Classification of gradient-like flows on dimensions two and three*

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Abstract

We consider here gradient-like flows of class C^r ($r \geq 1$) on a closed manifold M of class C^{r+1} and dimension two or three. We study the classification of these flows by the relation of topological equivalence. In this sense, the flows which are more relevant are the polar flows (only one source and only one sink).

To each polar flow X on M^3 it is univocally associated a classical Heegaard-diagram; we construct this diagram as the intersection of the invariant manifolds of the saddles of X with an intermediate level surface, and we get:

Theorem 1a – *The polar flows X_1, X_2 on M^3 are topologically equivalent if and only if their associated Heegaard-diagrams are topologically isomorphic.*

Theorem 2 – *Given a Heegaard-diagram E of M there exists a polar flow X on M, whose associate diagram is E.*

Then, the polar flows on M^3 are classified by the Heegaard-diagrams.

We define next parceled Heegaard-diagrams. To each parceled diagram E, we associate univocally a 2-dimensional simplicial complex $K(E)$, which has the following properties:

Theorem 3 – *M is S^3 if and only if K may be embedded – in a certain restricted way – in $\mathbb{R}^3$.*

Theorem 4 – *If E is a parceled Heegaard-diagram of M, then $\pi_1(M) = = \pi_1(K)$.*

To each polar flow X on M^2, we associate a circular distribution of a finite number of points; these points are colored, in the sense that the points

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are associated by pairs, and – if M is not orientable – each point has also a spin. We construct this distribution on an analogous way to the 3-dimensional case, and get:

Theorem 1b – *The polar flows X_1, X_2 on M^2 are topologically equivalent if and only if their associated circular distributions of colored points are isomorphic.*

For each M^2, we give a complete description of the circular distributions of points which can be associated to polar flows on M.

We must mention here that M. Peixoto [8] has a classification of these flows on M^2 (indeed Morse-Smale flows) from a point of view completely different to the one we adopt here.

To an arbitrary gradient-like flow on M^2 or M^3 we associate a stratification which generalizes the Heegaard-diagrams (on M^3) and the circular distributions of colored points (on M^2); then we prove Theorem 1c, which is a generalization of 1a and 1b.

We consider the first order bifurcations of gradient-like flows, in the sense of Sotomayor ([[7], [18]). We characterize each of these bifurcations by certain transformations T_F, T_P, T_S, T_T (and their inverses) of the associated stratifications.

A theorem of Newhouse-Peixoto ([7]) has then as corollary the following:

Theorem 5 – *If E_1, E_2 are two stratifications (read Heegaard-diagrams, or circular distributions of colored points) associated to the gradient-like flows X_1, X_2, then these flows are on the same M if and only if E_1 can be transformed into E_2 by a finite number of transformations of the types T_F, T_P, T_T, T_S, and their inverses.*

In the case of the Heegaard-diagrams, this theorem coincide with a classical theorem of J. Singer (Theor. 7 of [13]).

We thank C. Camacho, J. Palis, S. Newhouse, M. Peixoto and J. Sotomayor for helpful conversations.

Preliminaries

We recall here some definitions.

A vector-field (flow) on a compact manifold is said to be gradient-like [15] if:

i) It has a finite number of singularities, which are all hyperbolic.

ii) The stable and unstable manifolds of the singularities have transversal intersection.

iii) The α-limit (ω-limit) of every orbit is some of the singularities.

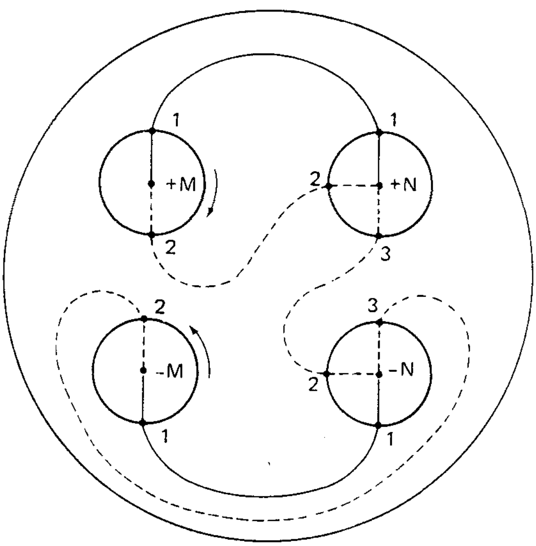
A flow is said to be polar [5] if it has only one source and only one sink.

Two flows X_1, X_2 on the manifolds M_1, M_2 are said to be topologically equivalent if there is a homeomorphism h: $M_1 \to M_2$ which send oriented orbits of X_1 onto oriented orbits of X_2.

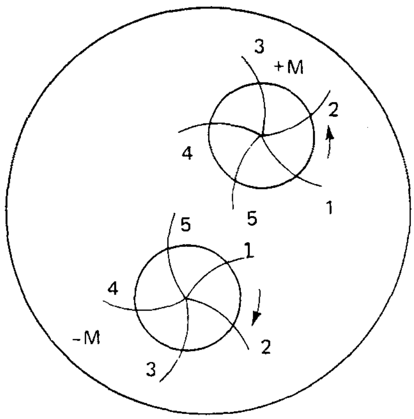
A Heegaard-diagram of order k consists of a sphere S^2 with:

i) $2k$ points associated by pairs $(+M, -M)$; for each of them a small circle with centre in it, and homeomorphisms between the boundaries of associated circles – from associated points – in such a way that if we remove the interior of the circles and indentify the boundaries by the homeomorphisms, we get a closed surface.

ii) A finite number of disjoint arcs, closed or with extremes in these points and intersecting transversally the boundaries of the circles, in such a way that they rejoin to k-topologically independent simple closed curves in the closed surface, i.e. they don't intersect and the complementary of their union is connected.



iii) Once an arc intersects an associated circle, it goes directly to the corresponding associated point.



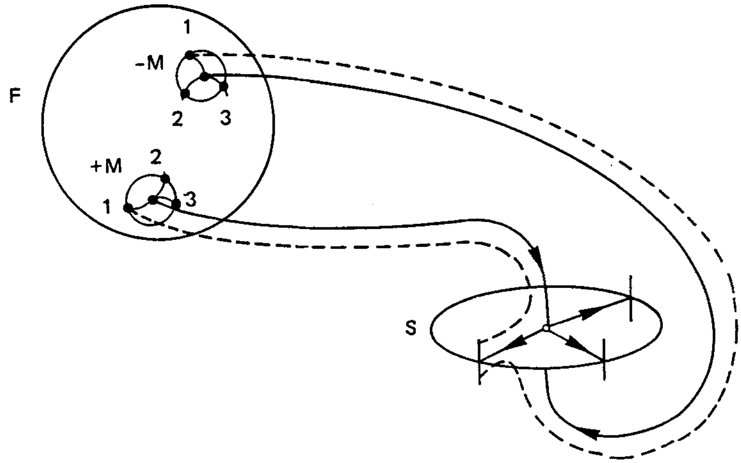
By cutting the surface by the closed topologically independent curves and attaching $2k$ circles to the boundaries of the so obtained sphere with $2k$ holes, we get a *dual Heegaard-diagram* with new arcs the boundaries of the older circles.

