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Existence of Multiple Positive Solutions for $-\triangle u = \lambda u + u^{\frac{N+2}{N-2}} + \mu f(x)$*

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Abstract. In this paper, we discuss the existence and nonexistence of solutions for the problem

$$-\Delta u = \lambda u + u^{\frac{N-2}{N+2}} + \mu f(x), \quad u > 0 \text{ in } \Omega \quad u|_{\partial\Omega} = 0, \quad n > 2. \qquad ((*)_\mu)$$

where Ω is a bounded smoothness domain in R^N, $\lambda \in R^1$, $\mu \geq 0$, $f(x)$ is a given non-negative function. Some interesting results have been obtained.

§1. Introduction

In this paper, we consider the existence of multiple solutions of semilinear elliptic boundary value problem

$$\begin{cases} -\Delta u = \lambda u + u^p + \mu f(x), & x \in \Omega, \quad N > 2, \\ u|_{\partial\Omega} = 0 \quad u > 0 \text{ in } \Omega, \end{cases} \qquad ((1.1)_\mu)$$

where Ω is a bounded smoothness domain in R^N, $\lambda \in R^1$, $\mu \geq 0$ are some given constants, $p = \frac{N+2}{N-2}$ is the critical Sobolev exponent and $f(x)$ is some given function in $C_0(\Omega) \cap C^{1+\alpha}(\overline{\Omega})$ such that $f(x) \geq 0$, $f(x) \not\equiv 0$ in Ω.

We are interested in the existence of solutions of $(1.1)_\mu$ because it exibits many interesting existence and non-existence phenomena related to some lack of compactness of the corresponding energy functional

$$I(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 - \lambda u^2 dx - \frac{1}{p+1}\int_\Omega |u|^{p+1} dx - \mu \int_\Omega f(x) u dx, \quad u \in H_0^1(\Omega).$$

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In the case $\mu = 0$, it is well-known that $(1.1)_0$ has no solution for $\lambda = 0$. In fact, Brezis and Nirenberg[2] have proved that $(1.1)_0$ has no solution for all $\lambda \le 0$ if Ω is a star-shaped domain and $(1.1)_0$ possesses at least one solution for $\lambda \in (0, \lambda_1)$, $N \ge 4$. Where λ_1 is the first eigenvalue of the eigenvalue problem $-\Delta u = \lambda u \quad \text{in} \quad \Omega, \quad u|_{\partial\Omega} = 0$. A interesting problem is whether the existence and non-existence phenomena still remain true if we give $(1.1)_0$ a small perturbations $g(x, u)$? If $g(x, u)$ is a lower order homogeneous function in the sense

$$\lim_{u\to\infty} \frac{g(x,u)}{u^p} = 0 \quad \text{and} \quad g(x,0) \equiv 0,$$

an elegent existence result have been obtained in [2]. For the inhomogeneous case ($g(x, 0) \not\equiv 0$), they discussed the following special problem

$$\begin{cases} -\Delta u = \lambda(1+u)^p, \\ u|_{\partial\Omega} = 0 \end{cases} \tag{1.2}$$

and obtained some existence and non-existence results. The main aim of this paper is to discuss the existence and non-existence of multiple solutions of $(1.1)_\mu$. The following results have been obtained.

Theorem 1. *For $\lambda \in (-\infty, \lambda_1)$ there exists a positive constant $\mu^* < +\infty$ such that $(1.1)_\mu$ has a minimal solution for all $\mu \in (0, \mu^*]$ and $(1.1)_\mu$ has no solutions for $\mu > \mu^*$.*

Theorem 2. *For $\lambda \in [0, \lambda_1)$, $\mu \in (0, \mu^*)$ problem $(1.1)_\mu$ possesses at least two solutions.*

Theorem 3. *For $\lambda \in (-\infty, 0)$, $\mu \in (0, \mu^*)$ we have*

i) *If $3 \le N \le 5$, then $(1.1)_\mu$ possesses at least two solutions.*

ii) *If $N \ge 6$, $\Omega = B_R(0) = \{x \in R^N \mid |x| < R\}$ and $f(x)$ is radial with $f'(r) < 0$, then there exists a positive constant $\mu^{**} \le \mu^*$ such that $(1.1)_\mu$ has only one solution for all $\mu \in (0, \mu^{**})$.*

We prove Theorem 1 by means of a standard barrier method and Theorem 2 by variational methods. Finally we obtain Theorem 3 by using an improved Pohozaev's identity.

