ON THE NUMBER OF LIMIT CYCLES WHICH APPEAR BY PERTURBATION OF SEPARATRIX LOOP OF PLANAR VECTOR FIELDS

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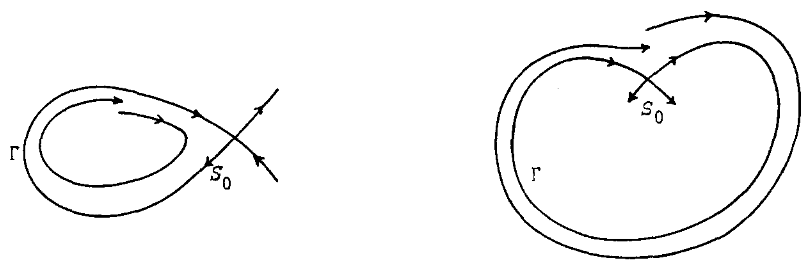
Consider a family of vector fields X_λ on the plane. This family depends on a parameter $\lambda \in \mathbb{R}^\Lambda$, for some $\Lambda \in \mathbb{N}$, and is supposed to be C^∞ in $(m,\lambda) \in \mathbb{R}^2 \times \mathbb{R}^\Lambda$.

Suppose that for $\lambda = 0$, the vector field X_0 has a *separatrix loop*. This means that X_0 has an hyperbolic saddle point s_0 and that one of the stable separatrix of s_0 coincides with one of the unstable one. The union of this curve and s_0 is the *loop* Γ. A return map is defined on one side of Γ.



Loops on the plane

Figure 1

We are interested in the number of limit cycles (isolated closed orbits) which may appear near Γ, for small values of λ. This problem was first studied by A.A. Andronov and others [A]. They showed that for 1-parameter families, with the condition that $\operatorname{div} X_0(s_0) \neq 0$, it appears at most one cycle. Next,

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L.A. Cherkas in [C], considered the question of the structure of the transition map near a saddle point, for a family of vector fields X_λ (below, I call it the "Dulac map" of the saddle). He derived from his study some results about the number of cycles. For example, he showed that if $\operatorname{div} X_0(s_0) = 0$ and if the Poincaré map of the loop is hyperbolic, then this number doesn't exceed 2.

I want to present a generalization of these results. Suppose that $\operatorname{div} X_0(s_0) = 0$. Then, it is known from Dulac [D], that the Poincaré map $P_0(x)$ of X_0, along the loop Γ has an expansion equal to: $\sum_{\substack{0\le j\le i\\ 1\le i}} a_{ij}x^i(Lnx)^j$. (This means that for each $k \in \mathbb{N}$, the Poincaré map is equal to a finite sum of the above serie for $0 \le j \le i \le i(k)$ and some $i(k) \in \mathbb{N}$, up to some C^k, k-flat function; k-flat means that all the derivatives are zero, at $x = 0$, up to the order k). In fact, if the function $P_0(x)-x$ is not C^∞-flat (i.e.: $a_{10} = 1$ and $a_{ij} = 0$ for $(i,j) \neq (1,0)$), then it is equivalent to $\beta_k x^k$ or $\alpha_{k+1}x^{k+1}Lnx$, β_k or $\alpha_{k+1} \neq 0$, for some $k \ge 1$. ($P_0(x)-x$ equivalent to $\beta_1 x$ means here that $P_0(x)$ is C^1 and hyperbolic). Now, the principal result is as follows:

Theorem A. Let X_λ, $\lambda \in \mathbb{R}^\Lambda$, a C^∞ family of vector fields on the plane, which has a separatrix loop Γ for $\lambda = 0$, at some hyperbolic saddle point s_0. Suppose that $\operatorname{div} X_0(s_0) = 0$. Let $P_0(x)$, the Poincaré map of X_0, relative to the loop Γ. Suppose that $P_0(x)-x$ is not flat. Then, for λ small enough, X_λ has an uniform finite number of limit cycles near Γ. More precisely, if $P_0(x)-x$ is equivalent to $\beta_k x^k$, with $\beta_k \neq 0$, then X_λ has at most $2k$ limit cycles for small λ, near Γ; if $P_0(x)-x$ is equivalent to $\alpha_{k+1}x^{k+1}Lnx, \alpha_{k+1} \neq 0$, then X_λ has at most $2k+1$ limit cycles. (Here, "near Γ, for λ small enough" means: there exist a neighborhood U of Γ in $\mathbb{R}^2$ and a neighborhood V of $0 \in \mathbb{R}^\Lambda$ such that X_λ has at most the specified finite number of limit cycles in U for $\lambda \in V$).

Remark. Recently, J.S. Il'Iasenko proved that, for any isolated loop of analytic vector field X_0 on the plane, the function $P_0(x)-x$ is not flat. (Isolated means here: isolated among the limit cycles) [I]. So, for analytic vector fields, the theorem A works in the following form:

Let X_λ an analytic vector field family on the plane, with an isolated loop Γ at $\lambda = 0$. Then, for λ small enough, has an uniform finite number of limit cycles near Γ.

Now I want to indicate why the non-flatness condition in the theorem A will be verified in any generic family of vector fields, depending on a finite number of parameters.

Definition: Let s an hyperbolic saddle point of a C^∞ vector field X, with $\operatorname{div} X(s) = 0$. Recall that the infinite-jet of X at s is C^∞-equivalent to:

$$J^\infty X(s) \underset{C^\infty}{\sim} x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \Big(\sum_{i\geq 1} \alpha_{i+1}(xy)^i\Big)y\frac{\partial}{\partial y}$$

(The C^∞-equivalence is the equivalence up a C^∞ diffeomorphism and multiplication by a positive C^∞ function). We say that is a *saddle of order* $k \geq 1$, if α_{k+1} is the first non zero coefficient α_i, in this expansion.

Remark: Let σ, τ, two transversal segments to the local stable and unstable manifolds of s, such that a transition map $D(x)$ is defined from σ to τ by the flow of X.

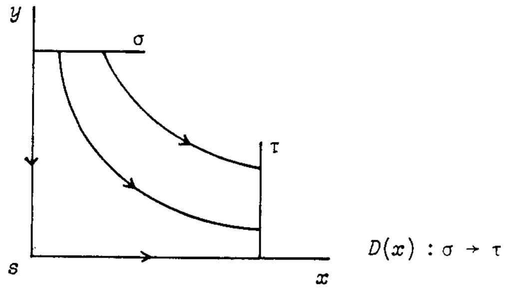

Figure 2

Then, it is easy to show that s is a saddle of order k if and only if $k+1$ is the order of the first unbounded derivative of $D(x)$ at $x = 0$. (In fact $D(x) \sim \alpha_{k+1} x^{k+1} Lnx$ in this case). So the notion of order does not depend on the above representation of $j^\infty X(s)$.

Now, we come back to a vector field X_0 with a saddle loop Γ at a saddle s_0, such that div $X_0(s_0) = 0$. Call $R(x)$ the Poincaré map of $- X_0$, from σ to τ:

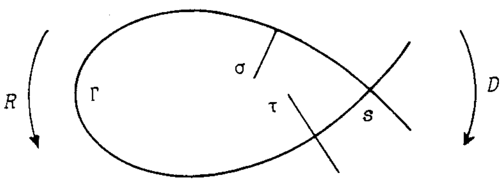

Figure 3

($R(x)$ is the Poincaré map above the regular part of Γ).

This map has a Taylor expansion equal to:

$$R(x) = x-\beta_0-\beta_1 x-\beta_2 x^2-\ldots\ldots-\beta_k x^k - \ldots$$

Clearly the coefficients $\alpha_1,\alpha_2,\ldots,\alpha_k,\ldots$ and $\beta_0,\beta_1,\ldots,\beta_k,\ldots$ are independent of each other. So, if X_0 belongs to a ℓ-parameter family of C^∞ vector fields, we can suppose *generically* that one of the $\ell+1$ first coefficients in the list: $\beta_0,\alpha_1,\beta_1,\alpha_2,\ldots,\beta_k,\alpha_k,\ldots$ is non zero. (Generically means: for X_λ in some open dense subset in the space of all ℓ-parameter families, with the compact-open C^∞ topology).

If β_k is this first non zero coefficient, then $P(x)-x \sim R^{-1}(x)-x$ is equivalent to $\beta_k x^k$. If α_{k+1} is the first one, $P(x)-x \sim D(x)-x \sim \alpha_{k+} x^{k+1} Lnx$ (As we will show in the following). So, we have the following generic corollary of the theorem A:

Corollary B: Let a C^∞ ℓ-parameter *generic* family of vector fields X_λ, $\lambda \in \mathbb{R}^\ell$, $\ell \geq 1$. Suppose that X_0 has a separatrix loop at a saddle point s_0. Then there exist at most ℓ limit cycles of X_λ near Γ, for λ small enough.