Two Heegaard-diagrams are said to be *isomorphic* if there is a homeomorphism from one sphere into the other, which send associated points into associated points, arcs into arcs, associated circles into associated circles, and is compatible with the homeomorphisms between associated circles.

Polar flows in 3-manifolds

We will consider polar flows in a closed 3-manifold M – i.e. flows with one source, one sink, and k saddles in each level and we show that the conjugacy types of these flows are in one-to-one correspondence the isomorphic classes of Heegaard diagrams of M,

We associate a Heegaard diagram to a given polar flow X on M, by considering the transversal intersections of the stable manifolds of the saddles with a sphere centred at the source. The intersection of the two components of the stable manifold of a first level saddle give two associated points ($+M$, $-M$), and the intersections of the stable manifolds of the second level saddles give the arcs of the diagram.



A homeomorphism between associated circles is given compatible with the transversal intersections of the stable manifolds of the second level saddles with the unstable manifold of a first level saddle.

In a similar way it can be obtained the dual diagram in a sphere centred at the sink.

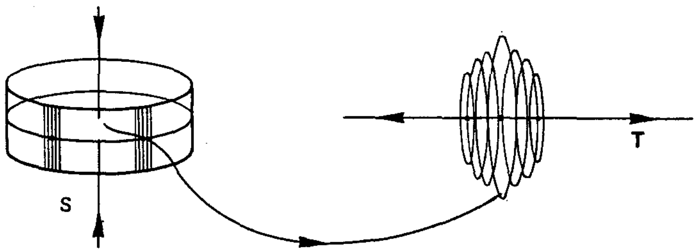
The assertion is then proved by the two following theorems.

Theorem 1a – *Let X_1, X_2 be polar flows on the closed 3-manifolds M_1, M_2, and let E_1, E_2 be their Heegaard diagrams at the sources. Then, X_1 and X_2 are topologically equivalent if and only if the two diagrams are isomorphic.*

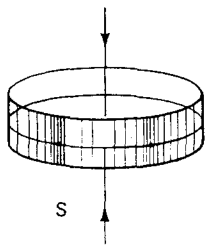
Proof: The condition is trivially necessary, and for the sufficiency we construct a homeomorphism $H: M_1 \to M_2$ which gives topological equivalence between X_1 and X_2, with notation and methods from [19], [11], [10].

a) The homeomorphism between E_1 and E_2 induces a one to one correspondence between the saddles of the two flows and homeomorphisms between fundamental neighbourhoods of its unstable manifolds.
b) Tubular families are constructed for each one of the second level saddles, as the transformed by the flows of fibres with basis in the points of the fundamental neighbourhood of the unstable manifolds; the homeomorphisms between the fundamental neighbourhoods induce a one to one correspondence between the fibres of the tubular families.

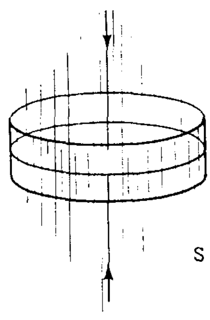
For each saddle of the first level, we consider a fundamental fence of its unstable manifold; the tubular families intersect transversally these fences in local fibrations (by the λ-lemma); the one to one correspondence between the fibres induce fibred homeomorphisms between these local fibrations.



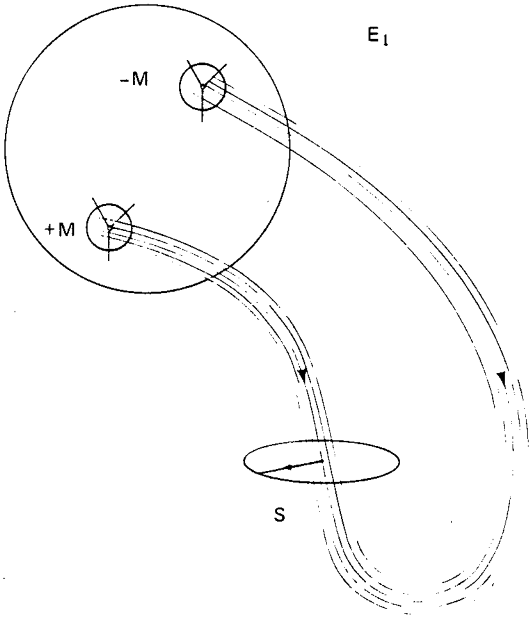
The local fibrations can be extended to global fibrations and the local homeomorphisms can be extended to fibred homeomorphisms between the whole fundamental fences.



Tubular families are then constructed for the first level saddles as the transformed by the flows of the global fibrations of the fences; we have again a one to one correspondence between fibres.

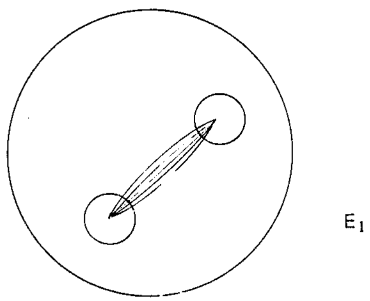


These tubular familes intersect transversally the spheres centered in the sources (fundamental neighbourhoods) in neighbourhoods of the associated points $+M$, $-M$; each fibre intersects in a point (by the λ-lemma); the correspondence between the fibres of the families induce local homeomorphisms between these neighbourhoods.



The tubular families of the second level intersect transversally the spheres in fibred neighbourhoods of the arcs of the diagram; the correspondence between them induces fibred local homeomorphisms between these neighborhoods, which are compatible with the homeomorphisms induced by the tubular families of the first level.

These local homeomorphism can then be extended to a homeomorphism between the whole spheres.



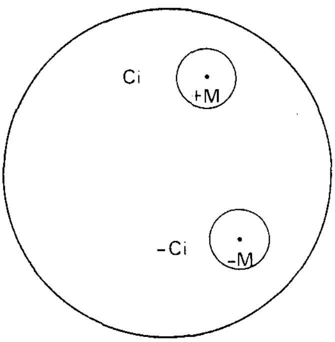
So new homeomorphisms are constructed between fundamental neighbourhoods of all the singularities.

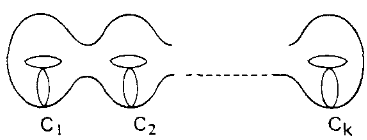
c) The homeomorphism H: $M_1 \to M_2$ which conjugates X_1 and X_2 is constructed in the following way: the orbit of a given point $x \in M$ intersects transversally the fundamental neighbourhood of the unstable manifold of a singularity afther a time t; this intersection is transformed by b) into a point on the fundamental neighbourhood of a singularity in the second flow, the time $-t$ of this point is $H(x)$.

It is easy to check that H is a homeomorphism. The checking can be found in [9].

Theorem 2 – *Each Heegaard diagram can be obtained as the transversal intersection of the stable manifolds of the saddles of a polar flow in a sphere centred in the source of this flow.*

Proof: We consider the closed surface T obtained from the diagram, with the boundaries of the associated circles as canonial meridians $c_1, \ldots, c_k$; let $d_1, \ldots, d_k$ be the topologically independent closed curves that the arcs of the diagram determine on T.