§2. The Existence of Minimal Solution

Let λ_1 be the first eigenvalue of the operator $-\Delta$ and $\varphi(x)$ be the first eigenfunction which is larger than zero in Ω and $\int_\Omega \varphi^2 dx = 1$. Then we have the following lemmas:

Lamma 2.1. *For any $\lambda \in (-\infty, \lambda_1)$ there exists a constant $C > 0$ such that $(1.1)_\mu$ has no solutions for $\mu > C$.*

Proof. From $p > 1$ we can choose $C_1 > 0$ such that $u^p \ge (\lambda_1 - \lambda)u - C_1$ for all $u > 0$. If u is a solution of $(1.1)_\mu$, then

$$\mu \int_\Omega f(x)\varphi(x)dx \le C_1 \int_\Omega \varphi(x)dx.$$

Where $\varphi(x)$ is the first eigenfunction of $-\Delta$. Taking $C = \frac{\int C_1\varphi(x)dx}{\int f(x)\varphi(x)dx}$ we obtain $\mu \le C$.

Lemma 2.2. *For $\lambda < \lambda_1$, $(1.1)_\mu$ possesses at least one solution which is a minimum of all solutions if μ small enough.*

Proof. For any $\delta > 0$, let $\tilde{u} = \delta\varphi$, it is easy to varify that there exists a $\delta_0 > 0$ such that $\tilde{u} = \delta_0\varphi(x)$ is a supersolution of $(1.1)_\mu$ if μ small enough, and $\underset{\sim}{u} = 0$ is a subsolution of $(1.1)_\mu$ for all $\mu > 0$. Using the methods of monotone interation and strong maximum principle it follows that there exists a solution u_μ of $(1.1)_\mu$ such that $\underset{\sim}{u} < u_\mu \leq \tilde{u}$ and u_μ is a minimal solution of $(1.1)_\mu$.

Lemma 2.3. *For $\lambda \in (-\infty, \lambda_1)$ there exists a positive constant $\mu^* < +\infty$ such that $(1.1)_\mu$ has a minimal solution for all $\mu \in (0, \mu^*)$, and $(1.1)_\mu$ has no solutions if $\mu > \mu^*$.*

Proof. For $\lambda \in (-\infty, \lambda_1)$, set

$$\mu^* = \sup\{\mu \in R^+, \ | \ (1.1)_\mu \text{ has at least one solution}\}.$$

From Lemma 2.1 and Lemma 2.2 we have $0 < \mu^* < +\infty$. For any $\mu \in (0, \mu^*)$ there exists a $\overline{\mu} \in (\mu, \mu^*)$ such that $(1.1)_{\overline{\mu}}$ has a minimal solution $u_{\overline{\mu}}$. We can easily varify that $u_{\overline{\mu}}$ is a supersolution and 0 is a subsolution of $(1.1)_\mu$. Using the method of monotine iteration and strong maximum principle it follows that there exists a solution u_μ of $(1.1)_\mu$ such that

$$0 < u_\mu \leq u_{\overline{\mu}} \text{ for } x \in \Omega, \ \ \mu \leq \overline{\mu} \tag{2.1}$$

and $u_\mu$is a minimal solution.