We are also interested to the case of a family which is a perturbation of an *Hamiltonian vector field*. This type of family has the following form:

$$X_\lambda = X_0 - \varepsilon\bar{X} + o(\varepsilon)$$

where $\lambda = (\varepsilon,\bar{\lambda})$ with ε near zero and $\bar{\lambda}$ in some finite dimensional space of parameters. We suppose also that X_0 is an hamiltonian vector field. This means that for some area-form Ω on $\mathbb{R}^2$, there exists a C^∞ function H, such that $X_0 \lrcorner \Omega = dH$. The vector field $\bar{X}$ depends on the parameter $\bar{\lambda}$ only. The term $o(\varepsilon)$ depends on $(m,\bar{\lambda},\varepsilon)$. We suppose that the level $\{H = 0\}$ contains a loop Γ at a saddle point s_0 of X_0 and that the levels $\{H = b\}$ for $b > 0$, near 0, contain closed curves Γ_b near $\Gamma = \Gamma_0$. We define the integral function $I(b,\bar{\lambda})$ by:

$$I(b,\bar{\lambda}) = \int_{\Gamma_b} \bar{\omega} \quad \text{where} \quad \bar{\omega} = \bar{X} \lrcorner \Omega.$$

It is known that this function is very interesting to study the limit cycles of X_λ for small $\varepsilon \neq 0$. In fact, if σ is a transversal segment to Γ, parametrized by the positive values of H, the Poincaré map P_λ of X_λ on σ, has the following expansion:

$$P_\lambda(b)-b = \varepsilon\int_{\Gamma_b} \bar{\omega} + o(\varepsilon).$$

It is easy to see that $I(b,\bar\lambda)$ admits an expansion equal to $\sum_{i\geq 0} [b_i(\bar\lambda)b^i + a_i(\bar\lambda)b^{i+1}Lnb]$ for C^∞ functions a_i, b_i in $\bar\lambda$. (The convergence is, as above, up to C^k, k-flat functions, for any k). The number of cycles near Γ is related to this expansion of I:

Theorem C: Let $X_\lambda = X_0 - \varepsilon\bar{X} + o(\varepsilon)$ a perturbation of an Hamiltonian vector X_0, defined as above. Suppose that $I(b,\bar\lambda_0) \sim b_k(\bar\lambda_0)b^k$ with $b_k(\bar\lambda_0) \neq 0$. Then X_λ has at most $2k$ cycles near Γ, for $\lambda = (\varepsilon, \bar\lambda)$ near $(0,\bar\lambda_0)$ and $\varepsilon \neq 0$. Suppose that $I(b,\bar\lambda_0) \sim a_k(\bar\lambda_0)b^{k+1}L \sim b$, with $a_k(\bar\lambda_0) \neq 0$. Then X_λ has at most $2k+1$ cycles near Γ, for λ near $(0,\bar\lambda_0)$ and $\varepsilon \neq 0$.

The proofs of theorems A and C are based on a structure theorem for the Dulac map of X_λ. Such a result was established by Cherkas in [C]. I present here alternative demonstration and formulation for the structure of the Dulac map, in finite class of differentiability, and not in analytical class as in [C]. I shall indicate also the relation between the coefficients of the normal form of X_λ at the saddle point, and the expansion of the Dulac map. Find this relation is important to obtain the precise bounds $2k$, $2k+1$ on the number of cycles, in the theorems A and C. We begin with the following:

Proposition D: Let X_λ a C^∞ family of vector fields, such that X_0 admits a saddle point s, with $\operatorname{div} X_0(s) = 0$. Then there exists a sequence $(\delta_N)_N$, $0<\ldots<\delta_{N+1} < \delta_N < \ldots < \delta_1$ and

C^∞ functions $\alpha_N(\lambda)$, defined on $W_N = \{\lambda \mid |\alpha_1(\lambda)| \leq \delta_N\}$ such that, for each N:

$$J^{2N+1}X_\lambda(s_\lambda) \underset{C^\infty}{\sim} x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (\sum_{i=0}^{N} \alpha_{i+1}(\lambda)(xy)^i)y\frac{\partial}{\partial y}$$

for $\lambda \in W_{N+1}$. Here, s_λ is the saddle point of X_λ near s_0 (s_λ is supposed to exist for $\lambda \in W_1$). The C^∞ equivalence, is the C^∞ equivalence of $(2N+1)$-jets: multiplication by positive C^∞ functions, and conjugacy by C^∞ diffeomorphisms, depending C^∞ on (x,y,λ). Of course the jets are taken only in the (x,y)-direction.

Now, it is known from S. Sternberg [S], that for each $K \in \mathbb{N}$, a given C^∞ vector field is always C^K-conjugate to its $(2N(K) + 1)$ polynomial jet, in a neighborhood of a given hyperbolic saddle, for some $N(K)$. The same resul is also availuable for λ-families, in a neighborhood of the saddle with conjugacies depending on the parameter. Combining this, with the proposition D, we obtain the following reduction of the family, in C^K class of differentiability:

Proposition E: Let a C^∞ family X, such that X_0 admits a saddle point s. Let some $K \in \mathbb{N}$. Then, in some neighborhood of the path $\{(s(\lambda),\lambda) \mid \lambda \in W_{N(K)+1}\}$ in $\mathbb{R}^2 \to \mathbb{R}^\Lambda$, the family is C^K-equivalent to the polynomial family of vector fields:

$$x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - (\sum_{i=0}^{N(K)} \alpha_{i+1}(\lambda)(xy)^i))y\frac{\partial}{\partial y}.$$

Here $s(\lambda)$ is the saddle of X_λ, near s_0, and the $\alpha_j(\lambda)$ are the functions defined in the proposition D. The C^K-equivalence is now the multiplication and conjugacy by functions and diffeomorphisms, depending C^K on (x,y,λ).

Remark: The C^K equivalence sends the saddle s_λ on the fixed point $0 \in \mathbb{R}^2$. Now an homothecy in $\mathbb{R}^2$ doesn't change the form of the polynomial vector field in the proposition E (It just modifies the values of the functions α_i). So, we can suppose that the image of the equivalence contains any given fixed neighborhood of $0 \in \mathbb{R}^2$ (For example the ball of radius 2).

So, it is sufficient to consider a polynomial family of vector fields:

$$X_\alpha = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \left(\sum_{i=0}^{N} \alpha_{i+1}(xy)^i\right) y\frac{\partial}{\partial y}$$

where $\alpha = (\alpha_1,\ldots,\alpha_{N+1})$. Let $\sigma = \{x \geq 0,\ y = 1\}$ and $\tau = \{y \geq 0,\ x = 1\}$, two transversal segments, in the same quarter $\{x,y \geq 0\}$ of the saddle. We call Dulac map D_α of X_α, relative to σ, τ, the transition map defined by the flow of X_α, from σ to τ (Of course we parametrize σ by x, and τ by y).

We suppose that we restrict α to the neighborhood of $0 \in \mathbb{R}^{N+1}$ defined by: $|\alpha_1| < \frac{1}{2}$, $|\alpha_i| < M$ for $2 \leq i \leq N+1$ and some $M > 0$. Then the Dulac map D_α is defined on some neighborhood of $0 \in \sigma$ independant of α. (We take $D_\alpha(0) = 0$). In fact $D_\alpha(x)$ is analytic in (x,α) for $x > 0$. We want to make precise the nature of D_α at $x = 0$. For this, we introduce the function:

$$\omega(x,\alpha_1) = \frac{x^{-\alpha_1}-1}{\alpha_1}.$$

Note that for each $k > 0$, $x^k\omega \to -x^k Lnx$ as $\alpha_1 \to 0$ (Uniformely for $x \in [0,X]$ for any $X > 0$). We are going to consider finite conbinations of the functions $x^i\omega^j$ with $i,j \in \mathbb{N}$ and $0 \leq j \leq i$. These functions $x^i\omega^j$ form a totally ordered set with the following order: $x^i\omega^j \prec x^{i'}\omega^{j'} \iff i' > i$ or $i = i'$ and $j > j'$ $(1 \prec x\omega \prec x \prec x^2\omega^2 \prec x^2\omega \prec x^2 \prec \ldots)$.

The notation $x^i\omega^j + \ldots$ means that after the sign + one finds a finite combination of $x^{i'}\omega^{j'}$ of order stricly greater than $x^i\omega^j$. Then, we have the following structure for D_α:

Theorem F. Let any $K \in \mathbb{N}$. Then the Dulac map D_α of X_α (relative to the segments σ, τ defined above) has the following expansion:

$$D_\alpha(x) = x + \alpha_1 \; [x\omega + \ldots] + \alpha_2 [x^2\omega + \ldots] + \ldots + \alpha_{N+1} [x^{N+1}\omega + \ldots] + \psi_K$$

where each term between brackets is a finite combination of $x^i\omega^j$ (with the above convention); the coefficients of the non written $x^i\omega^j$ after the signs + are C^∞ functions in α, which are zero for $\alpha = 0$. The remaining term ψ_K is a C^K-function in (x,α), which is K - flat for $x = 0$, and any $\alpha \cdot (\psi_K(0,\alpha) = \ldots =$
$= \ldots = \frac{\partial^K \psi_K}{\partial x^K}(0,\alpha) = 0)$.

Remark: The expressions in the brackets depend on K. But the ordered expansion of $D_\alpha(x)$ in term of the $x^i\omega^j$ is unique. Next, if we take $K \leq N$ (which is always possible), we can reduce the brackets up to the monomials $x^i\omega^j$ with $i \geq K+1$. (Because these nomomials are C^K and K-flat). So the expansion of $D_\alpha(x)$ reduces to:

$$D_\alpha(x) = x + \alpha_1 \; [x\omega + \ldots] + \ldots + \alpha_K [x^K\omega + \ldots] + \phi_K$$

with ϕ_K, C^K and K-flat, and the brackets depending only on the $x^i\omega^j$ for $0 \leq j \leq i \leq K$.

A natural generalization of loops are the singular hyperbolic cycles (made by hyperbolic saddles and separatrices). I think there are some difficulties to extend the above results to the perturbations of general such cycles. Of course, it would be very interesting to have results for non-hyperbolic singular cycles.
I wish also to emphasize that the expansion of the map D_α in term of functions $x^i\omega^j$ is of the type introduced by A. Hovansky in [H] and the proofs of the theorems A, C below use arguments similar to those used by A. Hovanski.

I - Normal form of a family of vector fields near a saddle point (Proof of the proposition D)

Let X_λ a family of vector fields as in the statemente of proposition D. One may suppose that X_λ is defined on some fixed neighborhood V of $0 \in \mathbb{R}^2$, which contains for each $\lambda \in W_1$, $W_1 = \{\lambda \mid |\alpha_1| < \delta_1\}$, a saddle point at $0 \in \mathbb{R}^2$ as unique singular point. We may also suppose that there exist coordinates (x,y) in V such that:

$$J^1 X_\lambda(0) = x\frac{\partial}{\partial x} - (1 - \alpha_1(\lambda))y\frac{\partial}{\partial y} \tag{1}$$

where $\alpha_1(\lambda)$ is a C^∞ function of $\lambda \in W_1$, with $\alpha_1(0) = 0$.

