FREE-ION LASERS WITH RADIATIVE ION COOLING

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(Submitted May 10, 1994)

Abstract

Free-ion lasers (FILs) are future machines which offer performance levels significantly beyond those presently available. These performance levels follow from the fact that the up-to-date technique permits one to obtain ion beams that possess much smaller emittances and much greater total stored energy than electron beams of the same relativistic factor. Using synchrotron radiation (SR) damping in future ultrahigh energy ($\gamma > 10^4$) ion storage rings or three-dimentional radiative ion cooling (RIC) in ion storage rings of arbitrary energy will permit one to increase the limiting current and to shorten the emittance of the ion beams and hence to increase the performance levels of FELs still further. The main objective of this work is to evaluate and discuss the possible machine parameters.

1. Introduction

Sources based on energy transformation of particle beams to coherent electromagnetic radiation can be named free-particle lasers (FPLs). Electrons are considered the most suitable active medium for FPLs since the electron mass is much smaller than the mass of other particles and hence electrons can be easily accelerated to the necessary relativistic factor $\gamma = \varepsilon/mc^2$ and will emit more powerful electromagnetic radiation in given external fields. That is why only free-electron lasers (FELs) have been investigated until now. However, in some cases free-ion lasers (FILs) may be preferable [1-3]. Indeed, the wavelength of the radiation emitted by a particle is $\lambda = \lambda_u(1 + p_\perp^2)/2\gamma^2$, where λ_u is the undulator period and $p_\perp$ is the deflecting parameter. The power of a particle beam is $P_b = mc^2\gamma i/e$, where m and e are the mass and charge of the particle and i is the current of the particle beam. Under the conditions of optimal generation, $p_\perp \sim 1$, the power can be presented in the form

$$P_b = \frac{mc^2 i}{e}\sqrt{\frac{\lambda_u}{\lambda}}.$$

The power P_b increases with m, i, and λ_u. Ion masses and ion beam currents are higher than electron ones. It is reasonable to adopt $\lambda_u \sim 1$ m for high power sources both of electron and ion beams. Under the condition of equal efficiencies, the limiting power of FILs can be three and more orders higher than the limiting power of FELs. For example, the electron beam of the APS possesses two orders higher emittance and five orders less stored energy than the proton beams of the LHC (under construction) and SSC (project) (see Table 1) [4]. The relativistic factors of the electrons and protons are nearly the same in these cases. The beam stored energy of the LHC storage rings will exceed 500 MJ.

Translated from a manuscript of the Lebedev Physics Institute, Russian Academy of Sciences.

1071-2836/94/1505-0403

Table 1.

Machine	Energy [GeV]	γ	Emittance [π m$\cdot$ rad]	Natural emittance [π m$\cdot$ rad]	Number of particles N	Beam stored energy
APS	7	$1.4 \cdot 10^4$	$8.2 \cdot 10^{-9}$	$7.5 \cdot 10^{-9}$	$8 \cdot 10^{12}$	9.2 kJ
Tevatron(p^+)	938	$1 \cdot 10^3$	$2.3 \cdot 10^{-9}$		$5 \cdot 10^{12}$	0.85 MJ
LHC(p^+)	$8.5 \cdot 10^3$	$9 \cdot 10^3$	$4.4 \cdot 10^{-10}$		$4.75 \cdot 10^{14}$	0.64 GJ
SSC	$2 \cdot 10^4$	$2.1 \cdot 10^4$	$4.7 \cdot 10^{-11}$	$7.1 \cdot 10^{-14}$	$1.27 \cdot 10^{14}$	0.46 GJ

In the case of protons and fully stripped ions of energy greater than 10 TeV/nucleus, the synchrotron radiation leads to a decrease in the emittances of the ion beams to natural values, and hence to the more unique quality of the proton and ion beams. By analogy with synchrotron radiation, a cooling method of nonfully stripped ion beams in storage rings of arbitrary energy based on the process of resonance Rayleigh scattering of a laser light by relativistic ions can be used at top energy of the rings [5]. It gives the possibility of storing the ultimately achievable high current, and production of low-emittance ion beams using multiple injection of ions in the storage ring.

A decrease in the limiting-from-below emittance and an increase in the limiting-from-above stored energy of the particle beams with increase in the particle mass means that FILs using both light and heavy ions can be considered as next-generation machines which can offer performance levels significantly beyond those presently available. This article is an attempt to adopt the developments in high-energy proton accelerator techniques to the case of the FILs.

2. Main consequences of the FPL theory

2.1. The minimum wavelengths λ_n of the electromagnetic radiation emitted by the relativistic charged particle in the undulator on the harmonic n in the free space and in the waveguide, accordingly, are defined by the undulator period λ_u, the relativistic factor γ, the cutoff wavelength of the waveguide mode λ_c, and the deflecting parameter $p_\perp$

$$\lambda_n^{fs} = \frac{\lambda_u}{2n\gamma^2}(1+p_\perp^2), \qquad \lambda_n^{wg} = \frac{\lambda_u\left(1 \pm \sqrt{1-(1+p_\perp^2)(1+\lambda_u^2/n^2\lambda_c^2)/\gamma^2}\right)}{n\,(1+\lambda_u^2/n^2\lambda_c^2)}, \tag{1}$$

where $\gamma = \varepsilon/mc^2$, $p_\perp = \left(\overline{|\vec{p}_\perp|^2}\right)^{1/2} = \left(\overline{|\vec{B}_\perp|^2}\right)^{1/2}/B_c$ is the root-mean-square relative transverse particle momentum, $\vec{p}_\perp = \gamma\vec{\beta}_\perp$, $\vec{\beta}_\perp = \vec{v}_\perp/c$, $\vec{v}_\perp$ is the transverse velocity of the particle, $\vec{B}_\perp$ is the transverse magnetic field strength of the undulator, $B_c = 2\pi mc^2/n^+e\lambda_u$, e is the electron charge, m is the mass of the particle, and n^+ is the number of ion charge states [6-8]. The spectral-angular energy emitted by the particle in the undulator attains the maximum value when the amplitude of the undulator magnetic field $B_m \simeq k_1 B_c$, where $k_1 \simeq 1$ depends on the variation law of the undulator field [6-8]. When the harmonic $n = 1$ is used, we have $p_\perp = B_m/\sqrt{2}B_c$, $k_1 \simeq 1$ for the plane harmonic undulator, and $p_\perp = B_m/B_c$, $k_1 = \sqrt{2}/2$ for the helical undulator. We have the coefficient $2\pi mc^2/e \simeq 19.7$ MG$\cdot$ cm when the protons are used.