Remark 2.1. For $\lambda \in (-\infty, 0)$, from the proof of Lemma 2.2 we can easily conclude that there exists a positive constant $\mu^{**} < +\infty$ such that the minimal solution u_μ satisfies

$$\lambda + pu_\mu^{p-1} \leq 0 \quad \text{for } \mu \in (0, \mu^{**}).$$

Indeed, we only need to choose the supersolution $\tilde{u} = \delta_0\varphi(x)$ satisfies $p(\delta_0\varphi(x))^{p-1} \leq -\lambda$.

Remark 2.2. There are no solutions of $(1.1)_\mu$ for all $\lambda \geq \lambda_1, \ \ \mu \geq 0$.

Indeed, if u is the solution of $(1.1)_\mu$, then

$$\lambda_1 \int_\Omega u\varphi dx = -\int_\Omega \Delta u\varphi dx = \int_\Omega (u^p\varphi + \lambda u\varphi + \mu f(x)\varphi)dx > \lambda \int_\Omega u\varphi dx$$

and thus $\lambda < \lambda_1$.

The proof of follow Lemma is similar to that of [6]. So we omit it.

Lemma 2.4. *Let u_μ is the minimal solution of $(1.1)_\mu$, for given $\lambda \in (-\infty, \lambda_1), \ \ \mu \in (0, \mu^*)$, the corresponding eigenvalue problem is*

$$\begin{cases} -\Delta\delta - \lambda\delta - pu_\mu^{p-1}\delta = \alpha\delta, \\ \delta|_{\partial\Omega}. \end{cases} \tag{2.2}$$

Then the first eigenvalue of (2.2) $\alpha_1 > 0$.

From Lemma 2.3, for any $\lambda \in (-\infty, \lambda_1)$ we can definite a set of minimal solution as follows:

$$A = \{u_\mu \ \ | \ \ \mu \in (0, \mu^*), u_\mu \ \ \text{is the minimal solution of} \ \ (1.1)_\mu\}. \tag{2.3}$$

Then we have:

Lemma 2.5. *There exists a positive constant C independent of μ such that $\|u\|_{H_0^1(\Omega)} \leq C$ for all $u_\mu \in A$.*

Proof. For any $u_\mu \in A$, from Lemma 2.4 we have

$$\int_\Omega |\nabla u_\mu|^2 dx = \lambda \int_\Omega u_\mu^2 dx + \int_\Omega u_\mu^{p+1} dx + \mu \int_\Omega f(x) u_\mu dx, \tag{2.4}$$

$$\int_\Omega |\nabla u_\mu|^2 dx - \int_\Omega (\lambda + p u_\mu^{p-1}) u_\mu^2 dx \geq \alpha_1 \int_\Omega u_\mu^2 dx \geq 0, \tag{2.5}$$

$$\int_\Omega |\nabla u_\mu|^2 dx \geq \lambda_1 \int_\Omega u_\mu^2 dx. \tag{2.6}$$

Using (2.4)–(2.6) we conclude that

$$(p-1)(\lambda_1 - \lambda) \int_\Omega u_\mu^2 dx \leq (p-1) \int_\Omega (|\nabla u_\mu|^2 - \lambda u_\mu^2) dx \leq p\mu \int_\Omega f u_\mu dx. \tag{2.7}$$

From $\mu \in (0, \mu^*)$ and Holder inequality and Young inequality we deduce

$$(p-1)(\lambda_1 - \lambda) \int_\Omega u_\mu^2 dx \leq p\mu^* C_1 \left(\frac{\delta}{2} \int_\Omega u_\mu^2 dx + \frac{1}{2\delta} \int_\Omega f^2 dx \right),$$

for any $\delta > 0$. Taking δ small enough such that $(p-1)(\lambda_1 - \lambda) - \frac{\delta p \mu^* C_1}{2} > 0$ we can obtain that there exists a positive constant C_2 independent of μ such that

$$\int_\Omega u_\mu^2 dx \leq C_2. \tag{2.8}$$

Using (2.8) and (2.7) we deduce that $\int_\Omega |\nabla u_\mu|^2 dx \leq C$ for some positive constant C independent of μ.