I want to establish the proposition D by an induction on N. The formula (1) is the first step of this induction for $N = 1$.

So, suppose that one has found $\delta_1 > \delta_2 > \ldots > \delta_{N+1} > 0$ and C^∞ functions $\alpha_1, \ldots, \alpha_{N+1}$, $\alpha_i : W_i \to \mathbb{R}$, such that for $\lambda \in W_N$:

$$J^{2N+1} X_\lambda(0) \underset{C^\infty}{\sim} x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \left[\sum_{i=0}^{N} \alpha_{i+1}(\lambda)(xy)^i\right] y\frac{\partial}{\partial y} \tag{N+1}$$

(The equivalence "$\underset{C^\infty}{\sim}$" being defined in the statement of prop. D).

Consider the $(2N+3)$-jet. The formula $(N+1)$ gives that:

$$J^{2N+3} X_\lambda(0) \underset{C^\infty}{\sim} X_\lambda^N + Y_{2N+2}(\lambda) + Y_{2N+3}(\lambda) \tag{$(N+2)_1$}$$

where X_λ^N is the right term of $(N+1)$ and $Y_{2N+2}(\lambda)$, $Y_{2N+3}(\lambda)$ are C^∞ maps of W_{N+1} in V_{2N+2}, V_{2N+3} respectively (V_L designates the space of homogeneous polynomial vector fields of degree L).

Let $\rho^L_{\alpha_1}$ the Lie bracket operator:

$$Z \in V_L \to [X_{\alpha_1}, Z] \in V_L$$

where X_{α_1} is the 1-jet: $X_{\alpha_1} = x\frac{\partial}{\partial x} - (1-\alpha_1)\, y\frac{\partial}{\partial y}$. For $\alpha_1 = 0$, ρ_0^{2N+2} is inversible. So, one may choose δ_{N+2}, $0 < \delta_{N+2} < \delta_{N+1}$, small enough to have $\rho^{2N+2}_{\alpha_1(\lambda)}$ inversible for each $\lambda \in W_{N+2}$. Then one can resolve the equation:

$$[X_{\alpha_1(\lambda)}, U_{2N+2}(\lambda)] = Y_{2N+2}(\lambda)$$

with $U_{2N+2}(\lambda)$ a C^∞ map of W_{N+2} in V_{2N+2}.

The diffeomorphism $Id - U_{2N+2}(\lambda)$ brings the jet $X^N_\lambda + Y_{2N+2} + Y_{2N+3}$ on a jet $X^N_\lambda + Y'_{2N+3}$, with Y'_{2N+3}, a C^∞ map of W_{N+2} in V_{2N+3}.

Let now: $N_0 = \operatorname{Ker} \rho^{2N+3}_0 = (xy)^{N+1}\{x\frac{\partial}{\partial x}, y\frac{\partial}{\partial y}\}$. This kernel is a supplement space of $B_0 = \operatorname{Image}(\rho^{2N+3}_0)$. So, ρ^{2N+3}_0 is an isomorphism of B_0 onto itself. By continuity the space $B_\lambda = \rho^{2N+3}_{\alpha_1(\lambda)}(B_0)$ is of codimension 2 in V_{2N+3}. Taking perhaps a smaller δ_{N+2}, we can suppose that B_λ is transversal to N_0 for each $\lambda \in W_{N+2}$

So, we can find (unique) C^∞ maps $V'_{2N+3}(\lambda)$ and $W'_{2N+3}(\lambda)$ of W_{N+2} in B_0 and N_0 respectively, such that:

$$Y'_{2N+3}(\lambda) = [X_{\alpha_1(\lambda)}, U'_{2N+3}(\lambda)] + W'_{2N+3}(\lambda).$$

The diffeomorphism $Id - U'_{2N+3}(\lambda)$ brings the jet $X^N_\lambda + Y'_{2N+3}(\lambda)$ on the jet $X^N_\lambda + W'_{2N+3}(\lambda)$. Now:

$$\begin{aligned} W'_{2N+3}(\lambda) &= \beta(\lambda)(xy)^{N+1}x\frac{\partial}{\partial x} + \gamma(\lambda)(xy)^{N+1}y\frac{\partial}{\partial y} \\ &= \beta(\lambda)(xy)^{N+1}(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + (\beta(\lambda) + \gamma(\lambda))(xy)^{N+1}y\frac{\partial}{\partial y} \end{aligned}$$

So we have:

$$X^N_\lambda + W'_{2N+3} = (1+\beta(\lambda)(xy)^{N+1})(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}) + (\sum_{i=0}^{N} \alpha_{i+1}\cdot(xy)^i)y\frac{\partial}{\partial y} + (\beta+\gamma)(xy)^{N+1}y\frac{\partial}{\partial y}$$

and, dividing by $1 + \beta(xy)^{N+1}$, we obtain:

$$\begin{aligned} J^{2N+3}\left(\frac{X^N_\lambda + W'_{2N+3}}{1+\beta(xy)^{N+1}}\right) &= x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + (\sum_{i=0}^{N} \alpha_{i+1}\cdot(xy)^i)y\frac{\partial}{\partial y} - \alpha_1\beta\cdot(xy)^{N+1}y\frac{\partial}{\partial y} \\ &+ (\beta+\gamma)\cdot(xy)^{N+1}y\frac{\partial}{\partial y} \end{aligned}$$

This jet is C^∞-equivalent to the initial one, in the formula $(N+2)_1$. So, we have proved that:

$$J^{2N+3}X_\lambda(0) \underset{C^\infty}{\sim} x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \Big(\sum_{i=0}^{N+1} \alpha_{i+1}(\lambda)\cdot(xy)^i\Big)y\frac{\partial}{\partial y} \qquad (N+2)$$

for $\lambda \in W_{N+2}$, with $\alpha_{N+2}(\lambda) = -\alpha_1(\lambda)\cdot\beta(\lambda) + \beta(\lambda) + \gamma(\lambda)$.

II - The structure of the Dulac map. (Proof of Th. F)

Let a given constant $M > 0$. We consider all the analytic families X in normal form:

$$X_\alpha = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \Big(\sum_{i=0}^{\infty} \alpha_{i+1}\cdot(xy)^i\Big)\, y\frac{\partial}{\partial y} \qquad (1)$$

where $P_\alpha(u) = \sum_{i=0}^{\infty} \alpha_{i+1} u^{i+1}$ is an *analytic entire function* of $u \in \mathbb{R}$, with $\alpha \in A$ where A is the set of a α defined by: $A = \{\alpha \mid |\alpha_1| < \frac{1}{2},\ |\alpha_i| < M \text{ for } i \geq 2\}$. Let the transversal segments σ, τ and the Dulac map $D_\alpha(x)$ defined as in the introduction. Observing the normal form above, it is natural to make the singular change of coordinates $(u = xy,\ x = x)$.

The differential equation for trajectories of X_α:

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y + \Big(\sum_0^{\infty} \alpha_{i+1}(xy)^i\Big)y \end{cases} \qquad (2)$$

is brought in the following equation:

$$\begin{cases} \dot{x} = x \\ \dot{u} = P_\alpha(u) = \sum_{i=1}^{\infty} \alpha_i\cdot u^i \end{cases} \qquad (3)$$

We see that in (3) the variables (x,u) are separated.

The first equation gives no trouble. So, we concentrate ourself on the second equation: $\dot{u} = P_\alpha(u)$ (4) which is analytic in $|u| \leq 1$ for each α as specified above. Call $u(t,u)$ the trajectory of this equation (solution of (4), such that $u(0,u) = u$).

This function is analytic for each t, in some neighborhood of $u = 0$. So we can expand $u(t,u)$:

$u(t,u) = \sum_{i=1}^{\infty} g_i(t)u^i$ (5), with $g_1(t) = e^{\alpha_1 t}$ and $g_i(0) = 0$ for all $i \geq 2$.

We want to study the form of the g_i and the convergence of the above series, in function of t. For this, we are going to compare $u(t,u)$ to the solution of the hyperbolic equation:

$$\dot{U} = \tfrac{1}{2} U + \sum_{i=1}^{\infty} M U^{i+1} \tag{6}$$

We have the following estimations:

Lemma 1: Let $U(t,u) = \sum_{i=1}^{\infty} G_i(t)u^i$ the power serie expansion of the trajectory of (6). Then for each $i \geq 1$ and $t \geq 0$:

$$|g_i(t)| \leq G_i(t) \quad \text{(for any } \alpha \in A).$$

Proof: Substituing (5) in the equation: $\frac{\partial u}{\partial t}(t,u) = P_\alpha(u(t,u))$ we obtain recurrent equations for the $g_i(t)$, the system E_g:

$$\begin{aligned}
\dot{g}_1(t) &= \alpha_1 g_1 \\
\dot{g}_2(t) &= \alpha_1 g_2 + \alpha_2 g_1^2 \\
\dot{g}_3(t) &= \alpha_1 g_3 + 2\alpha_2 g_1 g_2 + \alpha_3 g_1^3
\end{aligned}$$

and more generally:

$$\dot{g}_i = \alpha_1 g_i + P_i(\alpha_2, \ldots, \alpha_i, g_1, \ldots, g_{i-1}) \quad \text{for } i \geq 2$$

where P_i is a rational polynomial in $\alpha_2, \ldots \alpha_i, g_1, \ldots, g_{i-1}$ with positive coefficients.