2.2. The maximum gain α_m of the usual FPL that uses a thin homogeneous particle beam and a helical undulator with constant parameters in the case of $\alpha_m << 1$ is

$$\alpha_m' \simeq \frac{2\pi^2 f_m K_m^3 \lambda_u^2 p_\perp^2 i}{S\gamma^3 i_A,}|_{S\simeq\pi\sigma_{\gamma c}^2} \simeq \frac{8\pi^2K^2f_m}{\gamma}\frac{p_\perp^2}{1+p_\perp^2}\frac{i}{i_A}, \tag{2}$$

where i is the beam current, $i_A = mc^3/en^+$ (for protons $i_A \simeq 33$ MA), K is the number of undulator periods, S is the area of the amplified laser beam, $f_m \simeq 0.54$, and $\sigma_{\gamma c} = \sqrt{K\lambda_u\lambda_w/2\pi}$ is the radius of the *waist* of the amplified gaussian laser beam, when the Rayleigh length $l_R = \pi\sigma_{\gamma c}^2/\lambda_w = K\lambda_u/2$, [9,10]. Equation (2) is valid when the angular and energy spreads of the particle beams are $\Delta\alpha \ll \sqrt{1+p_\perp^2}/\gamma\sqrt{K}$ and $\Delta\varepsilon/\varepsilon \ll 1/K$. In the case of $K\lambda_u = 2l_R$ the area of the beam in the undulator is changed by less than twice [11].

2.3. The maximum power gain in the other limiting case, $\alpha_m \gg 1$, is

$$\alpha_m'' = \frac{1}{9}\exp\frac{2K\lambda_u}{l_g}, \tag{3}$$

where the e-folding gain length

$$l_g|_{\sigma_p>\sigma_{pc}} = \frac{2^{1/3}\gamma}{3^{1/2}\pi^{2/3}}\frac{(\lambda_u S_p)^{1/3}}{p_\perp^{2/3}}\left(\frac{i_A}{i}\right)^{1/3}, \qquad l_g|_{\sigma_p<\sigma_{pc}} = \frac{(1+p_\perp^2)^{1/2}\gamma^{1/2}\lambda_u}{2^{3/2}\pi p_\perp[\ln(\sigma_{pc}^2/\sigma^2)]^{1/2}}\left(\frac{i_A}{i}\right)^{1/2},$$

$\sigma_{pc}^2 = \lambda_u^2(1+p_\perp^2)^{3/2}(i_A/i)^{1/2}/16\pi^2\gamma^{3/2}p_\perp$, and S_p is the proton beam area [12].

2.4. The maximum rates of energy loss per particle in the parametric FPLs using the helical undulator and a beam consisting of a series of short ($< \lambda_1$) particle microbunches situated in sequence at distances of the emitted wavelength λ_1, or its subharmonic in free space, and in the circular waveguide (when $\lambda \sim r$, $\beta_{ph} \simeq \beta_g \simeq 1$), are

$$\left(\frac{d\varepsilon}{dy}\right)_{\max}^{fs} = \frac{\pi^2 mc^2}{\lambda_u}\frac{p_\perp^2}{1+p_\perp^2}\frac{i}{i_A}, \qquad \left(\frac{d\varepsilon}{dy}\right)_{\max}^{wg} = \frac{2mc^2p_\perp^2 l^d}{k_2\gamma^2 r^2}\frac{i}{i_A}, \tag{4}$$

where $\pi^2mc^2/i_A \simeq 0.3n^+$ MeV/kA, $2mc^2/i_A \simeq 0.061n^+$ MeV/kA, l^d is the e-folding damping length of the electromagnetic wave field strength in the waveguide, r is the radius of the waveguide, $k_2 = \int E^2 ds/\pi r^2 E_m^2 < 1$ is determined by the waveguide mode, E, E_m are the electric field strength and its maximum, s is the area, and β_{ph} and β_g are the phase and group velocities [13]. The rates of the particle energy loss ($d\varepsilon/dy$) in free space when the wide beam of the radius, $\sigma_p \gg \lambda\gamma/\sqrt{1+p_\perp^2}$, is used, and in the waveguide, tend to (4) at distances from the beginning of the undulator $l_p = 2\pi\sigma_p^2/\lambda_1$ and l^d, accordingly.

The FELs using prebunched beams are called parametric FELs [14]. The terms prebunched FELs, superradiant FELs, FELs based on the Dicke's radiation, and the spontaneous coherent undulator radiation sources can be used for this case. Parametric FELs (FPLs) represent the ultimate in FEL (FPL) capabilities [15].

2.5. In the case of the high-efficiency FPLs the particle microbunches of the beam must get the buckets, that is, the regions of the stable phase oscillations limited by the separatrix when they pass the distance $l_p < l_c = \lambda_u \cdot (\varepsilon/\Delta\varepsilon)$. The maximum energy spread of the particle beam must satisfy the condition $\Delta\varepsilon < \Delta\varepsilon_{\text{sep}}$, where $\Delta\varepsilon_{\text{sep}} = mc^2\sqrt{2B_m E_{wp}}/B_c$ is the maximum energy spread accepted by the bucket [2,8], $E_{wp} = 2\pi p_\perp A(y)i/c\lambda\gamma$ is the amplitude of the electric field strength of the wave emitted by the particle beam, and $A(y) = [\ln^2\sqrt{1+(y/l_p)^2} + \arctan^2(y/l_p)]^{1/2}$ [13]. That is why the beam current i in the case $l_c \geq l_p$, $A(y) \simeq 1$ must exceed the value

$$i_c = \frac{\gamma(1+p_\perp^2)i_A}{4p_\perp^2}\left(\frac{\Delta\varepsilon}{\varepsilon}\right)^2. \tag{5}$$

2.6. In order to prepare the modulated particle beam for the parametric FPL we can use special undulator modulators [8,10,14,16].

3. Main consequences of the accelerator theory

The main parameters which we want to optimize in FPLs are the gains (2), (3) and the maximum rate of energy loss per particle (4), which in turn require high current and low emittance beams.