The Proof of Theorem 1. Suppose $\{\mu_j\}_{j\geq 1}$ is a increasing sequence in $(0, \mu^*)$ satisfying $\lim_{j\to\infty} \mu_j = \mu^*$. The corresponding sequence of solutions is $\{u_j\}_{j\geq 1} \subset A$. From Lemma 2.5 we can choose a subsequence still denoted by $\{u_j\}_{j\geq 1}$, such that

$$u_j \rightharpoonup \overline{u} \quad \text{weakly in} \quad H_0^1(\Omega)$$

for some non-negative function $\overline{u} \in H_0^1(\Omega)$. It is easy to prove that $\overline{u}$ is a solution of $(1.1)_{\mu^*}$. We can find a minimal solution because 0 is a subsolution for all $\mu > 0$.

§3. Existence of the Second Solution

Let u_μ be the minimal positive solution of $(1.1)_\mu$ for $\mu \in (0, \mu^*)$. In order to find a second solution of $(1.1)_\mu$ we introduce the following problem:

$$\begin{cases} -\Delta v = \lambda v + (v + u_\mu)^p - u_\mu^p, \\ v|_{\partial\Omega} = 0, \quad v > 0 \text{ in } \Omega. \end{cases} \tag{$(3.1)_\mu$}$$

Clearly, we can get another solution $v_\mu = u_\mu + \overline{v}$ if $(3.1)_\mu$ possesses a positive solution $\overline{v}$. To solve $(3.1)_\mu$ we set $g(x, v) = \lambda v + (v + u_\mu)^p - u_\mu^p - v^p$, and $a(x) = \lambda + p u_\mu^{p-1}$. We define the corresponding variational function of $(3.1)_\mu$ by

$$J(v) = \frac{1}{2} \int_\Omega (|\nabla v|^2) dx - \frac{1}{p+1} \int_\Omega |v|^{p+1} dx - \int_\Omega G(x, v) dx, \tag{3.2}$$

where $G(x,v)=\int_0^v g(x,s)ds$ and $v\in H_0^1(\Omega)$. For convenience, we use "$\|.\|$", "$|.|_q$" to denote the norms in $H_0^1(\Omega)$, $L^q(\Omega)$ respectively. Applying Theorem 2.1 In [2] we obtain

Lemma 3.1. *Let $\lambda\in(-\infty,\lambda_1)$, $\mu\in(0,\mu^*)$, if there exists some $v_0\in H_0^1(\Omega)$, $v_0\geq 0$, $v_0\not\equiv 0$ in Ω such that*

$$\sup_{t\geq 0} J(tv_0)<\frac{1}{N}S^{\frac{N}{2}}, \tag{3.3}$$

then problem $(3.1)_\mu$ possesses a solution.

In the following, we shall verify that the crucial condition (3.3) naturally holds for different $\lambda\in(-\infty,\lambda_1)$, $\mu\in(0,\mu^*)$. To this end, we set

$$w_\epsilon(x)=(N(N-2)\epsilon)^{\frac{N-2}{4}}\left(\frac{1}{\epsilon+|x|^2}\right)^{\frac{N-2}{2}}, \tag{3.4}$$

and

$$\psi_\epsilon(x)=\psi(x)w_\epsilon(x), \tag{3.5}$$

where $\psi(x)\in C_0^\infty(\Omega)$ is a cut-off function. For $\rho>0$, let $\psi(x)\equiv 1$ if $|x|<\rho$; $\psi(x)\equiv 0$ if $|x|\geq 2\rho$.

For $a>0$, $a<\max_{x\in\Omega} u_\mu(x)$, set

$$\omega=\{x\in\Omega,\ \mid\ u_\mu(x)>a>0\}\not\equiv\emptyset. \tag{3.6}$$

Without loss of generality we may suppose $0\in\omega$. Choosing $\rho>0$ small enough such that $B_{2\rho}\subset\omega$.

