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BALANCED TWO-COLORINGS OF FINITE SETS IN THE SQUARE I

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Let $T(N)$ be the least integer such that one can assign ± 1's to any N points in the unit square so that the sum of these values in any rectangle with sides parallel to those of the square have absolute value at most $T(N)$. G. Tusnádi asked what could be said about the order of magnitude of $T(N)$. We prove

$$\log N \ll T(N) \ll (\log N)^4.$$

In contrast, if $T^*(N)$ denotes the corresponding quantity where rectangles of any possible orientation are considered, we have

$$N^{1/4-\varepsilon} \ll T^*(N) \ll N^{1/2+\varepsilon}$$

for any $\varepsilon > 0$.

1. Introduction

We shall say that a rectangle is *aligned* if its sides are parallel to those of the unit square.

Let us be given a finite subset X of the unit square and a two-coloration $f\colon X \to \{+1, -1\}$ of it. For a rectangle B, we define the *deviation* $d(B)$ to be $\left|\sum_{x\in B\cap X} f(x)\right|$. As a measure of imbalance of the two-coloration f, we introduce the *combinatorial discrepancy*

$$t(f) = \max_{B} d(B),$$

where the maximum is taken over all aligned rectangles. Finally, let

$$T(N) = \max_{\substack{X \\ |X|=N}} \min_{f} t(f).$$

G. Tusnádi, investigating the invariance principle for the multi-dimensional empirical distribution function (cf. Major [5]), raised the question of finding the true order of magnitude of $T(N)$. Tusnádi observed that the "probabilistic method" (cf. Erdős—Spencer [3]) easily gives

$$T(N) \ll (N \log N)^{1/2}.$$

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We use Vinogradov's notation $g(N) \ll h(N)$ to mean $g(N)=O(h(N))$, i.e. $|g(N)/h(N)|$ is bounded.

Tusnádi conjectured that the above upper bound is far from being best possible. Actually, he conjectured that $T(N) \ll (\log N)^c$ with a suitable absolute constant c. In the other direction, Tusnádi suspected that $T(N) \to \infty$ as $N \to \infty$. Our main objective is to prove these conjectures.

Theorem 1.1. $\log N \ll T(N) \ll (\log N)^4$.

In contrast, we shall prove that for any two-coloring of a suitable N-set in the unit square there exists a tilted (not aligned) rectangle with large deviation.

Let $T^*(N)$ be the least integer so that given any N-set X in the unit square there exists a two-coloration f of the points so that any (tilted) rectangle B has deviation $t(B)=\left|\sum_{x \in X \cap B} f(x)\right|$ at most $T^*(N)$.

Theorem 1.2. $N^{(1/4)-\varepsilon} \ll T^*(N) \ll N^{(1/2)+\varepsilon}$.

Here the upper bound is an immediate application of the "probabilistic method" (cf. Erdős—Spencer [3]).

The problems above belong to the pattern of *combinatorial discrepancy theory*. The basic problem of this theory is how to two-color a set so that the coloring be nearly balanced in each of the subsets considered. Though the problems above have purely discrete character, in order to prove the lower bounds in Theorems 1.1 and 1.2, we need "continuous" arguments, namely the *measure theoretic* (or *classical*) *discrepancy theory* (cf. W. M. Schmidt [10]).

Let $X=\{p_1, \ldots, p_N\}$ be a set of N points in the r-dimensional unit cube $[0, 1]^r$. If A is a measurable set with Lebesgue measure $\mu(A)$, set $Z(A)$ to be the number of those i, $1 \leqq i \leqq N$, for which $p_i \in A$, and set $D(A)=|Z(A)-N\mu(A)|$.

Let $\mathscr{A}$ be a non-empty class of measurable sets in $[0, 1]^r$ (e.g. the class of boxes (Cartesian products of intervals) with sides parallel to the coordinate axes). As a measure of non-uniformity of distribution of the set X, we introduce

$$\Delta(X, \mathscr{A}) = \sup D(A)$$

where the supremum is taken over all $A \in \mathscr{A}$. We call $\Delta(X, \mathscr{A})$ the *measure theoretic discrepancy with respect to* $\mathscr{A}$ of the set X.

In Section 3 we establish a link between the combinatorial and the classical discrepancy theories. We shall be able to employ results of the classical theory to prove our lower bounds.

We note that in order to obtain *lower* bounds for combinatorial discrepancy problems, we require both lower and *upper* bounds (occasionally new ones, cf. Theorem 4.1) for classical discrepancy problems.