The starting parameters of the particle beams are the number of particles per bunch N_b (or the bunch current), the total number of particles N (or the average current), the transverse dimensions σ_{px} and σ_{pz} along the x-, z- axes, the angular spreads α_{px} and α_{pz}, the longitudinal dimension σ_{py}, and the energy spread $\Delta\varepsilon$. These parameters determine the transverse and longitudinal emittances of the beam: $\epsilon_x = \pi\sigma_{px}\alpha_{px}$, $\epsilon_z = \pi\sigma_{pz}\alpha_{pz}$, and $\epsilon_y = \pi\sigma_{py}\Delta\varepsilon$. The emittance of the accelerated beam is determined either by the initial emittance of the beam injected into the accelerator or storage ring, or by their natural emittance $\epsilon^*_{x,z}$, which is the radiation damped equilibrium size of the phase-space area of the beam. In the case of the proton synchrotrons there are only two projects for storage rings, in which the essential synchrotron radiation takes place. They are the Superconducting Supercollider (SSC) in the USA, and Large Hadron Collider (LHC) at CERN (under construction). In the case of arbitrary energy storage rings, by analogy with synchrotron radiation damping, the radiative damping of non-fully stripped ion beams in storage rings of arbitrary energy based on the process of resonance Rayleigh scattering of a laser light by relativistic ions can be used at the top energy of the rings [5].

3.1. The transverse dimensions and angular spreads of the particle beam are defined by the emittances and the β-functions $\beta_{x,z}$ of the focusing or transport systems of the beams [17]

$$\sigma_{x,z} = \sqrt{\frac{\beta_{x,z}\epsilon_{x,z}}{\pi}} \qquad \sigma'_{x,z} = \sqrt{\frac{\epsilon_{x,z}}{\pi\beta_{x,z}}}. \tag{6}$$

In the case of the storage rings usually $\beta_{x,z} \simeq \overline{\beta}_{x,z} = R/\nu_{x,z}$, where R and $\nu_{x,z}$ are the averaged radius and betatron frequencies (tunes) of the ring.

3.2. The damping time of the vertical beam dimension in the storage ring determined by a synchrotron radiation is

$$\tau_z = \frac{3m^3c^5}{e^4n^{+4}\gamma\overline{|\vec{B}|^2}} = \frac{3m^4c^7}{e^5n^{+5}\overline{|\vec{B}||\vec{B}|^2\rho}}. \tag{7}$$

where $\vec{B}$ is the magnetic field strength of the bending magnets of the rings and ρ is the magnetic bending radius; for protons we have $3m^3c^5/e^4 \simeq 6.4\cdot 10^{18}$, $3m^4c^7/e^5 \simeq 2\cdot 10^{25}$ [14]. The damping times τ_x, τ_y of the horizontal and longitudinal beam dimensions accordingly are usually approximately equal to τ_z.

It follows from Eq. (7) that the damping time of the fully stripped heavy ions, such as ${}^{207}_{82}Pb$, is approximately half that of the proton.

3.3. The natural transverse emittances [4] and the natural energy spread squared of the particle beam are

$$\epsilon^*_{x,z} \simeq \frac{\hbar R\gamma^2}{mc\rho\nu^3_{x,z}}, \qquad \left(\frac{\Delta\varepsilon^*}{\varepsilon}\right)^2 = \frac{C_\gamma\gamma^2}{\rho J_s}. \tag{8}$$

where $C_\gamma = (55/2^5\sqrt{3})(\hbar/mc)$, and $J_s \sim 1$ to 2 is a constant determined by the magnetic structure of the accelerator [17]. For the protons, $C_\gamma \simeq 2.1\cdot 10^{-14}$ cm.

3.4. The limiting number of particles $N_{\rm lim}$ which can be stored in a ring is limited by the additional self-defocusing force. This is the so-called Laslett limit. In the ultrarelativistic case, $\gamma \gg 1$

$$N^L_{\rm lim} \simeq \frac{\pi h^2\nu_{x,z}|\delta\nu_{x,z}|\gamma}{Rr_0(\frac{h^2+\sigma_z(\sigma_x+\sigma_z)\varepsilon_1}{\sigma_z(\sigma_x+\sigma_z)b\gamma^2} + \varepsilon_1 + \varepsilon_2\frac{h^2}{g^2})}, \tag{9}$$

where $\delta\nu \simeq 0.25$ is the permitted shift caused by the self-defocusing force, $r_0 = n^{+2}e^2/mc^2 \simeq 1.54 \cdot 10^{-16}n^{+2}/A$ cm is the classical radius of the particle, ε_1 and ε_2 are the coefficients that depend on the geometry of the vacuum chamber and the magnetic pole faces accordingly, h and g are the half-heights of the vacuum chamber and the gap of the bending magnet of the ring, $b = \bar{i}/i$ is the bunching parameter, A is the mass number of the nucleus, and the constants ε_i in the case of flat, parallel, infinitely wide walls of the vacuum chamber and flat, parallel magnet pole faces are equal, $\varepsilon_1 = \varepsilon_2 = \pi^2/48$ [17]. The value (9) does not depend on the length of the particle beam or the bunching parameter, and tends to its maximum value $N_{\mathrm{lim}.m} = N_{\mathrm{lim}}|_{b=1}$ when $b \gg b_c = h^2/\sigma_z(\sigma_x + \sigma_z)\gamma^2$, $(h^2 \gg \sigma_z(\sigma_x + \sigma_z)\varepsilon_1)$.

When the bunching parameter $b < b_c$, the current $i = ecN_{\mathrm{lim}}/2\pi Rb$, tends to its maximum value

$$i^L_{\mathrm{lim}.m} = i_{\mathrm{lim}}|_{b\to 0} \simeq \frac{\nu_{x,z}|\delta\nu_{x,z}|\sigma_z(\sigma_x + \sigma_z)\gamma^3}{2R^2}\, i_A. \tag{10}$$

In order to store values of the number of particles and the total energy of the beam that are close to their maximum values, the total length of the bunches should exceed the value $l^{\mathrm{tot}} = 2\pi Rb_c = 2\pi Rh^2/\sigma_z(\sigma_x+\sigma_z)\gamma^2$.

3.5. The second particularly important limiting number of the particles which can be stored in a ring is determined by the Keil-Schnell limit of the longitudinal instability

$$i^{\mathrm{K-Sch}}_{\mathrm{lim}.m} = \frac{i_A\beta_s^2\gamma_s|\eta|}{c|Z_n|/n}, \tag{11}$$

where $\eta = d\ln\omega/d\ln p = 1/\gamma^2 - 1/\gamma_t^2 = 1/\gamma^2 - \alpha$ is the slippage parameter, α is the momentum compaction, γ_t is the transition energy, Z_n is the longitudinal impedance, and n is the harmonic number of the revolution frequency [18].