Now, $U(t,u)$ is the trajectory of $\dot{U} = P_\alpha(U)$ with $\alpha = (\frac{1}{2}, M, M, \ldots)$. So we have for the $G_i(t)$, the system E_G:

$$\begin{aligned} \dot{G}_1 &= \tfrac{1}{2} G_1 \\ \dot{G}_2 &= \tfrac{1}{2} G_2 + M G_1^2 \\ &\vdots \end{aligned}$$

and more generally:

$$\dot{G}_i = \tfrac{1}{2} G_i + P_i\ (M, \ldots, M, G_1, \ldots G_{i-1})$$

(with the same polynomial P_i as above).

We can resolve the system E_G by:

$$G_1(t) = e^{\frac{1}{2}t},\ G_2(t) = \psi_2(t) e^{\frac{1}{2}t} \quad \text{with} \quad \psi_2(t) = \int_0^t e^{-\frac{1}{2}\tau} \cdot M \cdot G_1^2\, d\tau$$

and more generally:

$$G_i(t) = \psi_i(t) e^{\frac{1}{2}t} \quad \text{with} \quad \psi_i(t) = \int_0^t e^{-\frac{1}{2}\tau} P_i(M, \ldots M, G_1(\tau), \ldots, G_{i-1}(\tau))\, d\tau$$

It follows easily from these formulas, that $G_i(t) > 0$ for $t > 0$.

Now, we are going to show the estimations $|g_i(t)| \leq G_i(t)$ for each $t \geq 0$. First, it is true for $i = 1$:

$$|g_1(t)| \leq e^{|\alpha_1| t} \leq e^{\frac{1}{2}t} = G_1(t).$$

Suppose now that we have shown that $|g_j(t)| \leq G_j(t)$ for each j: $1 \leq j \leq i-1$, and $t \geq 0$.

We compare the two equations:

$$\begin{cases} \dot{g}_i(t) = \alpha_1 g_i + P_i(\alpha_2, \ldots, \alpha_i, g_1, \ldots, g_{i-1}) \\ \\ \dot{G}_i(t) = \frac{1}{2} G_i + P_i(M, \ldots, M, G_1, \ldots, G_{i-1}). \end{cases}$$

Because the coefficients of P_i are positive, we have:

$$|P_i(\alpha_2,\ldots,\alpha_i,g_1,\ldots,g_{i-1})| \le P_i(|\alpha_2|,\ldots,|\alpha_i|,|g_1|,\ldots,|g_{i-1}|) \le$$
$$\le P_i(M,\ldots,M,G\ ,\ldots,G_{i-1}).$$

Now, for $t = 0$, we have $G_1(0) = 1$ and $G_i(0) = 0$ for $i \ge 2$. So, we have $G_i(0) = P_i(M,\ldots,M,G_1(0),\ldots,G_{i-1}(0)) = MG_1(0)^i = M$ and also $|\dot g_i(0)| \le |\alpha_i||g_1(0)|^i \le |\alpha_i| < M$.

So, for $t = 0$ we have $g_i(0) = G_i(0) = 0$ and $|\dot g_i(0)| < \dot G_i(0)$. This give,s by continuity, for t small enough:

$$|\dot g_i(t)| < \dot G_i(t).$$

We want to show that this inequality is availuable for $\forall\ t \ge 0$. (and so we will have: $|g_i(t)| \le G_i(t)$ for $\forall\ t \ge 0$). On the contrary, suppose that $t_0 > 0$ is the inferior bound of the values t, such that $|\dot g_i(t)| \ge \dot G_i(t)$. For all $t \in [0,t_0]$ we have: $|\dot g_i(t)| \le \dot G_i(t)$. So for all $t \in [0,t_0]$ we also have:

$$|g_i(t)| \le G_i(t).$$

Now, for $t = t_0$:

$$\dot g_i(t_0) = \alpha_1 g_i(t_0) + P_i(\alpha_2,\ldots,\alpha_i,g_1(t_0),\ldots,g_{i-1}(t_0))$$

$$\dot G_i(t_0) = \tfrac{1}{2}\, G_i(t_0) + P_i(M,\ldots,M,G_1(t_0),\ldots,G_{i-1}(t_0)).$$

By induction on i, we know that $G_j(t_0) \ge |g_j(t_0)|$ for $i \le j \le i-1$. By the choice of t_0, we have already notice that $G_i(t_0) \ge |g_i(t_0)|$. So the inequality $|\alpha_1| < \frac{1}{2}$ implies:

$$|\dot g_i(t_0)| < \dot G_i(t_0).$$

But, by continuity this strict inequality is availuable for the $t > t_0$, t near t_0: this last point contradicts the definition of t_0.

Next, we prove the following:

Lemma 2: There exists constants C, $C_0 > 0$ such that:

$$|g_i(t)| \le C_0\left[Ce^{t/2}\right]^i \text{ for any } i \ge 1,\ t \ge 0 \text{ and any } \alpha \in A.$$

Proof: Using the lemma 1, it is sufficient to show that $G_i(t) \le C_0|Ce^{t/2}|^i$ for some constants C_0, C, $i \ge 1$, $t \ge 0$, $\alpha \in A$. Recall that the function $U(t,u) = \sum_{i\ge 1} G_i(t)u^i$ is the trajectory of an hyperbolic vector field: $X = P(u)\frac{\partial}{\partial u}$ with $P(u) = \frac{1}{2}u + M\sum_{i=2}^{\infty} u^i$.

From a theorem of H. Poincaré on the analytic linearization, there exists an analytic diffeomorphism $g(u) = u+\dots$, converging for $|u| \le K_1$, for some $K_1 > 0$, such that:

$$g_*(P(u)\frac{\partial}{\partial u}) = \frac{1}{2}u\frac{\partial}{\partial u}.$$

This diffeomorphism sends the flow $U(t,u)$ of $P\frac{\partial}{\partial u}$ into the flow $U_0(t,u) = ue^{\frac{1}{2}t}$ of $\frac{1}{2}u\frac{\partial}{\partial u}$. This means:

$$U_0(t,g(u)) = g\ U(t,u) \quad \text{for } |u|,|U(t,u)| \le K_1.$$

Because $g(u)$ is inversible for $|u| \le K_1$, there exist constants a, $0 < a < A$ such that:

$$a|u| \le |g(u)| \le A|u| \quad \text{for } |u| \le K_1.$$

Suppose that $|u| \le \frac{a}{A}K_1e^{-\frac{1}{2}t}$. Then $|g(u)| \le A|u| \le aK_1e^{-\frac{1}{2}t}$

$|U_0(t,g(u))| = |g(u)|e^{\frac{t}{2}} \le aK_1$. Now $U(t,u) = g^{-1}\circ U_0(t,g(u))$.

This implies that: $|U(t,u)| \le \frac{1}{a}|U_0(t,g(u))| \le K_1$. Now, using inequalities of Cauchy for the coefficients $G_i(t)$, we find:

$$|G_i(t)| \le \frac{\operatorname{Sup}\{|U(t,u)|\ |\ |u|=R(t)\}}{|R(t)|^i} \le \frac{K_1}{|R(t)|^i} \text{ if } R(t) = \frac{a}{A}K_1e^{-\frac{t}{2}}.$$

So, we obtain:

$|G_i(t)| \leq K_1 (\frac{A}{a} K_1^{-1})^i e^{\frac{it}{2}}$ which is the desired estimation with $C_0 = K_1$ and $C = \frac{A}{a} K_1^{-1}$.

We will show below that the functions $g_i(t)$ are analytic functions of $t > 0$. For the moment, we notice that the formula: $\frac{\partial u}{\partial t}(t,u) = P_\alpha(u(t,u))$, shows that the series in u of $\frac{\partial u}{\partial t}$ has the same radius of convergence that $u(t,u)$. (Recall that $P_\alpha(u)$ is supposed to be an entire function). The same is true for any derivative $\frac{\partial^k u}{\partial t^k}(t,u)$, by an induction on k. This remark gives an estimate for the coefficients $\frac{d^k g_i}{dt^k}(t)$ of the derivative: $\frac{\partial^k u}{\partial t^k} = \sum_{i \geq 1} \frac{d^k g_i}{dt^k} u^i$, using the Cauchy inequality along the circle of radius $R(t) = \frac{a}{A} K_1 e^{-\frac{1}{2}t} = Ce^{-\frac{1}{2}t}$ as above:

$$|\frac{d^k g_i}{dt^k}(t)| \leq \frac{\text{Sup}\left\{|\frac{\partial^k u}{\partial t^k}(t,u)| \,\middle|\, |u| = R(t)\right\}}{|R(t)|^i}$$

which gives: $|\frac{d^k g_i}{dt^k}(t)| \leq C_k (Ce^{t/2})^i$ for some $C_k > 0$. So, we have:

Lemma 3: For each $k \geq 0$, there exists a constant $C_k > 0$ such that:

$$|\frac{d^k g_i}{dt^k}(t)| \leq C_k [C \cdot e^{t/2}]^i \quad \text{for any } i \geq 1,\ t \geq 0 \text{ and } \alpha \in A.$$

(Here C is the same constant as in lemma 2).

We will give now some precisions about the form of the functions $g_i(t)$. For this, we introduce the function:

$$\Omega(\alpha_1,t) = \frac{e^{\alpha_1 t}-1}{\alpha_1} \quad \text{for} \quad t \neq 0 \quad \text{and}$$

$\Omega(0,t) = t$. With this notation we have:

Proposition 4: For each $k \geq 1$, $g_k(t) = e^{\alpha_1 t} Q_k(t)$ where Q_k is a polynomial of degree $\leq k-1$ in Ω. The coefficients of Q_k are polynomials in $\alpha_1,\dots,\alpha_k$. More precisely:

$$Q_k = \alpha_k \Omega + \bar{Q}_k(\alpha_1,\dots,\alpha_k,\Omega)$$

where $\bar{Q}_k$ is a polynomial of degree $\leq k-1$ in Ω with coefficients in $J(\alpha_1,\dots,\alpha_{k-1}) \cap J(\alpha_1,\dots,\alpha_k)^2 \subset \mathbb{Z}[\alpha_1,\dots,\alpha_k]$ ($J(u,v,\dots)$: for the polynomial ideal generated by $u,v,\dots$).

