectivity of F. Now take J being open. We claim that $F(J \cap \Lambda)$ is an open subset of $B^+(A)$. Take $y \in J$. Given $\epsilon > 0$ there exists $N > 0$ such that $\theta(n) = h^{-1}(y)(n)$ for all $n \leq N$ implies $h(\theta) \in W^u_\epsilon(y)$. Therefore, since J is open, there exists $N > 0$ such that if $\theta \in B(A)$ satisfies $\theta(n) = h^{-1}(y)(n)$ for all $n \leq N$ then $h(\theta) \in J \cap \Lambda$. To prove that $F(J \cap \Lambda)$ is open we shall show that if $\overline{\theta} \in B^+(A)$ is close to $F(y)$ then $\overline{\theta} \in F(J \cap \Lambda)$. If $\overline{\theta}$ is close to $F(y) = \pi h^{-1}(y)$ then $\overline{\theta}(n) = h^{-1}(y)(n)$ for $0 \leq n \leq N$. Define $\theta \in B(A)$ by $\theta(n) = h^{-1}(y)(n)$ for $n \leq 0$ and $\theta(n) = \overline{\theta}(n)$ for $n \geq 0$. Observe that this definition is correct because $h^{-1}(y)(0) = (\pi h^{-1}(y))(0) = F(y)(0) = \overline{\theta}(0)$. Hence $\theta(n) = h^{-1}(y)(n)$ for all $n \leq N$ because it is true for $n \leq 0$ by definition and $\theta(n) = \overline{\theta}(n) = h^{-1}(y)(n)$ for $0 \leq n \leq N$. Then $h(\theta) \in J \cap \Lambda$. Hence $\overline{\theta} = \theta/\mathbb{Z}^+ = h^{-1}(h(\theta))/\mathbb{Z}^+ = \pi h^{-1}(h(\theta)) = F(h(\theta))$ completing the proof of the openess of $F(J \cap \Lambda)$. Since $J \cap \Lambda$ is a Cantor set we can take J such that $J \cap \Lambda$ is open and compact. Then $F\colon J \cap \Lambda \to F(J \cap \Lambda)$ is a homeomorphism. Define a measure μ on the Borel σ-algebra of $J \cap \Lambda$ by $\mu(S) = \nu(F(S))$ where $\nu = \nu_{\delta T \psi}$ is given by 1.1. Since $F(J \cap \Lambda)$ is open, $\mu(F(J \cap \Lambda)) > 0$. Hence $\mu(J \cap \Lambda)$ is positive and ≤ 1. To show that μ satisfies the inequalities of part (c) of Lemma 2.3, define, for $\mu \in J \cap \Lambda$,

$$S_\delta(y,n) = \{p \in J \cap \Lambda \,|\, \mathrm{d}(f^k(p), f^k(y)) \leq \delta \ \text{ for } \ 0 \leq k \leq n\}$$

and, if $\theta \in B^+(A)$ define $B(\theta, n) = \{\alpha \in B^+(A) | \alpha(j) = \theta(j) \text{ for } 0 \leq j \leq n\}$. Let us prove that there exist $\delta_1 > 0$ and $N > 0$ such that:

$$F(S_{\delta_1}(y,n)) \subset B(F(y), n) \subset F(S_{\delta_1}(y, n-N))$$

for all $y \in J \cap \Lambda$ and $n \geq N$. Choose any δ_1 satisfying $0 < \delta_0$ where δ_0 satisfies, as above, the property $\mathrm{d}(K_i, K_j) > \delta_0$ for all $1 \leq i < j \leq m$. Then, by the same arguments used before, if $p, y \in J \cap \Lambda$ and $\mathrm{d}(f^k(p), f^k(y)) \leq \delta_1$ for $0 \leq k \leq n$, it follows that $f^k(p)$ and $f^k(y)$ are contained in the same atom of the partition $\{K_1, \ldots, K_n\}$ for all $0 \leq k \leq n$. Hence $h^{-1}(p)(n) = h^{-1}(y)(n)$ for all $0 \leq k \leq n$ and then $F(p) \in B(F(p), n)$ for all $p \in J \cap \Lambda$ and $n \geq 0$.