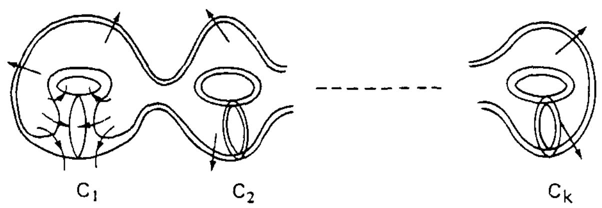


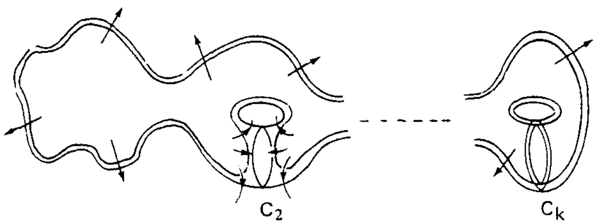
T

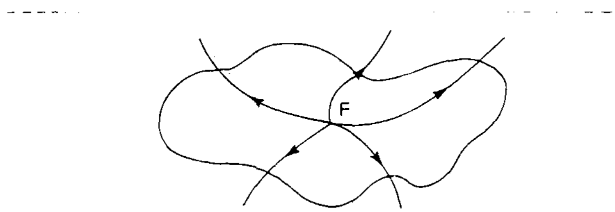
Let h be a homeomorphism between T and another surface T' which send $d_1, \ldots, d_k$ into the canonical meridians of T', and consider the two surfaces as the boundaries of handlebodies B, B'.

The manifold M is obtained from the two handlebodies by the h-identification of the boundaries.

In the first handlebody, we construct a gradient-like flow, transversal to the boundary, with a source and k saddles of the first level, s.t. the unstable manifolds of the saddles intersect the boundary in the canonical meridians $c_1, \ldots, c_k$.







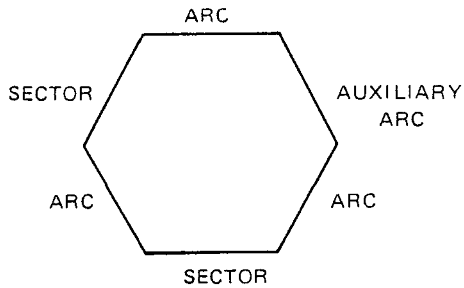
In similar way, a gradient-like flow is constructed in the other handlebody, with a sink and k saddles of the second level, whose stable manifolds intersect the boundary in the canonical meridians $d_1, \ldots, d_k$.

The two flows are glued together to a flow in M which satisfies the conditions of the theorem.

The given diagram is said to be a Heegaard diagram for the constructed manifold M [12].

Heegaard diagrams (polar flows) for S^3

A Heegaard diagram is said to be *parcelled* if there are drawn auxiliary disjoint arcs in the sphere minus the interiors of the associated circles, in such a way that they join transversally arcs of the diagram, divide the sphere minus the interior of the associated circles into simply connected parcels, and the sides of these parcels alternate arcs of the diagram (the whole arc or a part of an arc) and auxiliary arcs or sectors of the associated circles, with the following conditions:

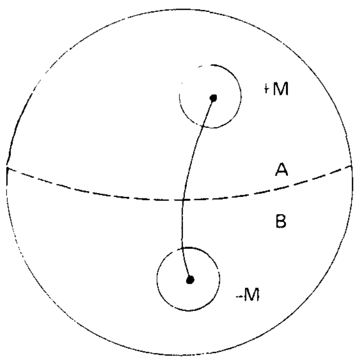


i) Each one of these parcels can only have one of its sides as intersection with the boundary of a given circle.

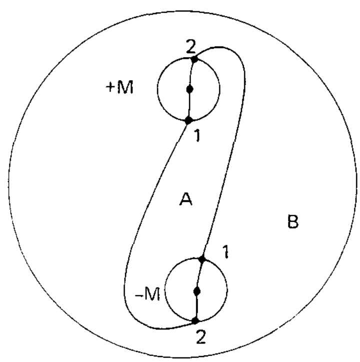
ii) Two of these parcels can have only one common auxiliarly arc.

iii) Two sides of a given parcel can't be associated sectors by the homeomorphism between the boundaries of two associated circles.

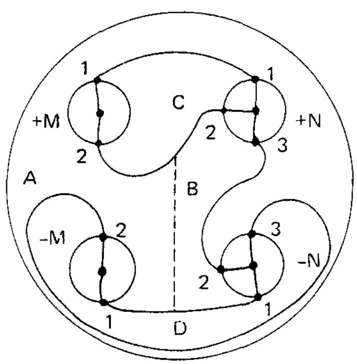
Ex. 1



Ex. 2

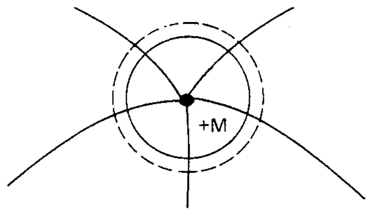


Ex. 3

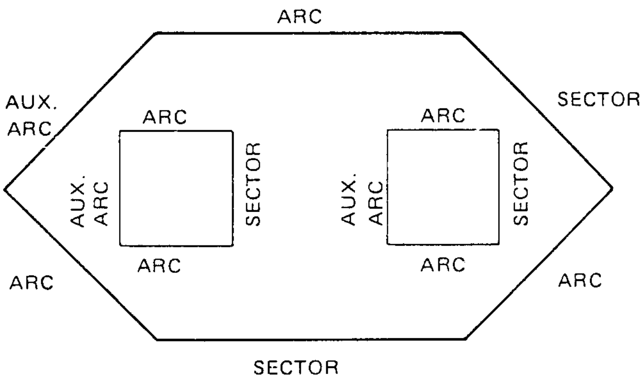


Proposition 1 – If every associated point is the extreme of some arc, the diagram can be parcelled.

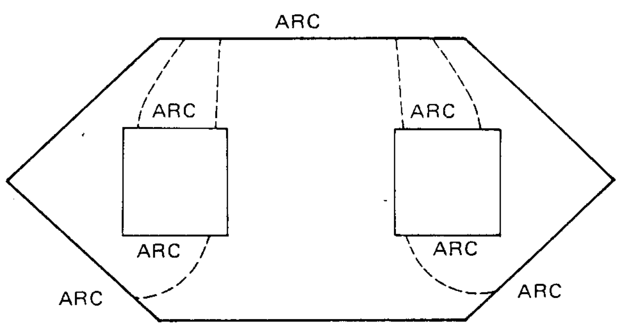
Proof: We draw auxiliary arcs formming a circle close to an associated one.



So, we have parcels with holes and the conditions i), iii).



We then draw as many auxiliary arcs as necessary to get parcels without holes, and with condition ii).



Proposition 2 — *If $\mathcal{D}$ is a Heegaard-diagram of S^3, every associated point of $\mathcal{D}$ is extreme of some arc.*

Proof: Consider the corresponding polar flow. If the proposition would not be true, this flow would have an invariant submanifold homeomorphic to S^2 and formed by the unstable manifold of the corresponding saddle and the sink; this is not compatible with the existence of only one source.