From [3] we have the following estimations

$$|\nabla\psi_\epsilon|_2^2=S^{\frac{N}{2}}+O\left(\epsilon^{\frac{N-2}{2}}\right), \tag{3.7}$$

$$|\psi_\epsilon|_{p+1}^{p+1}=S^{\frac{N}{2}}+O\left(\epsilon^{\frac{N^2}{2(N-2)}}\right), \tag{3.8}$$

$$|\psi_\epsilon|_2^2=\begin{cases} K_1\epsilon+O\left(\epsilon^{\frac{N-2}{2}}\right), & N\geq 5,\\ K_1\epsilon|\ln\epsilon|+O\left(\epsilon^{\frac{N-2}{2}}\right), & N=4,\\ O(\epsilon^{\frac{1}{2}}), & N=3,\end{cases} \tag{3.9}$$

where S is the best Sobolev constant and K_1 is a positive constant independent of ϵ.

Lemma 3.2. *Let ψ_ϵ be given by (3.5). Then there exists a constant $t_\epsilon>0$ such that*

$$\sup_{t\geq 0} J(t\psi_\epsilon)=J(t_\epsilon\psi_\epsilon) \tag{3.10}$$

and

$$J(t_\epsilon\psi_\epsilon)\leq\frac{1}{N}S^{\frac{N}{2}}-\int_{B_{2\rho}}\overline{G}(x,t_\epsilon\psi_\epsilon)dx-\begin{cases}\lambda K_1\epsilon+O\left(\epsilon^{\frac{N-2}{2}}\right), & N\geq 5,\\ \lambda K_1\epsilon|\ln\epsilon|+O(\epsilon), & N=4,\\ \lambda K_1\epsilon^{\frac{1}{2}}+O(\epsilon^{\frac{1}{2}}), & N=3,\end{cases} \tag{3.11}$$

where $\overline{G}(x,v)=\int_0^v\overline{g}(x,s)ds$ and $\overline{g}(x,s)=(s+u_\mu)^p-u_\mu^p-s^p$.

Proof. From the definition of J and Lemma 2.4 we can easily conclude that there exists a $t_\epsilon > 0$ such that (3.10) holds, like [4] we can prove that

$$0 < C_1 \le t_\epsilon \le C_2 < +\infty \tag{3.12}$$

as ϵ is small enough. Where C_1, C_2 are some constants independent of ϵ. Using (3.7)–(3.9) and (3.12) like [4] we may deduce (3.11).

Lemma 3.3. *The condition* (3.3) *naturally holds if one of the following assumption is satisfied:*

I) *For $\lambda \ge 0$ there exists some function $\overline{g}(v)$ such that $\overline{g}(x,v) \ge \overline{g}(v) \ge 0$ for $v \ge 0$, $x \in \Omega$ and*

$$\lim_{\epsilon\to 0^+} \epsilon \int_0^{\epsilon^{-\frac{1}{2}}} \overline{G}\left(\left(\frac{\epsilon^{-\frac{1}{2}}}{1+s^2}\right)^{\frac{N-2}{2}}\right) s^{N-1} ds = +\infty \tag{3.13}$$

for $N \ge 3$.

II) *For $\lambda < 0$, there exists some function $\overline{g}(v)$ such that $\overline{g}(x,v) \ge \overline{g}(v) \ge 0$ for $v \ge 0$, $x \in \omega$ and*

$$\lim_{\epsilon\to 0^+} \epsilon^{\frac{N-2}{2}} \int_0^{\epsilon^{-\frac{1}{2}}} \overline{G}\left[\left(\frac{\epsilon^{-\frac{1}{2}}}{1+s^2}\right)^{\frac{N-2}{2}}\right] s^{N-1} ds = +\infty \quad \text{for} \quad N \ge 5, \tag{3.14}$$