To prove the second inclusion, take $\epsilon > 0$ such that $W^u_\epsilon(x) \subset J$ for all $y \in J \subset \Lambda$ (recall that $J \cap \Lambda$ is compact and J is open). Take $N > 0$ so large that $\alpha(n) = \beta(n)$ for $n \leq N$ implies $h(\alpha) \in W^u_\epsilon(h(\beta))$. Moreover take ϵ smaller than δ_1, so that the last relation in particular implies $\mathrm{d}(h(\alpha), h(\beta)) < \delta_1$. Then, $\alpha(n) = \beta(n)$ for $n \leq N$ implies $\mathrm{d}(h(\alpha), h(\beta)) < \delta_1$. Given $\overline{\theta} \in B^+(F(y), n), n \geq N, y \in J \cap \Lambda$, define $\theta \in B(A)$ by $\theta(m) = \overline{\theta}(m)$ for $m \geq 0$ and $\theta(m) = h^{-1}(y)(m)$ when $m \leq 0$. Arguing as before, θ is well defined and $h(\theta) \in W^u_\epsilon(y)$. Since $W^u_\epsilon(y) \subset J$ it follows that $h(\theta) \in J \cap \Lambda$. If we show that $h(\theta) \in S_{\delta_1}(y, n - N)$ it will follow that $\overline{\theta}$ (that satisfies $\overline{\theta} = F(h(\theta))$) belongs to $F(S_{\delta_1}(y, n - N))$. Hence $\overline{\theta} \in F(S_{\delta_1}(y, n - N))$ thus proving the inclusion $B(F(y), n) \subset F(S_{\delta_1}(y, n - N))$. To prove that $h(\theta) \in S_{\delta_1}(y, n - N)$ observe that $f^k(h(\theta)) = h(\sigma^k(\theta))$ and $\sigma^k(\theta)(m) = \theta(m + k)$ for all m and k. Hence

$$\sigma^k(\theta)(j) = \overline{\theta}(j + k) = h^{-1}(y)(j + k)$$

when $0 \leq j + k \leq n$. Hence

$$\sigma^k(\theta)(j) = h^{-1}(y)(j + k) = h^{-1}(f^k(y))(j)$$

for $0 \leq j \leq n - k$. Then, if $n - k \geq N$:

$$\mathrm{d}(h(\sigma^k(\theta)), f^k(y)) = \mathrm{d}(h(\sigma^k(\theta)), h(h^{-1}(f^k(y))) \leq \delta_1.$$

Since $h(\sigma^k(\theta)) = f^k(h(\theta))$:

$$\mathrm{d}(f^k(h(\theta)), f^k(y)) \leq \delta_1$$

when $n - k \geq N$, or, what is the same, $k \leq N - n$. This means $h(\theta) \in S_{\delta_1}(y, n - N)$. This completes the proof of $B(F(y), n) \subset F(S_{\delta_1}(y, n))$. These inclusions can be written as

$$B(F(y), n) \subset F(S_{\delta_1}(y, s)) \subset B(F(y), n - N)$$

and then

$$\nu(B(F(y), n)) \leq \mu(F(S_{\delta_1}(y, n)) \leq \nu(B(F(y), n - N)).$$

Now recall that if $\varphi \in C^\gamma(B^+(A), \mathbb{R})$ then, if ν_φ is given by 1.1, then there exists $C_1 > 0$ such that for all $\theta \in B^+(A)$ and $n \geq 0$:

$$C_1^{-1}\lambda(\varphi)^{-n}\exp(S_n\varphi)(\theta) \leq \nu_\varphi(B(\theta, n)) \leq C_1\lambda(\varphi)^{-n}\exp(S_n\varphi)(\theta).$$

Then, if $\log\lambda(\delta T\psi) = P(\delta T\psi) = 0$, it follows that

$$C_1^{-1}\exp(S_n\delta T\psi)(\theta) \leq \nu(B(\theta, n)) \leq C_1\exp(S_n\delta T\psi)(\theta) \qquad (2)$$

for all $\theta \in B^+(A)$ and $n \geq 0$. From (1) and (2) it follows that there exists $C_2 > 0$ such that for all $y \in J$ and $n \geq 0$:

$$C_2^{-1} \exp(S_n \delta T\psi)(F(y)) \leq \mu(F(S_{\delta_1}(y,n))) \leq C_2 \exp(S_n \delta T\psi)(F(y)).$$

But since we can write

$$(T\psi) \circ \pi = \psi + u \circ \sigma - u$$

where $u \in C^0(B(A), \mathbb{R})$, it follows that there exists $A > 0$ satisfying

$$|(S_n T\psi)(\pi(\theta)) - (S_n \psi)(\theta)| \leq A$$

for all $n \geq 0$ and $\theta \in B(A)$. Since $F = \pi h^{-1}$, we obtain,

$$\left|(S_n T\psi)(F(y)) - (S_n \psi)(h^{-1}(y))\right| \leq A.$$

But clearly

$$(S_n \psi)(h^{-1}(y)) = -\log\left|(f^n)'(y)/E_y^u\right|.$$

Hence

$$C_2^{-1} \leq \frac{\mu(S_{\delta_1}(y,n))}{|(f^n)'(y)/E^u|^{-\delta}} \leq C_2$$

for all $n \geq 0$ and $y \in \Lambda$. Define $\rho(y,n) = \mathrm{d}(y, J - S_{\delta_1}(y,n))$. By well known arguments (that require f to be at least C^2), there exists $C_3 > 0$ such that

$$C_3^{-1} \leq \frac{\operatorname{diam} S_{\delta_1}(y,n)}{\left|(f^n)'(y)/E_y^u\right|^{-1}} \leq C_3 \tag{4}$$