Observation: Every Heegaard diagram can be transformed into another one with the conditions of Proposition 1, by a finite number of transformations of the type T_T^{-1}, defined later. As a consequence, for every manifold M^3 there exists a Heegaard diagram with the condicions of Proposition 1.

2-Dimensional canonical complex from a parcelled heegaard diagram

It has the following simplices:

Dimension 0): A vertex for:

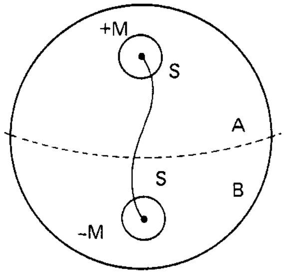
Each one of the associated points (i.e. associated circles) $+M$, $-M$.
Each pair of associated sectors (S)
Each parcel (A)
and two vertices F and P.

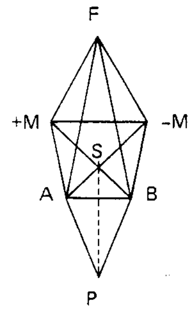
Dimension 1): 1-simplices joining:

Vertices from associated points ($+M$, $-M$) (i.e. associated circles).
Each sector S with the circles ($+M$, $-M$) to which it belongs.
Each parcel A with the sectors S and the circles $+M$ which it intersects.
All the parcels and circles with F.
All the parcels and sectors with P.
Two parcels with a common auxiliary arc.

Dimension 2): 2-simplices joining:

Each sector S with the circles $+M$, $-M$ to which it belongs.
Each parcel A to a sector S which it intersects and the corresponding circle $+M$.
Each parcel A with a circle $+M$ which it intersects and F.
Each parcel A with a sector S which it intersects and P.
Two parcels with a common auxiliary arc and F.
Two parcels with a common auxiliary arc and P.





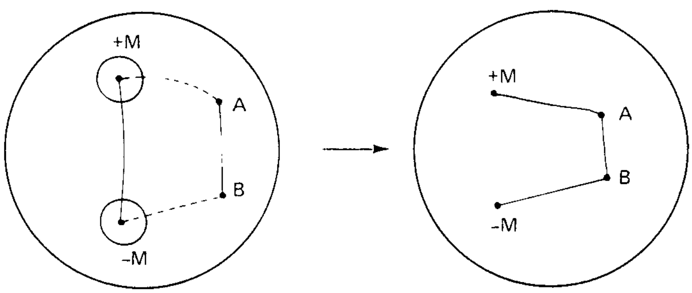
We call this complex K.

Canonical embedded graphs from a parcelled Heegaard diagram

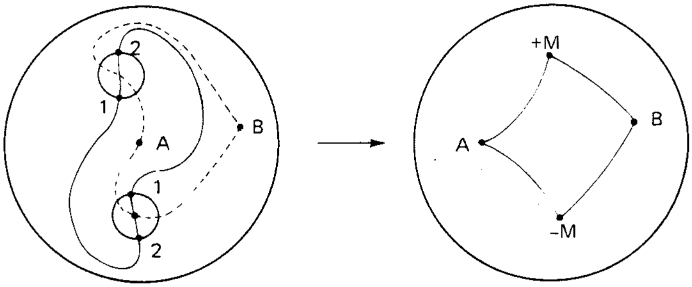
These graphs are embedded in S^2; G_F is embedded in the given diagram and G_P is embedded in the dual diagram.

Graph G_F): The vertices of G_F are the associated points of the diagram, and an interior point at any of the parcels. It has a 1-simplex joining a parcel and an associated point through a sector which is intersection of the two, and a 1-simplex joining two parcels through an auxiliary arc which is intersection of the two.

Ex. 1:

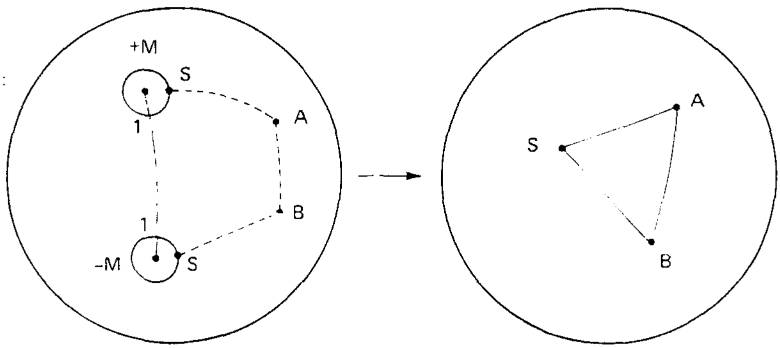


Ex. 2:

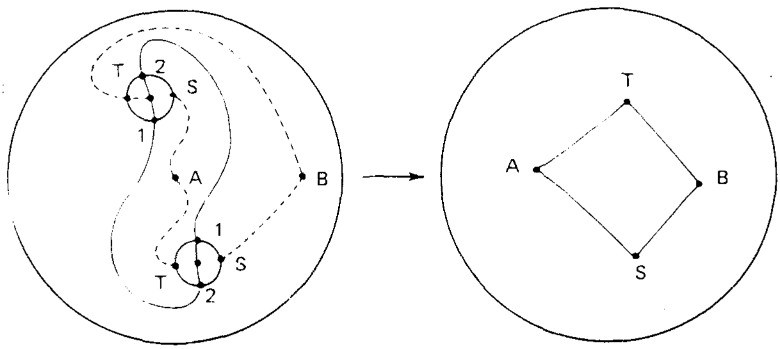


Graph G_F): In the sphere of the diagram, we consider a point interior to each sector, in such a way that the points of two associated sector are associated by the corresponding homeomorphism, and a point interior to each parcel; a 1-simplex joins a parcel and a sector in its boundary, or two parcels through an auxiliary arc which is intersection of the two. This graph is then carried to the dual Heegaard-diagram, to a graph G_P, where the points of two associated sectors are identified.

Ex. 1:



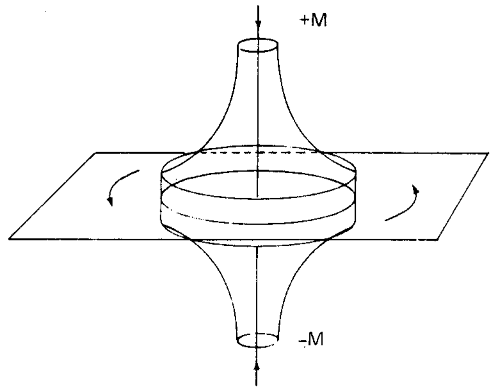
Ex. 2:



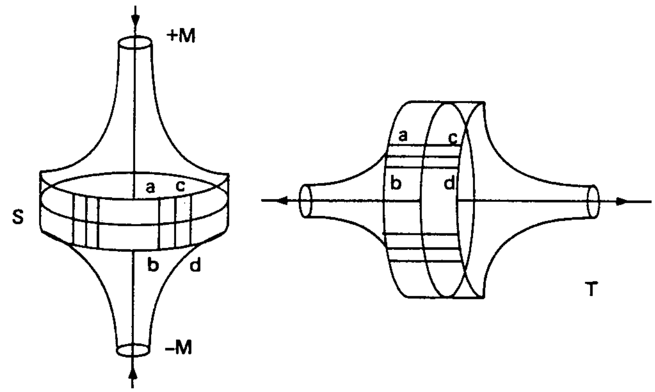
Theorem 3 – *$\mathscr{D}$ is a Heegaard diagram of S^3 if and only if for any parcellation of it, the canonical two complex K can be semilinearly embedded in R^3, in such a way that it intersects a small sphere centered at F in the canonical graph G_F and another one centered at P in the canonical graph G_P.*

Proof. Let X be the polar flow corresponding to the diagram $\mathscr{D}$.