3.6. In the ultrarelativistic case the electric field strength produced by the particle bunches on the surface of the vacuum chamber is $E \simeq \pi i/ch$ or E (V/cm) $\simeq 94i(\mathrm{A})/h(\mathrm{cm})$. At fields $E \sim 4 \cdot 10^6$ V/cm ($h \sim 1$ cm, $i \sim 5 \cdot 10^4$A) unwanted autoemission from the vacuum chamber walls is possible. It can impose an additional limit on the maximum current of the storage rings.

4. Radiative ion cooling

There are some effective methods of particle beam cooling in storage rings that permit one to decrease the phase volumes (emittances) of particle beams. They are a method based on the radiative reaction force that appears when particles emit synchrotron radiation [19], an electron cooling method [18], and a stochastic cooling method [20,21]. In [5] a three-dimensional radiative cooling method of non-fully stripped ion beams in storage rings was investigated where the process of resonance Rayleigh scattering of laser light by relativistic ions was used. That method can be considered as a development of a laser cooling method of nonrelativistic ions in storage rings [22,23].

Let us consider the following scheme of ion beam cooling. In the straight section of a storage ring, the laser photon beam is directed toward the ion beam and is scattered by the ions. The photon energy is of such value that in the coordinate system connected with the average ion velocity it was close to the transition energy between definite electron states in the ions. In this case, the cross-section of resonance Rayleigh photon scattering will be much greater than the Thomson one. In the relativistic case, the energy of scattered photons will be much greater than the energy of the laser photons. As a result, the power scattered by an ion can be rather high. Since the scattered radiation is directed mainly along the direction of ion velocity, it will be decelerate and lead to damping of vertical and horizontal betatron oscillations as well as phase oscillations of ions in the storage ring. The rate of damping can be tuned by adjusting the gradient of laser intensity in the radial direction.

In the moving coordinate system, the scattered radiation is spherically symmetrical and the energies of incident and scattered photons are about the same. In the laboratory coordinate system, the scattered radiation will be directed mainly in the ion velocity direction in the narrow interval of angles $\Delta\theta \sim 1/\gamma$, where γ is the ion relativistic factor. The energy of the scattered photons will be Doppler shifted. Their maximum energy is $\hbar\omega_{\max} \simeq (1+\beta)^2\gamma^2\hbar\omega_l$, where $\hbar\omega_l$ is the energy of laser photons, $\beta = v/c$, and v is the ion velocity. The average energy of the scattered photons $\hbar\overline{\omega} \simeq \hbar\omega_{\max}/(1+\beta)$, hence the scattered radiation can be presented in the form

$$P^s = (1+\beta)^2\gamma^2 \int \sigma_{\omega_l} I_{\omega_l} d\omega_l, \tag{12}$$

where $\sigma_{\omega l}$ is the cross-section of laser photon scattering by ions when the photons possess the frequency ω_l, and I_{ω_l} is the spectral intensity of the laser beam. In (12), one of the factors $(1+\beta)$ takes into account motion of ions toward the photon beam.

In the coordinate system connected with the moving ion, the scattering cross-section has the form

$$\sigma_{\omega l} = \frac{g_2}{g_1}\frac{\pi^2 c^2}{\omega_0'^2}\Gamma'_{2,1} g(\omega'_0, \omega'_l), \tag{13}$$

where $g_{1,2}$ are the statistical weights of the states 1 and 2, $\Gamma'_{2,1} = 2r_e\omega_0'^2 f_{1,2}g_1/cg_2 \ll \omega'_0$ is the probability of the spontaneous photon emission of the excited ion or the natural linewidth, $g(\omega'_0,\omega'_l) = (\Gamma'_{2,1}/2\pi)/[(\omega'_l - \omega'_0)^2 + \Gamma'^2_{2,1}/4]$ is the Lorentzian ($\int g(\omega)d\omega = 1$), $r_e = e^2/mc^2$ is the classical electron radius, e and m_e are its charge and mass, ω'_0 is the transition frequency between the states 1 and 2, ω'_l is the frequency of the scattered laser wave, and $f_{1,2}$ is the transition strength [24,25].

The cross-section (14) has a maximum $\sigma_{\max} = \sigma|_{\omega'_l=\omega'_0} = g_2\lambda_0'^2/2\pi g_1$, where $\lambda'_0 = 2\pi c/\omega'_0$ is the resonance wavelength in the oscillator coordinate system. The transition from the first excited level to the ground state of sodium-like ions corresponds to $f_{1,2} = 0{,}42$, $g_2/g_1 = 4$, $\lambda'_0 = 4/3Z^2_{\text{eff}}R'$, where $R' = e^4 m_e/4\pi c\hbar^3 = 109678$ cm^{-1} is the Rydberg constant, $Z_{\text{eff}} = Z_i + 1 \le Z$, Z_i is the ion charge, and Z is the nuclear charge.

When an ion moves at an angle θ to the axis of the straight section of the storage ring toward the laser wave, it experiences a force of frequency $\omega_i = \omega_l(1+\beta\cos\theta)$, where ω_l is the laser frequency in the laboratory coordinate system. In the ion coordinate system, the laser frequency will be γ times greater. This means that the laser frequency in this coordinate system is $\omega'_l = \omega'_i = \omega_i\gamma = \omega_l(1+\beta\cos\theta)\gamma$. In the relativistic case, $\beta = \sqrt{1-1/\gamma^2} \simeq 1$, $\gamma \gg 1$, $\theta \ll 1$

$$\omega'_l = 2\omega_l\gamma(1-\theta^2/4). \tag{14}$$

From Eqs. (13) and (14) it follows that the cross-section of laser photon scattering by ions strongly depends on ion energy and angle θ. It is close to its maximum value in the narrow frequency range $\Delta\omega'_l/\omega'_l = \Delta\omega_l/\omega_l \simeq 4\pi r_e f_{1,2}g_1/g_2\lambda'_0$. When the ion beam has a high angular spread $\Delta\theta_b$ and a high energy spread $\Delta\gamma_b$, then, for effective interaction of all ions with the photon beam, the bandwidth of the spectral line according to Eqs. (2) and (3) should exceed the value

$$\frac{\Delta\omega_l}{\omega_l} > \frac{\theta_b^2}{4} + \frac{\Delta\gamma_b}{\gamma}. \tag{15}$$

When condition (15) is satisfied, and assuming that the spectral intensity of the laser beam is distributed homogeneously in the frequency range $\Delta\omega_l$, the scattered power (12) can be represented in the form

$$P^s = 4\pi r_e\lambda'_0 f_{1,2}\left(\frac{\omega_l}{\Delta\omega_l}\right)I\gamma^2, \tag{16}$$

where I is the total laser intensity. In this case Eqs. (1) and (2) are valid in the spontaneous scattering conditions $4I\overline{\sigma}/\hbar\omega_l < \Gamma_{2,1}/\gamma$, where $\overline{\sigma} = (1/I)\int\sigma_{\omega,l}I_{\omega,l}d\omega_l = \pi^2 c r_e f_{2,1}I_{\omega,l}/\gamma I = \pi r_e\lambda'_0 f_{2,1}\omega_l/\Delta\omega_l$ (see Appendix 1).