The proofs of the lower bounds will be *non-constructive*. The problem of constructing finite point-sets with large combinatorial discrepancy will be treated in Part II of this paper.

In Section 2 we prove the upper bound in Theorem 1.1. In Section 3 we prove the lower bound in Theorem 1.1 by applying some results of the classical discrepancy theory. In Section 4 we show that a theorem of W. M. Schmidt concerning irregularities of distribution with respect to rectangles in arbitrary position is nearly best possible and deduce Theorem 1.2. Finally, in Section 5 we mention an r-dimensional generalization of Theorem 1.1 and outline the proof.

2. Proof of the upper bound in Theorem 1.1

We shall apply the following general result due to Fiala and Beck (see Theorem 1 in [2]).

Theorem 2.1. *Let us be given an arbitrary number of sets S_i of arbitrary size such that each element belongs to at most k sets (in standard terminology: the hypergraph $\{S_i\}_i$ has maximum degree $\leqq k$). Then it is possible to assign $+1$ and -1 to the elements so that all sets S_i have sum with absolute value at most $2(k-1)$.*

The following result is clearly equivalent to Theorem 1.1.

Theorem 2.2. *Let $A=[a_{ij}]$ be a $0-1$ matrix of size N by N. Then there exist "signs" $\varepsilon_{ij}=\pm 1$ so that*

$$\left|\sum_{i=1}^{s}\sum_{j=1}^{t}\varepsilon_{ij}a_{ij}\right| \leqq c(\log N)^4$$

for all $1\leqq s, t\leqq N$, where c is a universal constant.

Proof. We may assume $N=2^l$. Let M be the set of $\langle i,j\rangle$ for which $a_{ij}=1$. For $0\leqq p, q\leqq l$ we partition M into 2^{p+q} "submatrices", splitting the horizontal side of the matrix into 2^p equal pieces and the vertical side of the matrix into 2^q equal pieces. There are $(l+1)^2\sim(\log N)^2$ such partitions. Let us call a submatrix *special* if it occurs in one of these partitions. Theorem 2.1 implies the existence of an assignment of ± 1's so that the absolute value of the sum of the entries in each of the special submatrices is at most $2(l+1)^2$. But any submatrix containing the lower left corner is the union of at most l^2 special submatrices. More formally

$$[1, s]\times[1, t] = \bigcup_{1\leqq i\leqq w}\ \bigcup_{1\leqq j\leqq z}\Big[1+\sum_{r\leqq i-1}2^{u_r}, \sum_{r\leqq i}2^{u_r}\Big]\times$$

$$\times\Big[1+\sum_{r\leqq j-1}2^{v_r}, \sum_{r\leqq j}2^{v_r}\Big],$$

where $s=2^{u_1}+2^{u_2}+\ldots+2^{u_w}$, $u_1>u_2>\ldots>u_w\geqq 0$ and $t=2^{v_1}+2^{v_2}+\ldots+2^{v_z}$, $v_1>v_2>$ $>\ldots>v_z\geqq 0$. This completes the proof. ∎

3. Applications of the classical discrepancy theory

Given a set X of N points and a class $\mathscr{A}$ of measurable sets in the d-dimensional unit cube, let $T(X, \mathscr{A})$ be the least integer so that there exists a function $f\colon X\to\{+1, -1\}$ such that

$$\sup\Big|\sum_{p\in X\cap A}f(p)\Big| \leqq T(X, \mathscr{A}),$$

where the supremum is taken over all $A\in\mathscr{A}$.

Finally, set (for the definition of $\Delta(X, \mathscr{A})$ see Section 1)

$$\Delta(N, \mathscr{A}) = \inf\Delta(X, \mathscr{A}), \quad T(N, \mathscr{A}) = \sup T(X, \mathscr{A})$$

where the infimum and the supremum are taken over all N-element sets $X\subset[0, 1]^d$, respectively. It is easy to see that $T(M, \mathscr{A})\leqq T(N, \mathscr{A})$ if $M\leqq N$.

The following result provides the link between the combinatorial and the classical discrepancy theories mentioned in the Introduction.

Theorem 3.1.

$$\max_{k\geqq 1} \{\Delta(\lfloor N2^{-k}\rfloor, \mathscr{A})-\Delta(N, \mathscr{A})2^{-k}\} \leqq 2T(N, \mathscr{A})+1$$

($\lfloor z\rfloor$ *stands for the greatest integer* $\leqq z$).

The combinatorial core of the proof is the following simple lemma.