$$C_3^{-1} \leq \frac{\rho(y,n)}{\left|(f^n)'(y)/E_y^u\right|^{-1}} \leq C_3 \tag{5}$$

for all $y \in J \cap \Lambda$ and $n \geq 0$. Given a small $r > 0$ take $n > 0$ such that

$$\rho(y, n+1) \leq r \leq \rho(y,n).$$

Then, by (3) and (5),

$$\begin{aligned}
\mu(B_r(y)) &\leq \mu(S_{\delta_1}(y,n)) \\
&\leq C_2 \left|(f^n)'(y)/E^u_y\right|^{-\delta} \\
&\leq C_2 C_3^\delta \rho(y,n)^\delta \\
&= C_3 C_3^\delta r^\delta \left(\frac{\rho(y,n)}{r}\right)^\delta \\
&\leq C_2 C_3^\delta r^\delta \left(\frac{\rho(y,n)}{\rho(y,n+1)}\right)^\delta \\
&\leq C_2 C_3^\delta r^\delta \left(\frac{C_3\left|(f^n)'(y)/E^u_y\right|^{-1}}{C_3^{-1}\left|(f^{n+1})'(y)/E^u_y\right|^{-1}}\right)^\delta \\
&= C_2 C_3^{3\delta} r^\delta \left|f'(f^n(y))/E^u_{f^n(y)}\right|^{-\delta}
\end{aligned}$$

Hence, if C_u is an upper bound for $|f'(z)/E^u_z|^{-1}$, $z \in \Lambda$, it follows that

$$\mu(B_r(y)) \leq C r^\delta$$

with $C = C_2 C_3^{3\delta} C_4^{-\delta}$. In a similar way, but taking the maximum n such that

$$S_{\delta_1}(y,n) \subset B_r(y)$$

and using (4) instead of (5), a lower estimate of the form $\mu(B_r(y)) \geq C' r^\delta$ is obtained, completing the proof of Lemma 2.3.

Proof of Lemma 2.4. Let m be the dimension of the fibers of the unstable subbundle E^u of the hyperbolic set Λ. Let G be the Grassmannian bundle of m-dimensional subspaces of the fibers T_xM, i.e. G is the set of pair (x,E) with $x \in M$ and E being an m-dimensional subspace of T_xM endowed with its natural structure of smooth manifold. Associated to every $f \in \mathrm{Diff}^r(M)$ we have a diffeomorphism $F_f \in \mathrm{Diff}^{r-1}(G)$ defined by $F_f(p,E) = (f(p), f'(p)E)$. The map $\mathrm{Diff}^r(M) \ni f \to F_f \in \mathrm{Diff}^{r-1}(G)$ is C^∞. Given $0 \leq \gamma < 1$ and $g \in \mathrm{Diff}^r(M), r \geq 2$, define $\Phi_g : C^\gamma(\Lambda, G) \hookleftarrow$ by

$$\Phi_g(\xi)(x) = F_g(\xi(f^{-1}(x))).$$

When $\gamma = 0$ it is easy to check through standard techniques that the map $\xi_0 \in C^0(\Lambda, G)$ defined by $\xi_0(x) = (x, E^u_x)$ is a hyperbolic fixed point of the C^{r-1} map Φ_f. Moreover the map $\mathrm{Diff}^r(M) \times C^0(\Lambda, G) \ni (\xi, g) \to \Phi_g(\xi) \in C^0(\Lambda, G)$

is C^{r-1}. Hence there exists a C^r neighborhood $\mathcal{U}$ of f and a C^{r-1} map $\mathcal{U} \ni g \to \xi_g \in C^0(\Lambda, G)$ such that $\xi_f = \xi_0$ and $\Phi_g(\xi_g) = \xi_g$ for all $g \in \mu$. Let $\pi: G \to M$ be defined by $\pi(p, E) = p$. Then, well known methods show that $\pi \xi_g = h_g: \Lambda \to M$ is a topological equivalence between $f|\Lambda$ and $g|\Lambda_g$ and if $E^u(g)$ is the unstable subbundle of Λ_g, then $\xi_g(x) = (h_g(x), E^u_{h_g(x)}(g))$. From this it follows that the map $\mathcal{U} \ni g \to |(\det(g'/E^u(g)) \circ h_g| \in C^0(\Lambda, \mathbb{R})$ is C^{r-1}, thus proving (c). To prove (b) recall that it is well known that, since r is ≥ 2, taking $0 < \gamma < 1$ sufficiently small, the map ξ_0 is a hyperbolic fixed point of the C^{r-2} map $\Phi_g: C^\gamma(\Lambda, G) \hookleftarrow$. Then, if $\mathcal{U}$ is small enough, the map ξ_g obtained above is a C^{r-2} map $\mathcal{U} \ni g \to \xi_g \in C^\gamma(\Lambda, G)$. Then the map $\mathcal{U} \ni g \to |(\det(g'/E^u(g)) \circ h_g| \in C^\gamma(\Lambda, \mathbb{R})$ is C^{r-2}.

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