This means that the intensity of the laser must satisfy the condition

$$I < I_c = \frac{\pi c g_1 \hbar\omega_0'}{\gamma^2 g_2 \lambda_0'^3}\frac{\Delta\omega_l}{\omega_l}. \tag{17}$$

The damping times of betatron and phase oscillations are determined by the emitted (scattered) radiation. Let us consider as an example the simplest case of the vertical betatron oscillations [17,19]. In this case, the damping time is

$$\tau_z = \frac{2m_i c^2\gamma}{\overline{P}^s} = \frac{SR}{c n_{\rm int}\lambda_0' l_{\rm eff}\gamma f_{2,1}}\Big(\frac{\Delta\omega_l}{\omega_l}\Big)\Big(\frac{P_A}{P_l}\Big), \tag{18}$$

where $\overline{P}^s = n_{\rm int} l_{\rm eff} P^s/2\pi R$ is the mean scattered power, $P_A = m_e m_i c^5/e^2$, m_i is the ion mass, P_l is the laser power, S is the cross-sectional area of the laser beam, $n_{\rm int}$ and $l_{\rm eff}$ are the number of interaction regions and the effective interaction length of the laser and photon beams, and R is the mean radius of the storage ring orbit.

The area of the cross-section of the laser beam must exceed that of the ion beam. The length l_{eff} is determined as the minimum of the straight section length and double Rayleigh length $l_R = S/\lambda_l$, where $\lambda_l = 2\pi c/\omega_l$ is the laser wavelength (within the limits of this length, the laser area exceeds the minimum area by less than two times). For helium ions ${}^2_3\mathrm{He}^+$, the power $P_A \simeq 4.8\cdot 10^{13}$ W. In the case of $l_{\rm eff} = 2L_R$:

$$\tau^z = \frac{R}{c n_{\rm int} f_{1,2}}\Big(\frac{\Delta\omega_l}{\omega_l}\Big)\Big(\frac{P_A}{P_l}\Big). \tag{19}$$

The damping times of radial betatron oscillations and phase oscillations are

$$\tau_x = R/c\zeta_x = \tau_z, \qquad \tau_s = R/c\zeta_s = \tau_z/2. \tag{20}$$

where $\zeta_x =< (RP^s/2m_i c^3\gamma)[1-(1-2n_l)R\psi/\rho] >$, $\zeta_s =< (RP^s/2m_i c^3\gamma)[2+(1-2n_l)R\psi/\rho] >$, $(n_l = -(\rho/2)(\partial \ln I/\partial x))$, ρ is the instantaneous radius of the storage ring orbit, and ψ is the momentum compaction function [19].

The quantum nature of the photon scattering will lead to an equilibrium radial dimension and energy spread of the ion beam (see Appendix 2)

$$\sigma_x = \alpha R\sqrt{\frac{0.7(1+\beta)\hbar\omega_0'}{m_i c^2}}, \qquad \frac{\sigma_\gamma}{\gamma} = \sqrt{\frac{0.35(1+\beta)\hbar\omega_0'}{m_i c^2}}, \tag{21}$$

where $\alpha \simeq \nu_x^{-2}$ and ν_x are the momentum compaction factor and radial betatron oscillation tune of the storage ring, accordingly. The equilibrium vertical dimension of the ion beam $\sigma_z \ll \sigma_x$.

When a monochromatic laser beam and modulation of equilibrium relative energy γ_s in storage rings are used for the time and energy intervals $\Delta t < \tau_{z,x,s}$, $\Delta\gamma_s \simeq \Delta\gamma_b$ under the condition that $\gamma_s < \omega_0'/2\omega_l$, or if chirped laser pulses are used, the values of Eq. (21) can be made $\sqrt{m_i c^2/\hbar\omega_0'}$ times less.

Under these conditions the damping time of phase oscillations can be shorten. The use of synchrotron-betatron coupling resonances will permit one to transform the transverse ion oscillations to longitudinal ones, and hence to shorten the damping times of transverse betatron oscillations.

The ion can lose its electrons in a strong external electric field [26]. The decay time τ_d connected with the ionization of sodium-like ions by the external fields is

$$\tau_d = \frac{\hbar\xi n^2}{4\overline{\alpha}^2 m_e c^2 Z_{\rm eff}^2}\exp\Big(\frac{2}{3\xi}\Big), \tag{22}$$

where $\overline{\alpha} = e^2/\hbar c \simeq 1/137$, $\xi = E/E_i$, E is the electric field strength, $E_i = Z_{\text{eff}} e/a_i^2 = E_0 Z_{\text{eff}}^3/n^4$, $a_i = a_0 n^2/Z_{\text{eff}}$, $E_0 = e/a_0^2 = 1.72 \cdot 10^7 = 5.16 \cdot 10^9$ V/cm is the atomic field strength, $a_0 = \hbar^2/m_e e^2 = 0.529$ Å is the Bohr radius, and n is the main quantum number. We have $\tau_d|_{\xi=0.015} = 0.59$ sec and $\tau_d|_{\xi=0.012} = 3 \cdot 10^4$ sec or ~ 8 h. In the ion coordinate system an electric field of strength $E = \beta\gamma H$ appears, where H is the magnetic field strength of the storage ring. That is why the relative energy of the relativistic ions cannot exceed the value

$$\gamma = \gamma_c = \frac{E_0 Z_{\text{eff}}^3 \xi}{H n^4}|_{\xi=0.012} \simeq \frac{2.06 \cdot 10^6 Z_{\text{eff}}^3}{H n^4}. \tag{23}$$