Proof: Write again the system E_g for the g_i:

$$\dot{g}_1 = \alpha_1 g_1$$

$$\dot{g}_2 = \alpha_1 g_2 + \alpha_2 g_1^2$$

$$\vdots$$

$$g_k = \alpha_1 g_k + P_k(\alpha_2,\dots\alpha_k,g_1,\dots,g_{k-1})$$

The polynomial P_k is obtained from the coefficient of u^k in the expansion $\sum_{j\geq 2} \alpha_j \left[\sum_{i\geq 1} g_i u^i\right]^j$. It follows easily that P_k is homogeneous linear in $\alpha_2,\dots,\alpha_k$. Each monomial $g_1^{\ell_1}\dots g_{k-1}^{\ell_{k-1}}$ is such that:

$$\sum_{j=1}^{k-1} \ell_j \geq 2 \quad \text{and} \quad \sum_{j=1}^{k-1} j\cdot\ell_j = k. \qquad (*)$$

First we show that $g_k(t) = e^{\alpha_1 t} Q_k(t)$ with Q_k a polynomial in Ω of degree $\leq k-1$, with coefficients, polynomials in $\alpha_1,\ldots,\alpha_k$ (i.e.: $g_1(t) = e^{\alpha_1 t}$, $g_2(t) = \alpha_2 e^{\alpha_1 t}\cdot\Omega,\ldots$).

Look at the equation for g_k:

$$\dot{g}_k = \alpha_1 g_k + P_k(\alpha_2,\ldots,\alpha_k,\ g_1,\ldots,g_{k-1})$$

and use an induction in k. We suppose known that for each $j\leq k-1$: $g_j(t) = e^{\alpha_1 t} Q_j(t)$ with $\deg(Q_j) \leq j-1$. Notice that: $e^{\alpha_1 t} = \alpha_1\Omega+1$ So, each g_j is of degree $\leq j$ in Ω. Now, it follows from the first inequality in (*) that:

$P_k(\alpha_2,\ldots,\alpha_k,g_1,\ldots,g_k) = e^{2\alpha_1 t} X_k(\Omega)$, where X_k is a polynomial of degree $\leq k-2$ in Ω (To see this point, replace in each monomial $g_1^{\ell_1}\ldots g_{k-1}^{\ell_{k-1}}$ of P_k, a product of two factors $g_i g_j$ by $e^{2\alpha_1 t} Q_i Q_j$ and the other factors g_ℓ by $(\alpha_1\Omega+1)Q_\ell$).

Now, $g_k = e^{\alpha_1 t} Q_k$ with:

$$Q_k(t) = \int_0^t e^{-\alpha_1 t}\ P_k(\alpha_2,\ldots,\alpha_k,g_1,\ldots,g_{k-1})d\tau$$

$$Q_k(t) = \int_0^t e^{\alpha_1 t} X_k(\Omega)d\tau = \int_0^t X_k(\Omega)\dot{\Omega}\,d\tau$$

(Because $\dot{\Omega} = e^{\alpha_1 t}$).

So, we see that $Q_k(t)$ is a polynomial of degree $\leq k-1$ in Ω. From the induction it follows easily that the coefficients are polynomials in $\alpha_1,\ldots,\alpha_k$. To obtain the precise form of the statement, notice that for $k \geq 2$:

$$P_k(\alpha_2,\ldots,\alpha_k,g_1,\ldots,g_{k-1}) = \alpha_k g_1^k + \tilde{P}_k$$

where $\tilde{P}_k$ is linear homogeneous in $\alpha_2,\ldots,\alpha_{k-1}$ and each monomia in $\tilde{P}_k$ contains at least one of the g_i with $i \geq 2$. But, we know that the coefficients of such a g_i are divisible by $\alpha_1,\ldots,\alpha_i$. So, the coefficients in $\tilde{P}_k$ are in $J(\alpha_1,\ldots,\alpha_{k-1}) \cap J(\alpha_1,\ldots,\alpha_k)^2$.

Now: $Q_k = \alpha_k \int_0^t e^{(k-1)\alpha_1\tau} d\tau + \int_0^t e^{-\alpha_1\tau}\tilde{P}_k(\tau)d\tau$

Look first at the term $\int_0^t e^{(k-1)\alpha_1\tau} d\tau$:

$$\int_0^t e^{(k-1)\alpha_1\tau} d\tau = \frac{e^{(k-1)\alpha_1\tau}-1}{(k-1)\alpha_1}.$$

Use again: $e^{\alpha_1\tau} = \alpha_1\Omega+1$. We obtain:

$$e^{(k-1)\alpha_1 t} = 1 + (k-1)\alpha_1\Omega+\alpha_1^2 S(\Omega)$$

where $S(\Omega)$ is a polynomial in Ω.

So, we have: $\alpha_k \int_0^t e^{(k-1)\alpha_1 t} = \alpha_k\Omega + \frac{\alpha_k\alpha_1}{k-1} S(\Omega)$.

The term $\int_0^t e^{-\alpha_1\tau} \tilde{P}_k d\tau$ gives a polynomial in Ω, with coefficients in $J(\alpha \ldots\alpha_{k-1}) \cap J(\alpha_1\ldots\alpha_k)^2$. So, we obtain finally: $Q_k(t) = \alpha_k\Omega + \bar{Q}_k$ with $\bar{Q}_k$ as in the statement.

We go back to the map $D_\alpha(x)$. The time to go from σ to τ is equal to:

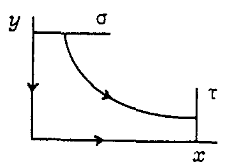

Figure 4

$t(x) = -Ln\ x$ (where $(x,1) \in \sigma$ is a given point on σ).

Now, we have $u|_\sigma = x$ and $u|_\tau = y$. So, we can calculate $D_\alpha(x)$ as the value $u(t,u)$ for $u = x$ and $t = t(x) = -Ln\, x$:

$$D_\alpha(x) = u(-Lnx,x) \quad \text{for} \quad x \geq 0.$$

(We extend D_α in 0, by $D_\alpha(0) = 0$).

There is no problem to see that D_α is well defined for $x \in [0,X]$, where X is some value greater than 0, and is analytic, for $x \neq 0$. We want to study its behavior in $x = 0$. For this, we notice that the lemma 2 implies that for each $t > 0$, the convergence radius of the serie $\sum g_i(t)u^i$ is greater than $\frac{1}{C}e^{-\frac{1}{2}t}$. So, for any x small enough, the serie $\sum_i g_i(t)x^i$ converges for each $t < -2Lnx$ and in particular for $t = -Ln\, x$. So we can utilise the expansion $\sum_i g_i(t)u^i$ to calculate $D_\alpha(x)$:

$$D_\alpha(x) = \sum_{i=1}^{\infty} g_i(-Lnx)x^i.$$

The convergence is normal on an interval $[0,X]$ for some $X > 0$. Now, we can utilize the estimates on g_i, $\frac{d^k g_i}{dt^k}$ of lemmas 2, 3 to obtain the following:

Proposition 5: Let any $k \in \mathbb{N}$. Then there exists a $K(k)$ such that:

$$D_\alpha(x) = \sum_{i=1}^{K(k)} g_i(-Lnx)x^i + \psi_k$$

where ψ_k is a C^k function in (x,α), k-flat at $x = 0$.

Proof: Given k, we want to find $K(k)$ such that:

$$D_\alpha^k(x) = \sum_{K+1}^{\infty} g_i(-Lnx)x^i \quad \text{is a } C^k,\ k\text{-flat function.}$$

We are going to see that the series D_α^K can be derived term by term. First, we have:

$$\frac{d}{dx}[g_j(-Lnx)x^j] = -g_j^{(1)}(-Lnx)x^{j-1} + jg_j(-Lnx)x^{j-1}$$

(where $g_j^{(1)} = \frac{dg_j}{dx}$).

Now, from the estimations of lemma 3 we have:

$$|g_j^{(1)}(-Lnx)| \leq C_1|C\cdot x|^{-\frac{j}{2}}$$

And from lemma 2:

$$|g_j(-Lnx)| \leq C_0|C\cdot x|^{-\frac{j}{2}}.$$

So, for some constant M_1, we have:

$$|\frac{d}{dx}(g_j(-Lnx)x^j)| \leq jM_1|C.x|^{\frac{j}{2}-1}$$

More generally, using lemma 3, we have for each $s \leq j$:

$$|\frac{d^s}{dx^s} g_j(-Lnx)x^j| \leq \frac{j!}{(j-s)!} M_s|C\cdot x|^{\frac{j}{2}-s}$$

for some constant M_s depending on s.

It follows from this, that if $K > 2k$ and if $0 \leq s \leq k$, the series:

$$\sum_{j\geq K+1} \frac{d^s}{dx^s} |g_j(-L_n(x))x^j|$$ converges and is equal to zero

for $x = 0$.

So, we obtain that the function $\sum_{j\geq K+1} \ldots = D_\alpha^{k\,\prime}$ is k-flat and C^k.

Suppose now that $P_\alpha(u) = \sum_{i=1}^{N+1} \alpha_i u^i$ *is a polynomial as in the introduction.* We show how to rearrange the expansion $D_\alpha(x)$ to derive the theorem F of the introduction from the propositions 4 and 5 above (with K replaced by k).