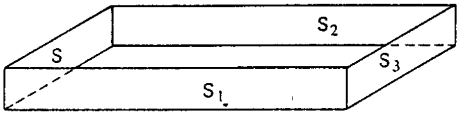
For each first level saddle of X, we consider a solid which covers part of its invariant manifolds, as shown in the figure, and intersects the diagram in the sphere at F, in two associated circles.

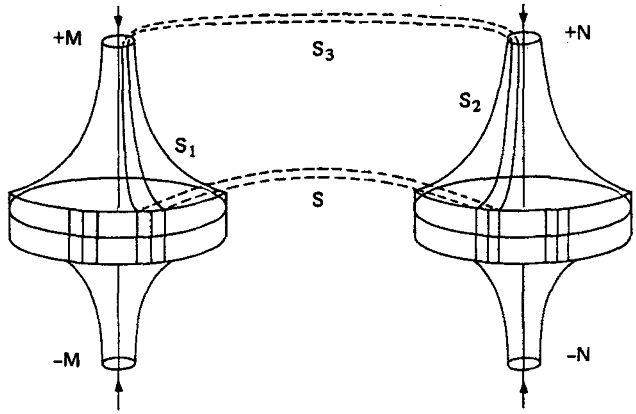


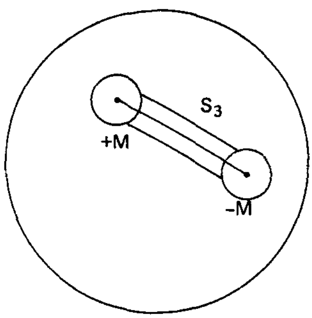
We consider similar solids for each second level saddle, intersecting the dual diagram in the sphere at P in two associated circles, s.t. they intersect the first solids in the dotted bands at the central ring, as shown in the figure.

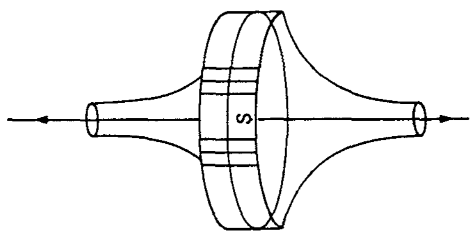


For each of the not dotted bands, we consider a box s.t. one of its basis is the considered band, the opposite basis is a neighborhood of the arc of the diagram which is transversal intersection of the corresponding invariant manifold, and with two other opposite faces in contact with solids of the other level.

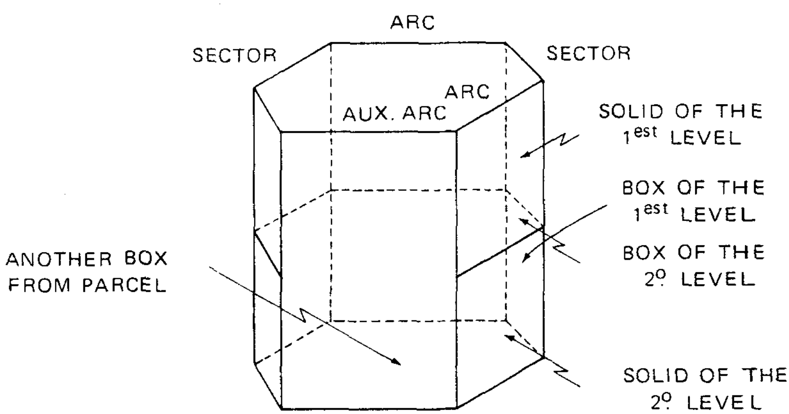




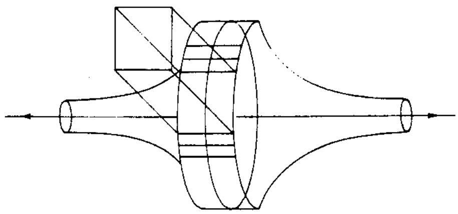




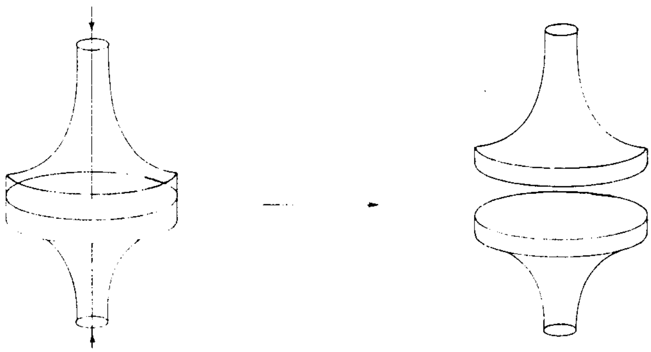
The remainder of the whole manifold consists of boxes with basis the parcels of the parcelled diagram.



By removing the solids of the second level ad the boxes with basis in their bands, it is obtained a PL 3-manifold X with boundary equal to k copies of S^2.



We divide each solid of the first level into two pieces $+M$, $-M$ by a central section in the ring.



So, we have obtained a brick decomposition of X, formed by the associated pieces $+M$, $-M$, the boxes S with basis in the solids of the first level, the boxes A with basis the parcels, and the balls F and P centered at the source and the sink and with boundary the given Heegaard-diagram and its dual.

The canonical complex K is the 2-dimensional nerve of this decomposition, and can be semilinearly embedded in X, intersecting transversally the two spheres F and P in the canonical graphs G_F and G_P.

So the necessary condition of the theorem is obvious. For the sufficiency it is enough to show that X can be semilinearly embedded in $\mathbb{R}^3$ (if K can), for M is obtained by attaching k solid spheres to the boundary of X, and then one applies the Shoenflies Theorem [2].

X can be embedded in R^3 by reproducing its brick decomposition as a decomposition of a neighbourhood of the embedded K:

The two balls F and P are small balls centered at the vertices F and P.

The solids of the first level are obtained from a neighbourhood of the 1-simplices $+M, -M$; $+M, F$ and $-M, F$

For each vertex from a sector, it is obtained a box of the first level as a neighbourhood of the simplices $+M, -M, S$ and S, P.

The other boxes are obtained from each vertex from a parcel, as a neighbourhood of its star.

So we have a brick decomposition of a subspace of R^3 which coincides with that of X, because of the condition on the graphs G_F and G_P.