Example. In a storage ring of energy 500 GeV ($\gamma \simeq 177$), a helium ion beam ${}_3^2\text{He}^+$ is accelerating. It has a circular cross-section of area $\simeq 1$ cm^2, angular spread $\theta_b = 10^{-2}$, and energy spread $\Delta\gamma_b/\gamma = 5 \cdot 10^{-3}$. The average radius of the storage ring is $R = 400$ m (superconducting version), the radial betatron oscillation *tune* $\nu_x \simeq 10$, and the average number of the stored particles $N_i = 2 \cdot 10^{12}$. For cooling, a laser beam is used that excites in an open resonator a reactive power of 1 kW at a laser beam area $S = 1$ cm^2, and a spectral bandwidth $\Delta\omega_l/\omega_l = 5 \cdot 10^{-3}$. Ten interaction points are used ($n_{\text{int}} = 10$). In this case the exciting energy of the ion is $\hbar\omega_0' \simeq 40.6$ eV, $\lambda_0' \simeq 3.04 \cdot 10^{-6}$ cm, $\Gamma_{2,1} = 7.57 \cdot 10^9$ sec $^{-1}$, and, according to (14), the laser wavelength must be about $\lambda_l \simeq 10.6 \cdot 10^{-4}$ cm, the effective interaction length $l_{\text{eff}} \simeq 2l_R \simeq 19$ m, the damping time of the vertical ion betatron oscillations (28) $\tau_z \simeq 161$ sec, the equilibrium radial dimension and energy spread of the ion beam $\sigma_x = 0.65$ mm and $\sigma_\gamma/\gamma = 1.15 \cdot 10^{-4}$, respectively, and the power of the quasimonochromatic spontaneous radiation scattered by the ion beam $N_i \cdot \overline{P}^s \simeq 4.2$ kW, at wavelength $\lambda \simeq 0.85$ Å. If the reflectivity coefficient of the laser mirrors is $\simeq 0.99$, the power of the laser is ~ 20 W. CO_2 lasers or free-electron lasers can be used [15,27-30].

In the case of ${}_3^2\text{He}^+$, according to Eq. (23), we have $\gamma_c = 174$ at $n = 2$ and $H = 5.82$ kG, or at $n = 1$ and $H = 93.2$ kG. This means that in the case $\gamma = 174$ the ${}_3^2\text{He}^+$ ions can be stored in storage rings with ordinary magnetic fields $H \sim 10^4$ G and short straight sections where the excited ions can move in the fields without decay, or in storage rings with superconducting fields $H \sim 10^5$ G, and long straight sections where the excited ions will have time to relax to the ground state before entering bending magnets with high magnetic fields. In the latter case the length of the straight section l_{ss} can be found from the condition $(cn_{\text{int}}\tau_z/2\pi R)\exp[-(l_{ss} - l_{\text{eff}})/l_e] \ll 1$, where $l_e = \gamma c/\Gamma_{2,1} \simeq 7$ m is the length of the ion relaxation. In the above example we must use a storage ring with straight sections of length $l_{ss} > 150$ m.

For cooling of heavy ions with $Z_{\text{eff}} \gg 1$, and moderate γ, where high excitation energies are necessary, spontaneous undulator radiation sources in short-wavelength regions [15] can be used. The excitation of the ions by means of strong magnetic fields that are not strong enough to cause ionizing decay can be used instead of laser excitation. Heavy ions, both positive and negative, can be cooled. The blackbody radiation of the vacuum chamber must be taken into account [31-34], and the chamber must be cooled of warm storage rings, if necessary, to avoid the background of the backward resonance Rayleigh scattered photons, or to use it, e.g., for beam diagnostics.

The estimates that are made in this study show the possibility of effective cooling of relativistic ion beams in storage rings. This method will permit one to store high-intensity and low-emittance ion beams for elementary particle physics, spectroscopy, ion thermonuclear fusion [30], free-ion lasers, quantum generators on moving ions [35,36], and high-power hard quasi-monochromatic spontaneous incoherent backward Rayleigh scattering sources [37]. Resonance Rayleigh interaction of small-emittance monoenergetic moving ions with two counterpropagating laser beams or with a laser beam and an undulator will permit one to transform effectively an ion beam in a series of short microbunches, and to realize a superradiant lasing regime.

In Table 2 the main parameters of the storage rings SSC and LHC are presented. In this table the value

$2\sigma_s$ is the length of the particle bunch, l_{bs} is the bunch spacing, l_{ss} is the length of the straight section of the storage ring, α is the momentum compaction factor of the storage ring, τ_s is the longitudinal damping time, and $\varepsilon^{\text{beam}}$ is the total stored energy of the beam. We assumed that the constant $J_s = 1.5$ when calculating the natural energy spread. These parameters correspond to work of the machines in the colliding beam regime that is optimal for the elementary particle physics, and far from optimal for the FPL technique. Nevertheless Table 2 permits one to evaluate the possible parameters of the particle beams that can be achieved after their transformation for more effective utilization in the FPL regime. The main purpose of this transformation is the production of high instantaneous peak currents in the beam bunches. This goal can be reached in different ways [39,40].

Table 2. Machine parameters

Value Units	ε [Tev]	γ	ϵ_x [π m$\cdot$ rad]	ϵ_x^* [π m$\cdot$ rad]	ρ [km]	R [km]	ν_x	τ_z h	τ_s h	h [cm]	B [kG]	α 10^{-4}
SSC[4]	20	$2.13 \cdot 10^4$	$4.7 \cdot 10^{-11}$	$7.1 \cdot 10^{-14}$	11.7	13.9	123	20		3.2	66	2.24
LHC[15]	8.5	$9 \cdot 10^3$	$4.4 \cdot 10^{-10}$		2.83	4.24		22	11		100	2.94

i [A]	i [mA]	$i_{\text{lim}.m}$ [kA]	$N_{\text{lim}.m}$ 10^{17}	N 10^{14}	N_b 10^{10}	$2\sigma_s$ [cm]	$\varepsilon^{\text{beam}}$ [MJ]	l_{bs} [m]	l_{ss} [km]	$\Delta\varepsilon/\varepsilon$ 10^{-4}	$\Delta\varepsilon^*/\varepsilon$ 10^{-6}	$\bar{\beta}$ [m]	σ_x [mm]	σ_x' rad
2.0	73	500	2.7	1.27	0.75	6	406	5.0	6.0	5	2.3	10^2	0.1	10^{-6}
45	0.85			4.75	10	11	640	4.5	0.8	5	2.0			

After the production of high-current particle bunches by using transverse or longitudinal phase space the emittances of the bunches will be increased. In this case it takes some damping time τ_z before the emittances acquire acceptable values. Both synchrotron radiation and Rayleigh scattering of a laser light by ions can be used. In this case $\varepsilon_x^* << \varepsilon_x^{**} < \varepsilon_x$.