First, as in the introduction, we introduce:

$$\omega(\alpha_1, x) = \frac{x^{-\alpha_1} - 1}{\alpha_1} = \Omega(\alpha_1, -Lnx).$$

The proposition 4 gives us the following:

$$g_k(Lnx) = e^{-\alpha_1 Lnx} Q_k(-Lnx)$$
$$= x^{-\alpha_1}[\alpha_k\omega + \bar{Q}_k(\alpha_1, \ldots, \alpha_k, \omega)]$$

with $\bar{Q}_k$ of degree $\leq k-1$ in ω, and coefficients in $J(\alpha_1, \ldots, \alpha_{k-1}) \cap J(\alpha_1 \ldots \alpha_k)^2$. So, the general term $g_k(-Lnx)x^k$ in $D_\alpha(x)$ is equal to:

$$g_k(-Lnx)x^k = x^{k-\alpha_1}(\alpha_k\omega + \bar{Q}_k).$$

This term can be rewrite as: (using $x^{-\alpha_1} = \alpha_1\omega+1$)

$$g_k(-Lnx)x^k = \alpha_k x^k\omega + \alpha_1\alpha_k x^k\omega^2 + x^k(1+\alpha_1\omega)\bar{Q}_k(\alpha_1, \ldots, \alpha_k, \omega)$$

for $k \geq 2$ and $xg_1(-Lnx) = x^{1-\alpha_1} = \alpha_1 x\omega + x$.

So, we have:

$$D_\alpha(x) = x + \alpha_1 x\omega + \alpha_2 x^2\omega + \alpha_1\alpha_2 x^2\omega^2 + x^2 + x^3(1+\alpha_1\omega)\bar{Q}_2 +$$
$$+ \alpha_3 x^3\omega + \alpha_1\alpha_3 x^3\omega^3 + x^3(1+\alpha_1\omega)\bar{Q}_3 + \ldots + \psi_k$$

where $+\ldots$ is for the expansion of the $x^s g_s(-Lnx)$ for $4 \leq s \leq K(k)$ (The coefficients α_i are taken to be zero for $i > N+1$).

Now, we rearrange the sum $\sum_{i=1}^{K(k)} g_i(-Lnx)x^i$ in the following way: first, we take all the terms whose coefficient is divisible by α_1. Next, all the remaining terms (not divisible by α_1) but divisible by α_2 and so on, until α_{N+1}. We obtain the following expansion:

$$D_\alpha(x) = x+\alpha_1 [x\omega+\alpha_2 x^2\omega+x^2\omega\bar{Q}_2+\alpha_3 x^3\omega^3+x^3\omega\bar{Q}_3+\ldots]$$
$$+ \alpha_2 [x^2\omega+\text{terms in } x^3\bar{Q}_3,\ldots x^K\bar{Q}_K \text{ divisible by } \alpha_2, \text{ not by } \alpha_1]$$
$$\vdots$$
$$+ \alpha_N [x^N\omega+\text{terms in } x^{N+1}\bar{Q}_{N+1},\ldots,x^K\bar{Q}_K \text{ div. by } \alpha_N, \text{ not by } \alpha_1,\ldots,\alpha_{N-1}]$$
$$+ \alpha_{N+1} x^{N+1}\omega + \psi_k.$$

From the above expansion it is clear that each term after $x^s\omega$ in the bracket relative to α_s is of order greater that $x^s\omega$ and has coefficients in $(\alpha_1,\ldots,\alpha_{N+1})$ (because it comes from a term with coefficients in $J(\alpha_1\ldots\alpha_{N+1})^2$, next divided by α_s). The sum is stopped at α_{N+1} because $\alpha_i = 0$ for $i > N+1$. The function ψ_k is C^k in (x,α), k-flat in x. So, we have verified all the statements of the theorem F.

III - Finiteness of the number of cycles in the generic case (Theorem A).

As in the statement of Theorem A, we suppose that X_λ, $\lambda \in \mathbb{P}^\Lambda$, is a C^∞ family of vector fields such that:

1) For $\lambda = 0$, X_0 has a loop (saddle connexion) Γ at some hyperbolic saddle point s.

2) div $X_0(s) = 0$.

3) The Poincaré map P_0 of X_0 around Γ, relative to some transversal segment σ parametrized by $x \geq 0$, is such that:

"Case β_k": $P_0(x)-x = \beta_k x^k+o(x^k)$ with $\beta_k \neq 0$ or

"Case α_{k+1}": $P_0(x)-x = \alpha_{k+1} x^{k+1} Lnx+o(x^{k+1} Lnx)$ with $\alpha_{k+1} \neq 0$, for some $k \geq 1$.

The proposition E (which is a direct consequence of the proposition D proved in part II) shows that for any $K \in \mathbb{N}$, we can choose a C^K change of coordinates around the saddle point s_λ of X_λ, bringing this vector field in the following normal form, defined in the ball V with coordinates (x,y), $x^2+y^2 \leq 4$:

$$X_\lambda = x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} + \left(\sum_{i=0}^{N(K)} \alpha_{i+1}(\lambda)(xy)^i\right)y\frac{\partial}{\partial y}$$

where the functions $\alpha_j(\lambda)$ are C^∞ on some neighborhood W of $0 \in \mathbb{R}^\Lambda$, and $N(K) \in \mathbb{N}$ is some number depending on K. For what follows, it will suffice to take $K > 2k+1$.

We can also suppose that the change of coordinates is chosen so that the Poincaré map P_0 is defined on $\sigma = \{y=1,\ x \geq 0\}$, near 0. Let also $\tau = \{x=1\}$.

For $\lambda \in W$, the Dulac map $D_\lambda(x)$ is defined from a neighborhood of $0 \in \sigma$ (parametrized by $x \geq 0$) to τ (parametrized by y). We can extend the chart V in a C^K-chart defined in a neighborhood of Γ. This chart is an union $V \cup V^1$ where V^1 is a neighborhood of the regular part of Γ, between σ and τ. The vector field X_λ is C^K on V^1.

Now, let $R_\lambda(x)$, the map from σ to τ defined, in a neighborhood of $0 \in \sigma$, by the flow of $-X_\lambda$. This map is differentiable of class C^K. So, we can write it:

$$R_\lambda(x) = x - [\beta_0(\lambda) + \beta_1(\lambda)x + \beta_2(\lambda)x^2 + \dots + \beta_K(\lambda)x^K + \phi_K]$$

with ϕ_K a C^K function in (x,λ), K-flat at $x = 0$. The functions $\beta_0,\dots,\beta_K$ are at least continuous. (In fact, $\beta_j(\lambda)$ is of class $K-j$).

Now, the Poincaré map relative to σ is equal to: $P_\lambda = R_\lambda^{-1} \circ D_\lambda$. It is clear that the case β_K is equivalent to:

$$\beta_0(0) = \dots\dots\dots = \beta_{k-1} = 0,\ \beta_k(0) = \beta_k \neq 0 \text{ and } \alpha_1(0) = \dots\dots\dots = \alpha_k(0) =$$

The case α_{k+1} is equivalent to:

$\beta_0(0) = \ldots\ldots = \beta_k(0) = 0, \quad \alpha_1(0) = \ldots\ldots = \alpha_k(0) = 0$ and $\alpha_{k+1}(0) = \alpha_{k+1} \neq 0$.

To look for the fixed points of P_λ we prefer to consider the map $\Delta_\lambda = D_\lambda - R_\lambda$: the fixed points of P_λ will correspond to the zeros of Δ_λ. Choosing $N(K) > K$ in the theorem F (which is always possible), we can write:

$$D_\lambda(x) = D_{\alpha(\lambda)}(x) = x + \alpha_1(\lambda)[x\omega + \ldots.] + \ldots + \alpha_K(\lambda)[x^K\omega + \ldots] + \psi_K.$$

So that:

$$\Delta_\lambda(x) = \beta_0(\lambda) + \alpha_1(\lambda)[x\omega + \ldots] + \beta_1(\lambda)x + \alpha_2(\lambda)[x^2\omega + \ldots] + \ldots$$

$$+ \beta_{K-1}(\lambda)x^{K-1} + \alpha_K(\lambda)[x^K\omega + \ldots] + \psi_K + \phi_K.$$

Using the remark after the statement of theorem F in the introduction we can write:

$$\Delta_\lambda(x) = \beta_0(\lambda) + \alpha_1(\lambda)[x\omega + \ldots] + \ldots + \beta_k(\lambda)x^k + \alpha_{k+1}(\lambda)\cdot x^{k+1}\omega + \ldots\ldots + \Phi_k$$

where the functions ψ_K, ϕ_K, Φ_K are C^K, K-flat in $x = 0$.
The precise meaning of the notation: $+\ldots$, *is given in the introduction*.

To study the number of zeros of Δ_λ, we have to extend somewhat the algebra generated by the $x^i\omega^j$. We introduce now the algebra of functions, continuous in (x,λ) which are finite combinations of the monomials $x^{\ell+n\alpha_1}\omega^m$, $\ell, n \in \mathbb{Z}$, $m \in \mathbb{N}$, $\alpha_1 = \alpha_1(\lambda)$, with coefficients, any continuous functions of λ. (We call it the algebra of *admissible functions*).

Of course, we consider also the monomials as functions of (x,α_1), but when we consider combinations of monomials, α_1 is always replaced by the function $\alpha_1(\lambda)$.
Now, we introduce between the monomials, the following *partial strict* order:

$$x^{\ell'+n'\alpha_1}\omega^{m'} \prec x^{\ell+n\alpha_1}\omega^{m} \Longleftrightarrow \begin{cases} \ell' < \ell \text{ or} \\ \ell' = \ell,\ n'=n \text{ and } m'>m \end{cases}$$

(Notice that $x^{\ell+n'\alpha_1}\omega^{m'}$ and $x^{\ell+n\alpha_1}\omega^{m}$ with $n \neq n'$, are not ordered).