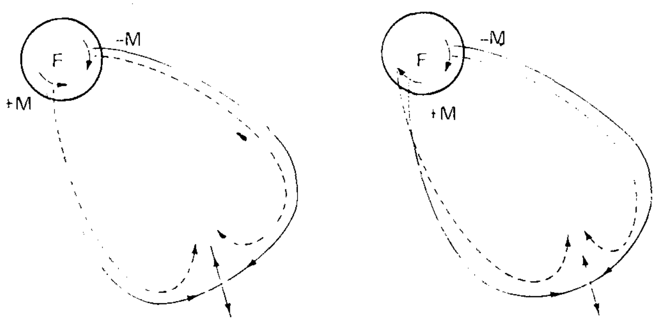
Theorem 4 — *If K is the 2-complex which corresponds to a given parcelled Heegaard-diagram of M, then* $\pi_1(K) = \pi_1(M)$.

Proof. As a consequence of a theorem of Boursuk [1, Corollay (14.4), page 190], X has the same homotopy type as K, and so (by the Van Kampen's theorem): $\pi_1(K) = \pi_1(M)$.

Polar flows on 2-dimensional manifolds

If the manifold has Euler characteristic k, then the flow has a source, a sink and $l = 2\text{-}k$ saddles.

The stable manifolds of the saddles intersect transversally a circle centered at the source in points associated (colored) by pairs. In the non-orientable case, arrows (spins) are attached at these points, for indicating the way they rejoin in a neighborhood of the saddle.



So circular distributions of colored points are associated to polar flows.

Two of these distributions are said to be isomorphic if there exists a homeomorphism of S^1 onto itself, which is compatible with the colors of the points (and with the spins, in the non-orientable case).

Then, the analogous of the Theorem 1 holds:

Theorem 1b – *Let* E_1, E_2 *be circular distributions of colored points associated to the polar flows* X_1, X_2 *on the 2-dimensional manifolds* M_1, M_2. *Then* X_1, X_2 *are topologically equivalent if and only if* E_1, E_2 *are isomorphic.*

Proof. Analogous to that of Theorem 1.

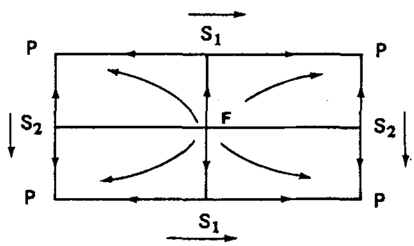
The remaining question is to get all the possible distributions for polar flows on a given 2-dimensional closed manifold.

We consider first the *orientable case.*

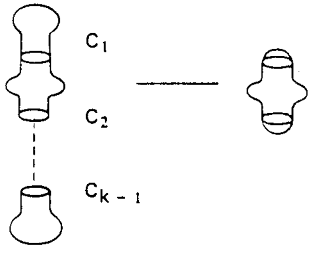
i) SPHERE: There is only one type of polar flow: source-sink.

ii) TORUS: The flow has two saddles, so there exists two possibilities:

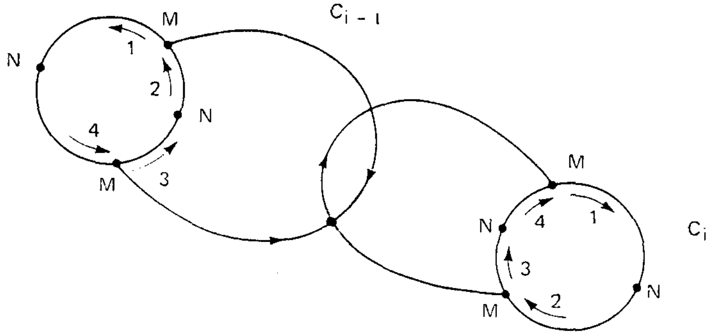
The first cannot hold, and the second is obtained from the following flow:



iii) k-TORUS: The flow has $2k$ saddles. Using the Smale filtration [14], [15], we cut the manifold by circles which are transversal to the flow and separate the saddles by pairs.



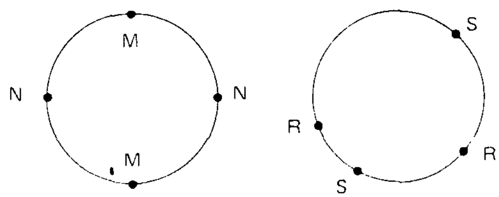
To each one of the pieces so obtained we attach disks with a source or a sink, and we get k polar flows in k different tori. So the stable manifolds of the saddles in the i-piece intersect the (i-1)-circle c_{i-1} as:

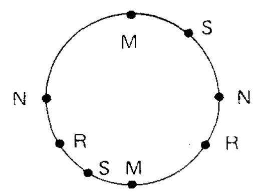


The stratifications of the C_i circle come back and joint to that of the C_{i-1}-circle as a superposition, with invertion of the orientation of the four intervals.

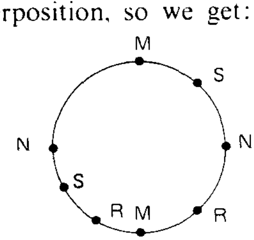
So, the stratifications of the k-torus are those obtained as a superposition of k stratifications of the torus; one over one, and inverting on each step the four intervals of the added stratification. (Orientation reversing rule).

As an example, if we supperpose the two stratifications of the torus

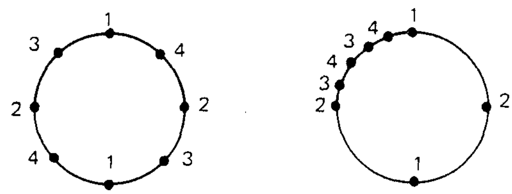




the result is not a stratification from a polar flow in the 2-torus. But we can obtain one of these stratifications if we reverse the intervals of the first stratification in the last supperposition, so we get:

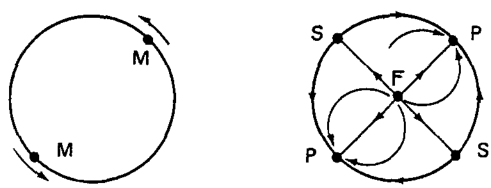


As another example, we show two non-isomorphics stratification in the 2-torus:

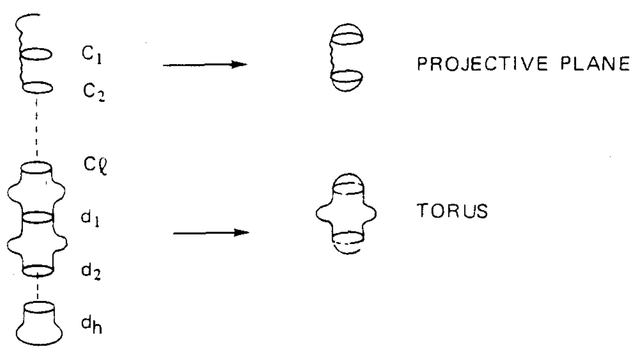


Consider now the *non-orientable case.*

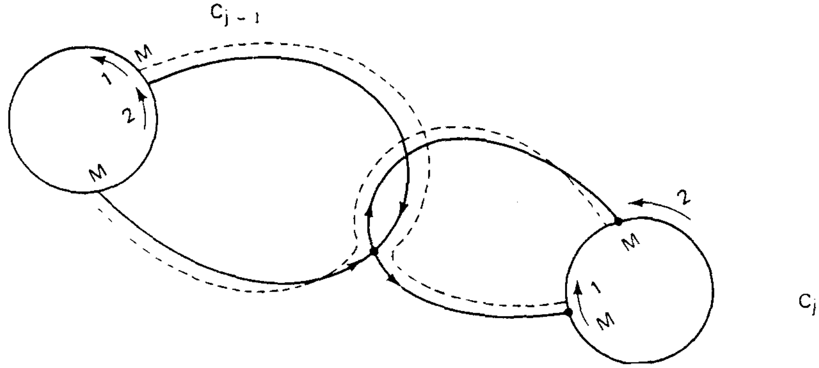
i) PROJECTIVE PLANE: It has only one polar flow, with a source, a sink and a saddle.



ii) SURFACE OF GENUS g: The filtration method gives now transversal circles $c_1, \ldots, c$ separating the saddles one after one, and transversal circles $d_1, \ldots, d_h$ separating the saddles by pairs. The pieces obtained from the first cuts are projective planes, and the pieces obtained from the second cuts are tori.