$$\lim_{\epsilon\to 0^+} \epsilon |\ln\epsilon|^{-1} \int_0^{\epsilon^{-\frac{1}{2}}} \overline{G}\left(\frac{\epsilon^{\frac{1}{2}}}{1+s^2}\right) s^3 ds = +\infty \quad \text{for} \quad N = 4, \tag{3.15}$$

$$\lim_{\epsilon\to 0^+} \epsilon^{\frac{1}{2}} \int_0^{\epsilon^{-\frac{1}{2}}} \overline{G}\left[\left(\frac{\epsilon^{-\frac{1}{2}}}{1+s^2}\right)^{\frac{1}{2}}\right] s^2 ds = +\infty \quad \text{for} \quad N = 3, \tag{3.16}$$

where ω is some nonempty open set in Ω and $\overline{G}(v) = \int_0^v \overline{g}(t)dt$.

Proof. The proof of case I) is the same as [2]. So we omit it. As for the case II) we can refer the article [4].

Lemma 3.4. *If $p \ge 2$ then*

$$(v+u_\mu)^p - v^p - u_\mu^p \ge p u_\mu v^{p-1} \quad \text{for all} \quad v \ge 0, \quad x \in \Omega. \tag{3.17}$$

Proof. For any $x \in \Omega$ set

$$h(s) = (u_\mu + s)^p - s^p - u_\mu^p - p u_\mu s^{p-1}, \quad s \in R^+.$$

Then $2h'(s) = (p-1)(p-2)u_\mu^2 \xi^{p-3} \ge 0$, where $\xi \in (s, s+u_\mu)$. Hence $h(s) \ge 0$ for all $s \in R^+$ because $h(0) = 0$.

Lemma 3.5. *If $p > 1$, then there exists a small constant $\delta > 0$ and a large $B > 0$ independent of x such that*

$$(v+u_\mu)^p - v^p - u_\mu^p \ge v^\delta \quad \text{for all} \quad v \ge B, \quad x \in \omega \subset \Omega, \tag{3.18}$$

where ω is given by (3.6).

Proof. Taking $0 < \delta < p-1$, let $m = \inf\{u_\mu(x), \mid x \in \omega\} > 0$, $M = \sup\{u_\mu(x), \mid x \in \omega\}$, then

$$\lim_{v\to+\infty} \frac{(v+u_\mu)^p - v^p - u_\mu^p}{v^\delta} \geq \frac{p}{\delta} \lim_{v\to+\infty} \frac{(p-1)(\xi+v)^{p-2}m}{v^{\delta-1}},$$

where $\xi \in (0, m)$. From Lemma 3.4 we may suppose $p < 2$, hence

$$\lim_{v\to+\infty} \frac{(v+u_\mu)^p - v^p - u_\mu^p}{v^\delta} \geq \frac{p}{\delta} \lim_{v\to+\infty} \frac{(p-1)v^{1-\delta}m}{(m+v)^{2-p}} = +\infty.$$

Thus there exists a constant $B > 0$ such that

$$\frac{(v+u_\mu)^p - v^p - u_\mu^p}{v^\delta} \geq 1 \text{ for all } v \geq B,\ x \in \Omega$$

which gives (3.18).

Lemma 3.6. *The condition* (3.3) *holds if* $\lambda \geq 0$, $N \geq 3$.

Proof. Form Lemma 3.5 there exists some constant $\delta > 0$ and $B > 0$ such that

$$(v+u_\mu)^p - u_\mu^p - v^p \geq v^\delta \text{ for } x \in \Omega,\ v \geq B.$$