5. Possible parameters of the FILs

In Table 3 examples of FPLs based on proton storage rings such as the superconducting version of the ISR, Tevatron, LHC, SSC are presented. FPLs No. 1, 2, 6 operate in the parametric regime, and the others are oscillators and amplifiers. In the examples the energy loss per particle, the laser radiation energy, and the laser power were calculated from the expressions $\Delta\varepsilon^l = (d\varepsilon/dy)_{\text{max}} K\lambda_u$, $\Delta\varepsilon^r = \varepsilon^b\eta$, and $i\Delta\varepsilon^l/e$, respectively.

In examples 1 and 2 the wavelengths of the emitted radiation correspond to to the back (+) and forward (-) propagation of the radiation [38]. We assumed that the H_{11} modes ($\lambda_c = 3.41r$) are excited, $k_2 = 0.5$, $r = 1$ cm and 0.3 cm, $l^d = 30$ m and 3 m, and, accordingly, h=3 cm, $2\pi R = 150$ m, $\nu_{x,z} = 20$. The field strengths in the waveguides $E_m = (d\varepsilon/dy)^{wg}_{\text{max}}\gamma/ep_\perp$ are equal to ~ 0.46 and 1.28 MV/cm, and, accordingly, the limiting current $i^{\text{lim}}_{\text{max}} = 2.2$ kA. Use of high-quality waveguide resonators is possible.

It follows from examples 3 to 6 that the length of the undulator, $\sim 2 - 5$ km, is enough to reach the FPL gain $\sim 10^3$ at wavelengths $\sim 2.5 - 50$ Å. This length is a small part of the storage ring dimensions. A system of undulators with dispersive sections similar to the optical klystron can be used to increase the gain, or to shorten the length of the undulator.

To obtain the high instantaneous peak currents in the bunches of the storage rings for the FPL regime, some transformations of the storage ring beam can be done [39,40]. At the moment when the minimum dimensions of the bunches in the storage rings are reached, the bunches can be directed to the bypass of the storage ring where the undulator is installed, or extracted from the storage ring and switched to the undulator

Table 3. List of FPLs parameters

FPL Regime	No.	1 Parametric	2 Parametric	3 Amplifier	4 Amplifier	5 Amplifier	6 Parametric
Proton energy	ε [TeV]	0.03	0.03	2	20	20	20
Gamma-factor	$\gamma/10^3$	$32/10^3$	$32/10^3$	2.13	21.3	21.3	21.3
Peak current	i [kA]	0.8	2	10	100	100	100
Beam size	σ_p [mm]	3	3	0.5	0.1	0.1	0.1
Energy spread	$10^{-3}\Delta\varepsilon/\varepsilon$	3	3	3	0.5	0.5	0.5
Number of particles	$N \cdot 10^{14}$	3	3	3	1.6	1.6	1.6
Beam energy	ε^b [MJ]	1.5	1.5	96	512	512	512
Bunch length	l^t [m]	1.9	0.75	1.5	7	7	7
....	....	....	....	....	....	....	....
Undulating period	λ_u [m]	0.2	0.2	2	2	0.2	2
Number of periods	K	150	15	300	10^3	$25 \cdot 10^3$	$3 \cdot 10^3$
Undulating field	B [kG]	100	100	100	100	100	100
Deflection parameter	$p_\perp$	0.1	0.1	1	1	0.1	1
....	....	....	....	....	....	....	....
Wavelength	λ1.2 cm	0.1 mm	4400 Å	44 Å	2.2 Å	44 Å	
Small gain	α'_m			0.3			
Gain length	l''_g [km]				0.42	1.05	0.42
High gain	α''_m				1520	1520	
Characteristic radius	σ_{pc} [mm]			0.12	0.68	0.13	0.68
Characteristic current	i_c [kA]						81
Light radius	$\sigma_{\gamma c}$ [mm]			0.65	1.2	1.34	
Growth length	l_p [m]			14.3	14.3	286	
....	....	....	....	....	....	....	....
Energy loss	$(\frac{d\varepsilon}{dy})^{fs}_m$ [MeV m]						7.8
Energy loss	$(\frac{d\varepsilon}{dy})^{wg}_m$ [MeV m]	0.29	0.8				
Energy loss per particle	$\Delta\varepsilon^l$ [GeV]						46.8
Laser efficiency	$\eta = \frac{\Delta\varepsilon^l}{\varepsilon}$[%]						0.23
Laser power	P	6.91 GW	4.8 GW				4.7 PW
Laser energy	$\Delta\varepsilon^r$	43.7 J	12 J				0.9 MJ

of the single-pass FPL. After the production of the high-current particle bunches by using longitudinal phase-space the emittances of the bunches will be increased. It is necessary to monitor transverse or the increase of the bunch emittances, which must not exceed the acceptable values. In the case of ultrahigh energy storage rings or in the case of storage rings using radiative ion cooling we can wait for some damping times τ_z at which the bunch emittances will be decreased to the desirable values.

The requirements of high peak bunch current of the proton and heavy ion storage rings of FPLs, superhigh gradient linear colliders [39,41], and proton and heavy-ion inertial fusion installations [40,42-44] are similar.

A magnetic field of $\sim 10^5$ G is widely used in accelerator technology. Fields of $4 \cdot 10^5$ G are achievable, but rather expensive [45]. The main problem at present is production of high currents of $\sim 10 - 100$ kA with high quality in existing or planned accelerators and storage rings. Synchrotron radiation of SSC, LHC, and future storage rings with an energy of ~ 50 TeV or greater permits one to overcome this problem.

In the oscillator regime of the X-ray FPLs, Bragg reflecting ring resonators or multilayer mirrors can be used [46,47]. FPL amplifiers based on self-amplified spontaneous emission (SASE) and regenerative amplifiers are attractive [47].

X-ray FELs or a system of lasers placed in sequence, serving as harmonic generators and amplifiers, can be used in undulator modulators of parametric FPLs as driver lasers. The subharmonic modulation of the proton beams can be used.

6. Conclusion

The estimates that were made in this study show the possibility of constructing the high-power high-efficiency FPLs in the centimeter to X-ray wavelength region when using the intense high-quality beams of large proton accelerators and storage rings. Such sources may be attractive for many applications. One of these applications is in nuclear and elementary particle physics when the colliding X-rays of the FPLs and proton or heavy ion beams are used. The energy of the SSC is enough for $e^{\pm}$ pair-production and for photo-nuclear physics. With increase in energy of the storage rings, photo-production of other particles can be observed [2].