The stable manifold of the saddle in the j-piece ($1 \leq j \leq l$) intersects transversally C_{j-1} as

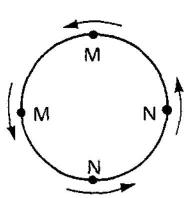


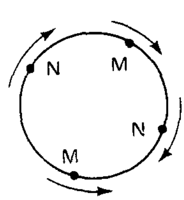
The distributions in C^j come back to C_{j-1} as a superposition, but inverting the orientation of the spins of one of the two arcs (*orientation reversing rule*).

The stable manifolds of the two saddles in the i-piece ($1 \leq i \leq h$) intersect transversally d_{i-1} ($d_o = c_l$) as in pag. 24.

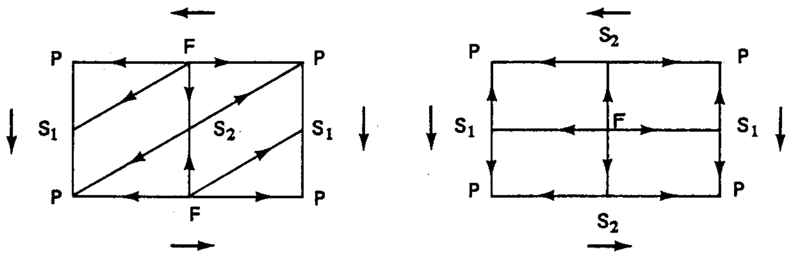
So, the distributions of polar flows in a genus g, closed, non-orientable surface, are those obtained as a superposition of $m\,(= h + 1)$ distributions of type II and l distributions of type I, taking into account the orientation reversing rule, and for m, l such that $2m + l = g$, and $l \neq 0$.

The distributions of the Klein bottle are then:





They are obtained from the flows:



Remark: In the genus 3, non-orientable surface, there are distributions obtained as a superposition of one of the type II and another of the type I, and distributions obtained from three of the type II; these two classes have no stratification in common.

Classification and bifurcation of gradient-like flows

We consider here gradient-like flows on closed manifolds of dimension two or three. For each of these flows, a set of labelled stratifications is given by the transversal intersection of the stable manifolds of the saddles with spheres centered at each one of the sources.

In dimension three, the set of labelled stratifications consists of associated points (which can be situated in different spheres), associated circles centered in these points, with homeomorphisms between them, and arcs which are closed or with extremes in the points. Two sets of labelled stratifications are said to be isomorphic if each of them has the same number of spheres $S_1, \ldots, S_n$; $S'_1, \ldots, S'_n$ and there exist homeomorphisms h_i: $S_i \to S'_i$ ($^i = 1, \ldots, n$), mapping associated points into associated points, arcs into arcs, and being compatible with the homeomorphisms between associated circles.

In dimension two, the set of labelled stratifications is given by associated points (may be in different circles), labelled as in the non-orientable polar case. Two of these sets are said to be isomorphic if they have the same number of circles $S_1, \ldots, S_n$; $S'_1, \ldots, S'_n$ and there exist homeomorphisms h_i: $S_i \to S'_i$ ($i = 1, \ldots, n$), mapping associated points into associated points, and compatible with the labels.

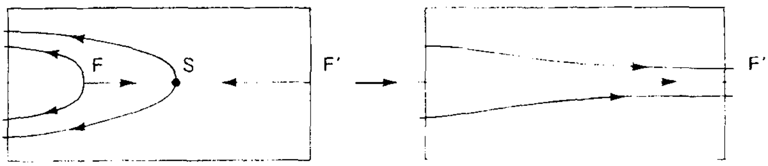
In both cases, the analogous of the Theorems 1 and 1a, holds:

Theorem 1c – *Two gradient-like flows X_1, X_2 are topologically equivalent if and only if their sets of stratifications at the sources are isomorphic.*

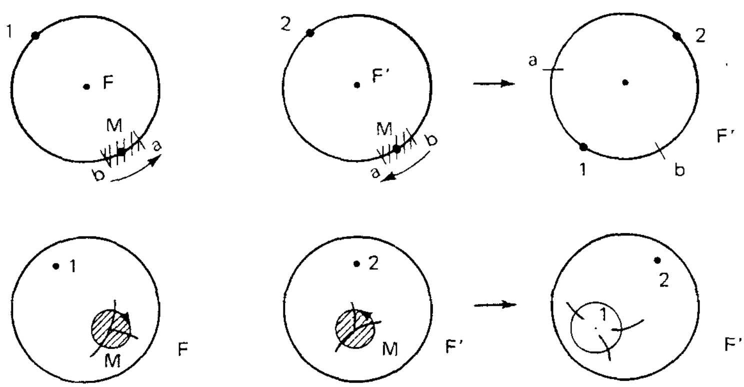
We will describe now the transformations of these stratifications that are obtained from some first order bifurcations [17], [18], [6], of the gradient-like flow.

Transformation T_F, T_F^{-1}

If a flow has a saddle (a first level saddle, in the 3-dimensional case) with each component of its stable manifold coming from different sources F and F', then the saddle and one of the sources F cancel as a first order bifurcation [4], [5].



The orbits with the source F as α-limit will change it by F', and the stratifications in F and F' reduce to a new one in F' obtained by removing nei ghborhoods of the associated points from the saddle, and attaching the boundaries. The other spheres remain unmodified (transformation T_F).



All the inverse transformations T_F^{-1} are induced by the other direction of a bifurcation of the same type.

Transformations T_p, T_p^{-1}

From the set of stratifications in the sources, we can get the number of sinks in the flow, as follows:

In dimension 2, by cutting the circles by the associated points and attaching the arcs so obtained by the associated extremes, taking into account the labels, a number of new circles equal to the number of sinks is got. Two arcs are said to be in the same component if they go to the same sink.

In dimension 3, by removing the interior of the associated circles and attaching the boundaries, a closed surface is obtained, in which the arcs of the stratifications join to closed disjoint circles which now divide the surface in as many components as there are sinks.

Consider the bifurcation which cancels a sink with a saddle; the corresponding transformation T_p is as follows:

In dimension 2, a pair of associated points bounding arcs in different components disappears.

In dimension 3, a set of arcs which join to the same circle bounding different components disappears.