It is easy to verify that

$$(v+u_\mu)^p - u_\mu^p - v^p \geq 0 \text{ for } x \in \Omega,\ v \geq 0.$$

Taking $\overline{g}(v) = B^\delta X_I(v)$ then

$$\overline{g}(x,v) = (v+u_\mu)^p - v^p - u_\mu^p \geq B^\delta X_I(v) = \overline{g}(v) \geq 0, \tag{3.19}$$

where $X_I(v)$ denote the characteristic function of $I = (B, +\infty)$. Thus

$$\overline{G}(v) = \int_0^v \overline{g}(s)dx \geq \beta > 0,$$

for some constant $\beta > 0$, and $V \geq B_1 > B$. So we have

$$\overline{G}\left(\left(\frac{\epsilon^{-\frac{1}{2}}}{1+s^2}\right)^{\frac{N-2}{2}}\right) \geq \beta,$$

for all s such that $\left(\left(\frac{\epsilon^{-\frac{1}{2}}}{1+s^2}\right)^{\frac{N-2}{2}}\right) \geq B_1$ and in particular for all $s \leq C\epsilon^{-\frac{1}{4}}$, where C is some constant and ϵ is small. Thus we have, for small ϵ

$$\epsilon \int_0^{\epsilon^{-\frac{1}{2}}} \overline{G}\left(\left(\frac{\epsilon^{-\frac{1}{2}}}{1+s^2}\right)^{\frac{N-2}{2}}\right) s^{N-1}ds \geq \beta\epsilon \int_0^{C\epsilon^{-\frac{1}{4}}} s^{N-1}ds = C'\epsilon^{1-\frac{N}{4}}.$$

The right hand side tends to $+\infty$ as $\epsilon \to 0^+$ if $N \geq 5$.

For $N = 4$, from Lemma 3.4 we have

$$(v+u_\mu)^p - u_\mu^p - v^p \geq Cv^2 \text{ for } x \in \Omega,\ v \geq 0.$$

Taking $\overline{g}(v) = Cv^2$ we can similarly deduce

$$\lim_{\epsilon\to 0^+} \epsilon \int_0^{\epsilon^{-\frac{1}{2}}} \overline{G}\left(\frac{\epsilon^{-\frac{1}{2}}}{1+s^2}\right) s^3 ds = +\infty.$$

For $N = 3$ we have $p = 5$. From Lemma 3.4 we have

$$(v+u_\mu)^p - u_\mu^p - v^p \geq Cv^4 \text{ for } x \in \Omega, \quad v \geq 0.$$

Taking $\overline{g}(v) = Cv^4$ we have

$$\lim_{\epsilon\to 0^+} \epsilon \int_0^{\epsilon^{-\frac{1}{2}}} \overline{G}\left(\left(\frac{\epsilon^{-\frac{1}{2}}}{1+s^2}\right)^{\frac{1}{2}}\right) s^2 ds = +\infty.$$

Using Lemma 3.3 we can immediately obtain that the condition (3.3) holds with $v_0 = \psi_\epsilon$ and ϵ small.

Lemma 3.7. *The condition* (3.3) *holds if* $\lambda < 0, \quad 3 \leq N \leq 5$.

Proof. By Lemma 3.4 we can choose $\delta > 0$ such that

$$(v+u_\mu)^p - u_\mu^p - v^p \geq Cv^{1+\delta} \text{ for } x \in \Omega, \quad v \geq 0.$$

If $N = 4,\ 5$ we can take $\overline{g}(v) = Cv^{(1+\delta)}$, then we may verify that $\overline{g}(v)$ satisfies the conditions (3.14), (3.15).

If $N = 3$, taking $\overline{g}(v) = Cv^4$, by Lemma 3.4 we may verify that $\overline{g}(x,v) = (v+u_\mu)^p - v^p - u_\mu^p \geq Cv^4 = \overline{g}(v)$ and (1.16) holds.

Applying Lemma 3.3 we can immediately get our lemma.