Later on, *the notation: $f+\ldots$ where f is a monomial will mean that after the sign + there exists a (non precised) finite combination of monomials g_i, with $g_i \succ f$.* (This notation extends the one defined in the introduction). *We also use the symbol $*$ to replace any continuous function of λ, non zero at $\lambda=0$, and we write $\dot{\phi}$ for the derivation in x: $\dot{\phi} = \frac{\partial\phi}{\partial x}$.*
With these conventions, we indicate now some easy properties of the algebra of admissible functions.

a) Let g, f two monomials with $g \succ f$; then $\frac{g}{f}(x,\alpha_1) \to 0$ for $(x,\alpha_1) \to (0,0)$. This follows from the two following observations: $\omega \geq \mathrm{Inf}(\frac{1}{|\alpha_1|}, -Lnx)$ and $x^{s(\alpha_1)}\omega^m \to 0$ (for any continuous function $s(\alpha_1)$, with $s(0) > 0$), if $(x,\alpha_1) \to (0,0)$, and $m \in \mathbb{N}$.

b) Let a monomial $f \succ 1$. Then $f(x,\alpha_1) \to 0$ for $x \to 0$ (uniformely, for α_1 bounded): $f \succ 1$ means that $f = x^{\ell+n\alpha_1}\omega^m$ with $\ell \geq 1$, and we can use the same argument as in a).

c) $f_1 \succ f_2$ and any $g \Longrightarrow gf_1 \succ gf_2$.

d) Let $f = x^{\ell+n\alpha_1}\omega^m$. Then:

$$\dot{f} = [\ell+(n-m)\alpha_1]x^{\ell-1+n\alpha_1}\omega^m - mx^{\ell-1+n\alpha_1}\omega^{m-1}.$$

From this formula follows easily:

e) Let $f = x^{\ell+n\alpha_1}\omega^m$ with $\ell \neq 0$, and g any monomial such that $g > f$. Then $\dot{g}$ is a combination of two monomials g' and g'' and $\dot{f} = *f' + \ldots$ with $f' \prec g'$, $f' \prec g''$.

We shall also use rational functions of the algebra of the following type: $\frac{f+\ldots\ldots}{1+\ldots\ldots}$. (*The admissible rational functions*). For them, we have:

$$\text{f)}\quad \left(\frac{x^{\ell+n\alpha_1}\omega^m+\ldots\ldots}{1+\ldots\ldots\ldots\ldots}\right)^{\cdot} = \star\,\frac{x^{\ell-1+n\alpha_1}\omega^m+\ldots\ldots}{1+\ldots\ldots\ldots\ldots\ldots}\quad \text{if}\quad \ell\neq 0.$$

We can give now a proof of theorem A. We shall consider successively the two cases α_{k+1} and β_k.

A. Proof of Theorem A in the case α_{k+1}

Recall that:

$$\Delta_\lambda(x)=\beta_0+\alpha_1[x\omega+\ldots]+\beta_1 x+\alpha_2[x^2\omega+\ldots]+\ldots+\alpha_k[x^k\omega+\ldots]+\beta_k x^k+\alpha_{k+1}x^{k+1}\omega+\ldots+\psi_K.$$

where α_i, β_j are continuous functions; ψ_K is a C^K function of (x,λ), K-flat in x, with $K > 2k+1$. Next, we suppose that $\beta_0(0) = \ldots = \beta_k(0) = 0$, $\alpha_1(0) = \ldots = \alpha_k(0) = 0$ and $\alpha_{k+1}(0)\neq 0$.

From the property d) above it follows:

$$(x^j\omega)^{\cdot} = (j-\alpha_1)x^{j-1}\omega+\ldots \quad \text{if}\quad j\neq 0 \quad\text{and}\quad \dot\omega = x^{-1-\alpha_1}.$$

So, deriving Δ_λ, we obtain, using also property e):

$$\dot\Delta_\lambda = \alpha_1[\star\omega+\ldots]+\beta_1+\alpha_2[\star x\omega+\ldots]+\ldots+\star\alpha_{k+1}x^k\omega+\ldots+\dot\psi_K$$

(For the notations $\star$, $+\ldots$, see the conventions introduced above). If we derive Δ_λ, $k+1$ *times*, we find:

$$\Delta_\lambda^{(k+1)}(x) = \alpha_1[\star x^{-k-\alpha_1}+\ldots]+\alpha_2[\star x^{-(k-1)-\alpha_1}+\ldots]+\ldots+\star\alpha_{k+1}\omega+\ldots+\psi_K^{(k+1)}$$

All the monomials $\beta_j x^j$, for $j\leq k$, have disappeared. Multiplying by $x^{k+\alpha_1}$, we obtain (use property c)):

$$x^{k+\alpha}\Delta_\lambda=\alpha_1[\star 1+\ldots]+\alpha_2[\star x+\ldots]+\ldots+\star\alpha_{k+1}x^{k+\alpha_1}\omega+\ldots+x^{k+\alpha_1}\psi_K^{(k+1)} \qquad (1)$$

(Above and afterwards each bracket designates an admissible function).

Locally (in some neighborhood of $\lambda=0$, $x=0$), the zeros of $\Delta_\lambda^{(k+1)}$ are zeros of the following function $\xi_1 = \dfrac{x^{k+\alpha_1}\Delta_\lambda^{(k+1)}}{[*]+\ldots\ldots]}$ where the denominator is the function with coefficient α_1 in (1).

$$\xi_1=\alpha_1+\alpha_2\frac{*x+\ldots\ldots}{*]+\ldots\ldots} + \alpha_3\frac{*x^2+\ldots}{*]+\ldots} +\ldots+\alpha_k\frac{*x^{k-1}+\ldots}{*]+\ldots\ldots} + \frac{*\alpha_{k+1}x^{k+\alpha_1}\omega+\ldots}{*]+\ldots\ldots\ldots\ldots} + \phi_1$$

Here, $\phi_1 = \dfrac{x^{k+\alpha_1}\psi_K^{(k+1)}}{*]+\ldots\ldots\ldots}$ is a C^{K-k-1} function, at least $K-k-1$ flat in $x=0$. Using the property f), we have:

$$\dot{\xi}_1 = \alpha_2\frac{*]+\ldots}{*]+\ldots} + \ldots + \alpha_k\frac{*x^{k-2}+\ldots}{*]+\ldots\ldots} + \frac{*\alpha_{k+1}x^{k-1+\alpha_1}\omega+\ldots}{*]+\ldots\ \ldots\ldots\ldots} + \phi_2$$

where $\phi_2 = \dot{\phi}_1$ is C^{K-k-2}, $K-k-2$ flat in $x = 0$; $\dot{\xi}_1 = \alpha_2 u_1+\ldots$ where u_1 is inversible as an rational admissible function. Let $\xi_2 = u_1^{-1}\dot{\xi}_1$ and derive again ξ_2:

$$\dot{\xi}_2 = \alpha_3\frac{*]+\ldots}{*]+\ldots} + \ldots + \dot{\phi}_2.$$

We write it $\dot{\xi}_2 = \alpha_3 u_2 + \ldots$ where u_2 is inversible as admissible rational function. We define $\xi_3 = u_2^{-1}\dot{\xi}_2$, and so on. By this way, we find a sequence of functions: $\xi_1,\xi_2,\ldots,\xi_k$ such as ξ_j is the product of $\dot{\xi}_{j-1}$ by some inversible admissible rational function. For the last one ξ_k, we have:

$$\xi_k = \alpha_k + \frac{*\alpha_{k+1}x^{1+\alpha_1}\omega\ \ldots}{*]+\ldots\ldots\ldots\ldots} + \phi_k$$

where ϕ_k is C^{K-2k}, $(K-2k)$-flat.

Deriving a last time, we obtain:

$$\dot{\xi}_k = \frac{*\alpha_{k+1} x^{\alpha_1} + \dots}{*1 + \dots\dots\dots\dots} + \dot{\Phi}_k.$$

Then, using the fact that $\dot{\Phi}_k$ is C^{K-2k-1}-flat, with $K-2k-1>0$ and the property a), we obtain that:

$$x^{-\alpha_1}\omega^{-1}\dot{\xi}_k = *\alpha_{k+1} + o(1).$$

(Where the term $o(1)$ is continuous). The assumption $\alpha_{k+1}(0)\neq 0$ implies that locally $x^{-\alpha_1}\omega^{-1}\dot{\xi}_k$ and also $\dot{\xi}_k$ are non zero for small (λ,x) $(x \geq 0)$. So, the function ξ_k has at most one zero, for small (λ,x), ξ_{k-1}, at most 2 zeros, and so on: ξ_1 has at most k most k zeros locally. Now ξ_1 has at least the same number of zeros as $\Delta_\lambda^{(k+1)}$, so finally we obtain that the map Δ_λ has at most $2k+1$ zeros for small (λ,x).

B. Proof of Theorem A in the case β_k

We derive the map Δ_λ only k times:

$$\Delta_\lambda^{(k)}(x) = \alpha_1\left[*x^{-k+1-\alpha_1} + \dots\right] + \dots + \alpha_k\left[*\omega + \dots\right] + *\beta_k + \dots + \psi_K^{(k)}$$

and introduce, next:

$$\xi_1 = \frac{\Delta_\lambda^{(k)}(x)}{\left[*x^{-k+1-\alpha_1} + \dots\right]} = \alpha_1 + \alpha_2\frac{*x + \dots}{*1 + \dots} + \dots + \frac{*\alpha_k x^{k-1+\alpha_1}\omega + *\beta_k x^{k-1+\alpha_1} + \dots}{+1 + \dots\dots\dots\dots\dots\dots} + \Phi_1$$

where Φ_1 is C^{K-k}, $(K-k)$-flat in $x = 0$.

As in paragraph A, we define a sequence of functions $\xi_1, \dots, \xi_{k-1}$ with ξ_j equal to $\dot{\xi}_{j-1}$ multiplied by an inversible admissible rational function. The last function ξ_{k-1} is equal to:

$$\xi_{k-1} = *\alpha_{k-1} + \frac{*\alpha_k x^{1+\alpha_1}\omega + *\beta_k x^{1+\alpha_1} + \ldots}{*1 + \ldots\ldots\ldots\ldots\ldots\ldots} + \Phi_{k-1}$$

and then:

$$\dot{\xi}_{k-1} = \frac{*\alpha_k x^{\alpha_1}\omega + *\beta_k x^{\alpha_1} + \ldots}{*1 + \ldots\ldots\ldots\ldots\ldots} + \dot{\Phi}_{k-1}$$

where $\dot{\Phi}_{k-1}$ is of classe C^{K-2k+1}, $(K-2k+1)$-flat.