The inverse transformations T_p^{-1} are all induced by the other direction of a bifurcation of the same type.

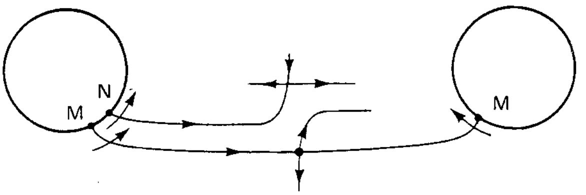
If a gradient-like flow has more than one source (sink) it must have one saddle with each component of its invariant manifold coming from (going to) different sources (sinks), so [5]:

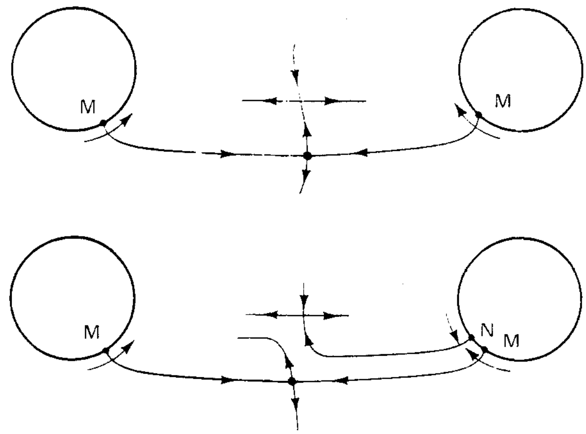
The set of labelled stratifications at the source of every gradient like flow can be transformed into another one from a polar flow by a finite number of transformations of the types T_F, T_P.

Transformations T_T, T_T^{-1}

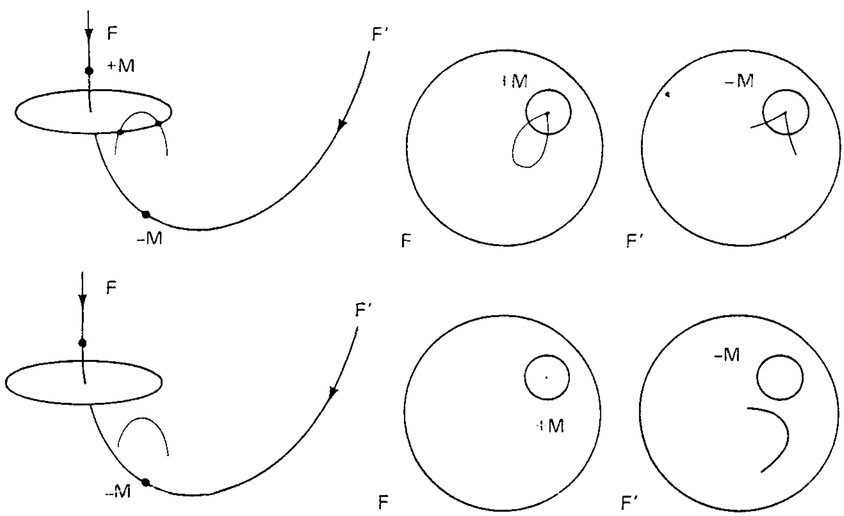
Another first order bifurcation is the first order tangency of the invariant manifolds of the saddles.

In dimension 2, this bifurcation induces a shift of a point contiguous to another M, to a point contiguous to the associated to the latter, taking into account the labels.



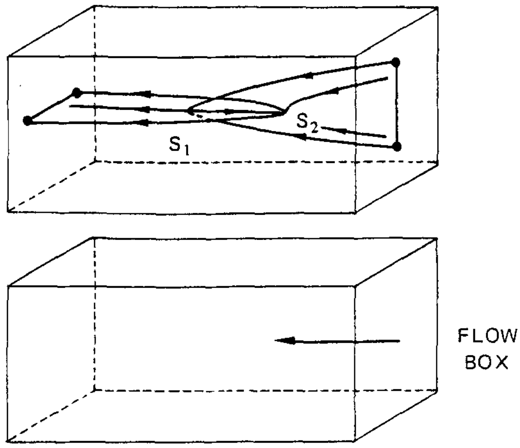


In dimension 3, the bifurcation pushes an arc with ends at the same associate point and bounding a cell without another arcs or points, in such a way it disappears.



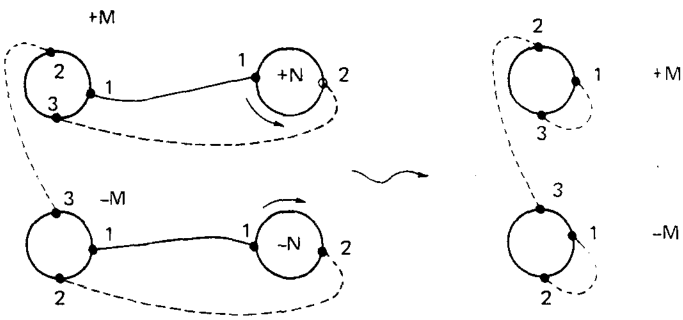
Transformations T_S, T_S^{-1}

In dimension 3, there is another bifurcation, holding when the invariant manifolds of two saddles intersect transversally in a single orbit; the bifurcation make the two saddles disappear:



The induced transformation T_S in the set of stratifications make disapper a pair of associated points and arcs joining to a same circle, one and only one of them ending at the two associated points; the other arcs with end in these points are prolonged to run the way the cancelled one did.

Example:



Also here, every inverse transformation T_P^{-1} is induced by the other direction of a bifurcation of the same type.

As a consequence of [7] (two Morse-Smale flows can be joined by an arc with a finite number of bifurcations, all of first order), the following theorem holds:

Theorem 5 – *Let E_1, E_2 be two sets of labelled stratifications in the sources, from two gradient-like flows X_1, X_2 in the same manifold M. Then, E_1 can be transformed into E_2 by a finite number of transformations of the types T_F, T_P, T_T, T_S and its inverses.*

On other way, we can apply a theorem of J. Singer (theorem 7 of [13]) and get in the case of Heegaard-diagrams an arc of transformations which has more information than the last one: i) there appears no transformation T_T; ii) If p, q are the two singularities which cancels in a transformation T_S, its invariant manifolds do not cut the invariant manifold of any other saddle.

Final Remarks: There exists a classification of some type of flows – in the spirit we adopt here – implicitly contained in some known results about classifications of manifolds. It seems to hold the following two consequences from Smale [16] and Wall [20] about $(n-1)$-connected $2n$-manifolds $(n>2)$:

i) Each gradient-like flow on one of these manifolds can be reduced by a finite number of first order bifurcations to a simpler one whose singularities are: one source, one sink and saddles only in the intermediate level, i.e., saddles with n-dimensional stable and unstable manifolds (Smale's handlebody theorem). The stratification in the source of one of these reduced flows is then given by k $(n-1)$-spheres embedded in a $(2n-1)$-sphere (k = number of saddles). [4]

ii) The labbels in the stratifications which suffice to prove some analogous of theorem 1 are the diffeotopy class of the embedding of the k-spheres (= linking number) and the diffeotopy class of the embedding of a tubular neighborhood of each sphere (= element of $\pi_{n-1}(SO_n)$).

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