In the following we discuss the non-existence results about $(3.1)_\mu$ for $\lambda < 0, \quad N \geq 6$. To this end, we suppose that $\Omega = B_R(0)$ and $f(x)$ is a radial function with $f'(r) < 0$ for all $r \in (0, R)$. We first prove a Pohozave identity. Let

$$g(u_\mu, v) = (v+u_\mu)^p - u_\mu^p + \lambda v,$$

$$G(u_\mu, v) = \int_0^v g(u_\mu, s)\, ds.$$

Lemma 3.8. *If* $v \in H_0^1(\Omega)$ *is a solution of* $(3.1)_\mu$ *then*

$$\int_\Omega |\nabla v|^2 dx = \frac{2N}{N-2}\int_\Omega G(u_\mu, v)dx + \frac{2}{N-2}\int_\Omega \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx - \frac{R}{N-2}\int_{\partial\Omega}\left|\frac{\partial v}{\partial n}\right|^2 ds.$$

Proof. The Proof is the same as the proof of well-known Pohozaev's identity (see [1] for example). So we omit it.

Lemma 3.9. *Let* $N \geq 6$. *then for* $\lambda < 0$ *there exists a constant* $\mu^{**} (0 < \mu^{**} < \mu^*)$ *such that* $(3.1)_\mu$ *has no solution if* $\mu \in (0, \mu^{**})$.

Proof. If $(3.1)_\mu$ possesses a solution v, by Lemma 3.8 and using $(3.1)_\mu$ we deduce

$$\begin{aligned}&\int_\Omega v((v+u_\mu)^p - u_\mu^p + \lambda v)dx\\ &\le \frac{2N}{N-2}\int_\Omega \left(\frac{1}{p+1}((v+u_\mu)^{p+1} - u_\mu^{p+1}) - u_\mu^p v + \frac{1}{2}\lambda v^2\right)dx\\ &+\frac{2}{N-2}\int_\Omega \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx\end{aligned}$$

Thus

$$\begin{aligned}0\le\ & \int_\Omega \left[u_\mu((v+u_\mu)^p - u_\mu^p - pu_\mu^{p-1}v) + \left(\frac{N}{N-2}-1\right)\lambda v^2\right]dx\\ &+\frac{2}{N-2}\int_\Omega \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx\\ =\ & \int_\Omega \frac{1}{6}p(p-1)(p-2)(\xi+u_\mu)^{p-3}v^3dx + \frac{2}{N-2}\int_\Omega (pu_\mu^{p-1}+\lambda)v^2dx\\ &+\frac{2}{N-2}\int_\Omega \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx,\end{aligned}$$

where $\xi \in (0,v)$. Because $N \ge 6$ and hence $p-2 \le 0$, we have

$$0 \le \int_\Omega \frac{2}{N-2}(pu_\mu^{p-1}+\lambda)v^2dx + \frac{2}{N-2}\int_\Omega \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx.$$

It follows from Remark 2.1 that

$$\int_\Omega \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx > 0. \tag{3.20}$$

On the other hand, by $f(x) = f(r)$ and $f'(r) < 0$ we know that u_μ is a radial function and $u'_\mu(r) < 0$ $(r = |x|)$ in $(0,R)$ (see [5]). Thus

$$(\nabla u_\mu \cdot x) = u'_\mu(r)r < 0 \quad \text{for} \quad r \in (0,R).$$

It is easy to verify that $\frac{\partial G}{\partial u_\mu} \ge 0$ for all $u_\mu \ge 0$, $v \ge 0$. Hence

$$\int_\Omega \frac{\partial G}{\partial u_\mu}(\nabla u_\mu \cdot x)dx \le 0.$$

This is contradictory to (3.20).

Proof of Theorem 2. From Lemma 3.1 and Lemma 3.6 we can conclude that $(3.1)_\mu$ has a solution $\bar{v}$ for $\lambda \in [0,\lambda_1)$ and $\mu \in (0,\mu^*)$. We can obtain the second solution v_μ of $(1.1)_\mu$ by taking $v_\mu = u_\mu + \bar{v}$. Combining Lemma 2.3 we can complete our proof.

Proof of Theorem 3. The first part of Theorem 3 come from Lemma 3.1, Lemma 3.7 and Lemma 2.3 . The second part of Theorem 3 come from Lemma 3.9 and Lemma 2.3 .

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