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Classification of 2-Transitive Symmetric Designs

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To Prof. Noboru Ito, to commemorate his 60th birthday

Abstract. All symmetric designs are determined for which the automorphism group is 2-transitive on the set of points.

This note contains a proof of the following result.

Theorem. *Let D be a symmetric design with $v > 2k$ such that Aut D is 2-transitive on points. Then D is one of the following*:

(i) *a projective space*;

(ii) *the unique Hadamard design with $v = 11$ and $k = 5$*;

(iii) *a unique design with $v = 176$, $k = 50$ and $\lambda = 14$*; *or*

(iv) *a design with $v = 2^{2m}$, $k = 2^{m-1}(2^m - 1)$ and $\lambda = 2^{m-1}(2^{m-1} - 1)$, of which there is exactly one for each $m \geq 2$.*

The designs in (iv) are discussed in detail in [3].

The theorem will be proved as a simple consequence of the classification of finite simple groups. The proof is easier that that of the analogous result [5] for designs with $\lambda = 1$. These two papers clarify the extent to which [4] is now obsolete.

Proof. Let G be a subgroup of Aut D that is 2-transitive on points. Then G is also 2-transitive on blocks, and these two 2-transitive permutation representations are inequivalent; in particular, the stabilizer G_x of a point x is not conjugate to the stabilizer G_B of a block B. Note that we may replace G by any 2-transitive subgroup of G.

A list of 2-transitive groups is contained in [5]; compare [1]. We only need to check whether a group on the list has two inequivalent 2-transitive permutation representations of the same degree (and having the same permutation character). When G has a nonabelian simple normal subgroup, this is, in effect, already contained in [1], and leads to (i)–(iii).

Assume that G does not have a nonabelian simple normal subgroup. Then $G \leq AGL(d, p)$ for some prime p, and G contain the translation group V. We can identify V with the set of points of D, and then let $x = 0$.

Now $G = VG_0 = VG_B$, so that G_0 and G_B are nonconjugate complements to V in G. If $Z(G_0) \neq 1$ then $G_0 = N_G(Z(G_0))$ is conjugate to $G_B = N_G(Z(G_B))$ (since $Z(G_0)$

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and $Z(G_B)$ are conjugate in $VZ(G_0) = VZ(G_B)$). Thus, $Z(G_0) = 1$. (Compare [6], p. 9, (α).) This eliminates many of the cases in [5], and leaves us with the following possibilities (for some e).

(i) $G_0 \leq \Gamma L(1, p^e)$.
(ii) $G_0 \trianglerighteq SL(k, p^e)$, $d = ke$, $k \geq 2$.
(iii) $G_0 \trianglerighteq Sp(2k, 2^e)$, $d = 2ke$, $k \geq 2$.
(iv) $G_0 \trianglerighteq G_2(2^e)'$, $d = 6e$.
(v) $G_0 = A_6$ or A_7 inside $GL(4, 2)$.

Case (i) is eliminated exactly as above, using $G_0 \cap GL(1, p^e)$ in place of $Z(G_0)$. In the remaining cases, note that $H^1(G_0, V) \neq 0$ since G_0 and G_B are nonconjugate complements to V. These cohomology groups are described in [6, (2.14)] for (ii), (iii) and (v), and in [2] and the lemma at the end of the present note for (iv). The only times $H^1(G_0, V) \neq 0$ are (iii), (iv), and (v) with $G = A_6$; and in each case $H^1(G_0, V)$ has dimension 1 over $GF(2^e)$ (where $2^e = 2$ in (v)). This means that $\mathrm{Aut}\, G = \mathrm{Aut}(G_0 V)$ is 2-transitive on the set of conjugacy classes of complements to V in G. (The induced 2-transitive group is just $A\Gamma L(1, 2^e)$.) Thus, G can only produce one design D up to isomorphism.

On the other hand, the group $V \cdot Sp(d, 2)$ does produce a symmetric design, called $\mathscr{S}^+(d/2)$ in [3]. Since $G \leq V \cdot Sp(d, 2)$, G acts 2-transitively on the points of that design. Thus, $D \cong \mathscr{S}^+(d/2)$. □

In the above proof we needed the following technical result. I am grateful to G. Mason both for a helpful discussion concerning the following lemma and for providing a different proof of it.

Lemma. *Let V be the natural 6-dimensional module for $K = G_2(2)'$ over $GF(2)$. Then* $\dim H^1(K, V) = 1$.

Proof. Let $KV = K_1 V$ with $K_1 \cong K$ and K_1 not conjugate to K. By Sylow's theorem we may assume that $K \cap K_1 \geq N_K(T) = TA$, where T is a Sylow 3-subgroup of K and A is cyclic of order 8. Note that $K_1 = \langle TA, N_{K_1}(A) \rangle$.

Since A fixes only 2 points in the natural 2-transitive representation of K_1, $|N_{K_1}(A)| = 16$. On the other hand, K has a unique conjugacy class of cyclic subgroups of order 3, so that $N_{KV}(A)$ is 2-transitive on $C_V(A)$. Thus, $|C_V(A)| = 2$ and $N_{KV}(A) = N_K(A) \times C_V(A)$. Now $|N_{KV}(A)/A| = 4$, and there are only two subgroups of $N_{KV}(A)$ isomorphic to $N_K(A)$. Thus, $N_{K_1}(A)$ is uniquely determined, and there are at most two conjugacy classes of complements to V in KV. Since there are at least two such classes, this completes the proof. □

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