We take now ξ_k as:

$$\xi_k = x^{-\alpha_1}\omega^{-1}\cdot[*1+\ldots]\ \dot{\xi}_{k-1} = *\alpha_k + *\beta_k \frac{*1+\ldots}{*1+\ldots}\cdot\frac{1}{\omega} + \Phi_k$$

where the bracket is the denominator in the expression of $\dot{\xi}_{k-1}$.
The function Φ_K is C^{K-2k}, $(K-2k)$-flat.

If we derive ξ_k, we obtain:

$$\dot{\xi}_k = *\beta_k \frac{x^{-1-\alpha_1} + \ldots}{*1+\ldots\ldots} \cdot \frac{1}{\omega^2} + \dot{\Phi}_k.$$

and:

$$\omega^2 \frac{*1+\ldots\ldots}{*x^{-1-\alpha_1}+\ldots}\cdot\dot{\xi}_k = *\beta_k + \omega^2 \frac{*1+\ldots\ldots}{*x^{-1-\alpha_1}+\ldots}\cdot\dot{\Phi}_k.$$

The rest is $o(1)$. So, because $\beta_k(0) \neq 0$, we have that $\dot{\xi}_k \neq 0$ from (λ,x) small enough. It follows easily that the map Δ_λ has at most $2k$ zeros for small (λ,x).

IV - Finiteness of the number of cycles for a perturbed Hamiltonian vector field (Proof of Theorem C)

As in the statement of Theorem C, we suppose that the family takes the special form:

$$X_\lambda = X_0 + \varepsilon\bar{X} + o(\varepsilon) \quad \text{where} \quad \lambda = (\varepsilon,\bar{\lambda}).$$

For $\varepsilon=0$, the hamiltonian vector field X_0 is C^∞ equivalent to $x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y}$. It follows from this that the functions $\alpha_i(\lambda)$ in the normal form are divisible by ε: $\alpha_i(\lambda) = \varepsilon\bar{\alpha}_i(\varepsilon,\bar{\lambda})$ for some C^∞ function $\bar{\alpha}_i$. So, the proposition E gives a C^K-normal form equal to:

$$x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y} - \varepsilon\left[\sum_{i=0}^{N(K)} \bar{\alpha}_{i+1}(\lambda)(xy)^i\right] y\,\frac{\partial}{\partial y}.$$

It suffices now to consider a polynomial family X_α with $\alpha = \varepsilon\bar{\alpha}$, $\bar{\alpha} = (\bar{\alpha}_1,\dots,\bar{\alpha}_{N+1})$. From the proof of theorem F in the part II, it is clear that the function $D_\alpha(x)-x$ is also divisible by ε. This means that there exists some C^K function $\bar{\Psi}_K(x,\alpha)$, K-flat in $x=0$, such that:

$$D_\alpha(x) = x+\varepsilon(\bar{\alpha}_1[x\omega+\dots]+\dots\bar{\alpha}_K[x^K\omega+\dots] + \bar{\psi}_K)$$

where $\omega = \dfrac{x^{-\alpha_1}-1}{\alpha_1}$ with $\alpha_1 = \varepsilon\bar{\alpha}_1$. (We choose $N(K) > K$).

Return now to the initial family X_λ. As in the part III, we can choose some C^K-chart around of the loop Γ, transversal segments σ, τ for which, the transition maps are respectively, the Dulac map : $D_\lambda(x) = D_{\alpha(\lambda)}(x)$ and a map $R_\lambda(x)$ such that $R_\lambda(x)-x$ is also divisible by ε:

$$R_\lambda(x) = x-\varepsilon(\bar{\beta}_0+\bar{\beta}_1 x +\dots+\bar{\beta}_K x^K+\bar{\Phi}_K)$$

where the $\bar{\beta}_j$ are continuous functions of λ and $\bar{\Phi}_K$ a C^K function of (x,λ) which is K-flat in $x=0$.

Now, the map $\Delta_\lambda = D_\lambda-R_\lambda$ is equal to $\Delta_\lambda = \varepsilon\tilde{\Delta}_\lambda$ with:

$$\tilde{\Delta}_\lambda = \bar{\beta}_0+\bar{\alpha}_1[x\omega+\ldots]+\ldots+\bar{\alpha}_K[x^K\omega+\ldots]+\bar{\beta}_K x^K+\Phi_K$$

for some C^K, K-flat function Φ_K.

As in the part III, we say that we are in the $\bar{\beta}_k$ or $\bar{\alpha}_{k+1}$ case at $\bar{\lambda}_0$ if $\bar{\beta}_k(0,\bar{\lambda}_0)$ or $\bar{\alpha}_{k+1}(0,\bar{\lambda}_0)$ is the first non zero coefficient in the expansion of $\tilde{\Delta}_{(0,\bar{\lambda}_0)}$. The zeros of the map Δ_λ pour $\varepsilon \neq 0$ are the zeros of $\tilde{\Delta}_\lambda$, and if $(\varepsilon,\bar{\lambda}) \to (0,\bar{\lambda}_0)$, $\alpha_1(\lambda) \to 0$. So, the study of the part III allows the following conclusion: in the case $\bar{\beta}_k$, the map Δ_λ has at most $2k$ zeros for $(\varepsilon,\bar{\lambda})$ near $(0,\bar{\lambda}_0)$, $\varepsilon \neq 0$; in the case $\bar{\alpha}_{k+1}$, the map Δ_λ has at most $2k+1$ zeros for $(\varepsilon,\bar{\lambda})$ near $(0,\bar{\lambda}_0)$, $\varepsilon \neq 0$.

It remains to show how the two cases $\bar{\alpha}_{k+1}$, $\bar{\beta}_k$ are related to the *expansion of the integral* I. Recall that:

$$I(b,\bar{\lambda}) = \int_{\Gamma_b} \bar{\omega}, \quad \bar{\omega} = \bar{X}\lrcorner\Omega \qquad dH=X_0\lrcorner\Omega$$

where Γ_b is a cycle of the Hamiltonian function H, near the loop. We suppose that these cycles are defined for $b > 0$. ($\{b=0\}$ corresponds to the loop). To compare $I(b,\bar{\lambda})$ to the Δ_λ-map we change the parametrization b by the parametrization x. ($b(x)$ is a diffeomorphism of the segment σ, preserving 0). So we take: $I(x,\bar{\lambda}) = I(b(x),\bar{\lambda})$.

Now, notice that:

$$\Delta_\lambda(x) = P_\lambda(x)-x + o(\varepsilon). \quad \text{So:}$$

$$P_\lambda(x)-x = \varepsilon\tilde{\Delta}_\lambda + o(\varepsilon).$$

If we compare this expression to the one using I, given in the introduction, we obtain that:

$\bar{\Delta}_\lambda(x) = I(x,\bar{\lambda}) + \phi(x,\bar{\lambda},\varepsilon)$ where ϕ is some function tending to 0, for $\varepsilon \to 0$. It follows from this that, for each λ:

$$I(x,\bar{\lambda}) = \bar{\Delta}_{\bar{\lambda}}(x) \quad \text{where} \quad \bar{\Delta}_{\bar{\lambda}}(x) = \tilde{\Delta}_{(0,\bar{\lambda})}(x).$$

(In fact, we have to notice that $\tilde{\Delta}_\lambda(x)$ is continuous in ε, because $x^i\omega^j \to x^i(Lnx)^j$, uniformely in x, when α_1 and also $\varepsilon \to 0$, for each $i > 0$). Return to the map $\tilde{\Delta}_\lambda$:

$$\tilde{\Delta}_\lambda = \bar{\beta}_0+\bar{\alpha}_1[x\omega+\ldots]+\bar{\beta}_1 x+\ldots + \bar{\beta}_k x^k+\bar{\alpha}_{k+1}x^{k+1}\omega+\ldots + \Phi_K.$$

In each bracket $[x^i\omega+\ldots]$, $i \leq k$, the term $+\ldots$ is zero for $\alpha_1\ldots =\ldots=\alpha_k = 0$. So, this term is divisible by ε. It follows that:

$$\bar{\Delta}_{\bar{\lambda}}(x)=\bar{\beta}_0(0,\bar{\lambda})+\bar{\alpha}_1(0,\bar{\lambda})xLnx+\bar{\beta}_1(0,\bar{\lambda})x+\ldots+\bar{\beta}_k(0,\bar{\lambda})x^k+\bar{\alpha}_{k+1}(0,\bar{\lambda})x^{k+1}Lnx+o(x^{k+1}Lnx).$$

Now, if $I(b,\bar{\lambda}_0) \sim b_k(\bar{\lambda}_0)b^k$ with $b_k(\bar{\lambda}_0) \neq 0$, we have in the x-coordinate:

$$I(x,\bar{\lambda}_0) = \bar{\Delta}_{\bar{\lambda}_0}(x) \sim \bar{\beta}_k(0,\bar{\lambda}_0)x^k \quad \text{with} \quad \bar{\beta}_k(0,\bar{\lambda}_0) \neq 0.$$

So we are in the "case $\bar{\beta}_k$". Also, if $I(b,\bar{\lambda}_0) \sim a_k(\bar{\lambda}_0)b^{k+1}Lnx$, then $I(x,\bar{\lambda}_0) \sim \bar{\alpha}_{k+1}(0,\bar{\lambda}_0)x^{k+1}Lnx$ with $\bar{\alpha}_{k+1}(0,\bar{\lambda}_0) \neq 0$, if $a_k(\bar{\lambda}_0)\neq 0$ and we are in the case $\bar{\alpha}_{k+1}$.

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