It is interesting to note that the energy, hardness, directivity, and monochromaticity of the gravitational radiation of particles such as the electromagnetic radiation emitted in the external fields are strongly increased with the relativistic factor γ. This means that electromagnetic FPLs are simultaneously gravitational FPLs. Gravitational FPLs can be attractive in fundamental physics [2]. The intensity of gravitational FPLs is much greater than the intensity of gravitational synchrotron radiation considered in [48].

FPLs programs at the storage rings can start in a parasitic mode like the programs of first-generation synchrotron and undulator radiation sources in electron and proton synchrotrons and storage rings. Dedicated FPLs based on storage rings of energy of ~ 50 TeV or greater could become monochromatic, continuously tunable powerful sources of hard X-radiation.

Appendix 1

Denoting, as usual, the relative probabilities (populations) of the atomic states belonging to levels $|1>$ and $<2|$ as n_1 and n_2, respectively, we can write down the system of equations for them:

$$\begin{cases} dn_2/dz = [(1+\beta)\dot{n}_\gamma\overline{\sigma}/\beta c](n_1 - n_2) - n_2/\beta c\tau_{2,1}, \\ n_1 + n_2 = 1, \end{cases} \tag{24}$$

where $\dot{n}_\gamma = I/\hbar\omega_l$ is the photon flow density and $\tau_{2,1} = \gamma/\Gamma'_{2,1}$ is the decay time. The solution of the system in the case $n_2|_{z=0} = 0$ is

$$n_2(z) = \begin{cases} n_{2m}(1 - e^{-z(1+D)/l_e}), & 0 \le z \le l_{\text{eff}} \\ n_{2m}(1 - e^{-l_{\text{eff}}(1+D)/l_e})e^{-(z-l_{\text{eff}})(1+D)/l_e}, & l_{\text{eff}} \le z \le \infty, \end{cases} \tag{25}$$

where $n_{2m} = 0.5D/(1+D) < 0.5$, $l_e = \beta c\tau_{2,1}$ is the length of the ion relaxation, and $D = 2(1+\beta)\dot{n}_\gamma\overline{\sigma}\tau_{2,1} = I/I_c$ is the saturation parameter.

The number of photons scattered by an ion in one straight section is

$$\Delta n_\gamma = \int \frac{n_2(z)}{l_e}dz = \frac{n_{2m}}{l_e}\{l_{\text{eff}} + \frac{Dl_e}{1+D}[1 - e^{-(1+D)l_{\text{eff}}/l_e}]\}. \tag{26}$$

The power of the scattered radiation can be presented in the form

$$\overline{P^s} = \hbar\overline{\omega}\Delta n_\gamma c/2\pi R = \frac{c\gamma\hbar\omega_0' n_{\text{int}} D}{4\pi R l_e(1+D)}\{l_{\text{eff}} + \frac{Dl_e}{1+D}[1 - e^{-(1+D)l_{\text{eff}}/l_e}]\}. \tag{27}$$

As a result, the damping time of the vertical betatron oscillations is

$$\tau_z|_{l_{\text{eff}} \gg l_e} = \frac{8\pi m_i c^2 R l_e(1+D)}{c\hbar\omega_0' l_{\text{eff}} n_{\text{int}} D}. \tag{28}$$

When $D << 1$ and $l_e << l_{\text{eff}}$, Eq. (28) transforms to Eq. (19).

Appendix 2

The vector of the radiational force $\vec{F}^s = -P^s\vec{v}/c^2$. The equation of the vertical betatron oscillations of particles has the form

$$\ddot{z} + \omega_0^2\nu_z^2 z = F_z^s/m_i\gamma = 2\dot{z}/\tau z, \tag{29}$$

where ω_0 is the frequency of revolution of the particle, $\vec{F}_z^s$ is the z-component of the the force $\vec{F}^s$, and $\tau_z = 2m_i c^2/P^s$ [17,19].

In the case of radial and longitudinal movement it is important to know the dependence of the power P^s emitted by a particle on the radial displacement x, and on the energy deviation $\Delta\gamma/\gamma$. In our case

$$P^s = P_s^s(1 + \frac{1}{I}\frac{\partial I}{\partial x}x + 2\frac{\Delta\gamma}{\gamma}), \tag{30}$$

where P_s^s is the power emitted by the equilibrium particle, ρ is the bending radius of the particle's trajectory, and $n_e = -(\rho/2I)(\partial I/\partial x)$. This dependence coincides with the case of synchrotron radiation if we replace the magnetic field index of the storage ring n by n_l.

The rate of increase of the square amplitudes of radial and vertical oscillations $A_{x,z}$ is defined by the expression

$$\frac{dA_{x,z}^2}{dt} = \int \frac{\partial n_\gamma}{\partial\omega\partial t}(\Delta A_{x,z})^2 d\omega, \tag{31}$$

where $\partial n_\gamma/\partial\omega\partial t = P_\omega^s/\hbar\omega$, $P_\omega^s = 3P^s\xi(1-2\xi+2\xi^2)/\omega_{\max}$ is the spectral distribution of the emitted photons, and $\xi = \omega/\omega_{\max}$. The increase of the square amplitudes of radial (both betatron and phase) and vertical oscillations after emission of one photon are equal, respectively to $\Delta(A_x)^2 = (\alpha R\hbar\omega/m_i c^2\gamma)^2$ and $\Delta(A_z)^2 = \beta_z(\hbar\omega)^2/8\gamma^2(m_i c\gamma)^2$ [17,19]. The equilibrium value is $A_{eq}^2 = (\partial A^2/\partial t)\tau$. The equilibrium dimensions of the ion beam are $\sigma_{eq\,x,z} = \sqrt{A^2_{eq\,x,z}/2}$. The radial dimension of the ion beam defined by the energy spread is equal to $\sigma_s = R\alpha\Delta\gamma/\gamma$. From the above, Eq. (21) follows for the equilibrium radial dimension and energy spread, and the expression

$$\sigma_z = \frac{R}{\gamma\nu_z}\sqrt{\frac{7(1+\beta)\hbar\omega_0'}{20m_ic^2}}. \tag{32}$$

for the equilibrium vertical dimension.

The natural emittance of the ion beam in this case is

$$\epsilon_x^{**} = \frac{0.7\pi(1+\beta)R}{\nu_x^3}\frac{\hbar\omega_0'}{m_ic^2}. \tag{33}$$

Note that the natural emittance ϵ_x^* [Eq. (8)] is transformed to ϵ_x^{**} [Eq.(33)] when the critical energy of the synchrotron radiation quanta $\hbar\omega_c$ is equal to the energy of the scattered Rayleigh quanta $\hbar\omega_{\max}$.

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