Lemma 3.2. *Let X be a finite set and let us be given a system $\mathscr{B}$ of subsets of X with $X\in\mathscr{B}$. Let T be the least integer such that given any subset $Y\subseteq X$, one can find a function f: $Y\to\{+1, -1\}$ so that*

$$\max_{B\in\mathscr{B}} \Big|\sum_{x\in Y\cap B} f(x)\Big| \leqq T.$$

Then for every $k\geqq 0$, there exists a subset Y_k of X such that $|Y_k|=\lfloor |X|2^{-k}\rfloor$ and

$$\big||Y_k\cap B|-|B|2^{-k}\big| \leqq 2T$$

for all $B\in\mathscr{B}$ ($|H|$ denotes, as usual, the number of elements of the set H).

Proof. We are going to prove the following statement:

(1) *For every $k\geqq 0$, it is possible to partition X into 2^k parts $Y_{k,j}$, $1\leqq j\leqq 2^k$, such that $\big||Y_{k,j}\cap B|-|B|2^{-k}\big|\leqq(1-2^{-k})T$ for all $B\in\mathscr{B}$.*

We prove (1) by induction on k. For $k=0$ the statement is trivial: let $Y_{0,1}=X$. Assume now that the statement is true for some $k\geqq 0$. By the hypothesis of the lemma there is a function $f_{k,j}$: $Y_{k,j}\to\{+1, -1\}$ such that

$$\Big|\sum_{x\in Y_{k,j}\cap B} f_{k,j}(x)\Big| \leqq T \quad \text{for all} \quad B\in\mathscr{B}. \tag{2}$$

Set

$$Y_{k+1,2j-1} = \{x\in Y_{k,j}\colon f_{k,j}(x) = +1\},$$

$$Y_{k+1,2j} = \{x\in Y_{k,j}\colon f_{k,j}(x) = -1\}$$

for $j=1, \ldots, 2^k$. From (2) it follows that

$$\big||Y_{k+1,2j}\cap B| - |(Y_{k,j}\setminus Y_{k+1,2j})\cap B|\big| \leqq T,$$

that is,

$$\big|2|Y_{k+1,2j}\cap B| - |Y_{k,j}\cap B|\big| \leqq T \quad \text{for all} \quad B\in\mathscr{B}. \tag{3}$$

We obtain similarly

$$\big|2|Y_{k+1,2j-1}\cap B| - |Y_{k,j}\cap B|\big| \leqq T \quad \text{for all} \quad B\in\mathscr{B}. \tag{4}$$

By the induction hypothesis,

$$\big||Y_{k,j}\cap B| - |B|2^{-k}\big| \leqq (1-2^{-k})T.$$

Thus by (3),

$$\left||Y_{k+1,2j}\cap B|-|B|2^{-k-1}\right| \leqq \left||Y_{k+1,2j}\cap B|-|Y_{k,j}\cap B|/2\right|+$$

$$+\frac{1}{2}\left||Y_{k,j}\cap B|-|B|2^{-k}\right| \leqq T/2+\frac{1}{2}(1-2^{-k})T = (1-2^{-k-1})T.$$

Using (4) we get similarly

$$\left||Y_{k+1,2j-1}\cap B|-|B|2^{-k-1}\right| \leqq (1-2^{-k-1})T,$$

completing the induction step, and thereby the proof of (1).

Now we finish the proof of the lemma as follows. Clearly there must exist an index j_0, $1\leqq j_0\leqq 2^k$, such that $|Y_{k,j_0}|\geqq|X|2^{-k}$. Let Y_k be a subset of Y_{k,j_0} of cardinality $\lfloor |X|2^{-k}\rfloor$. Since $X\in\mathscr{B}$, by (1)

$$0 \leqq |Y_{k,j_0}|-|X|2^{-k} \leqq T.$$

Therefore

(5) $$0 \leqq |Y_{k,j_0}|-|Y_k| \leqq T.$$

By (5) and (1)

$$\left||Y_k\cap B|-|B|2^{-k}\right| \leqq \left||Y_k\cap B|-|Y_{k,j_0}\cap B|\right|+$$

$$+\left||Y_{k,j_0}\cap B|-|B|2^{-k}\right| \leqq T+T = 2T,$$

for all $B\in\mathscr{B}$. The lemma follows. ∎

Proof of Theorem 3.1. Let us be given an N-element set X and a class $\mathscr{A}$ of measurable sets in $[0, 1]^d$ and let

$$\mathscr{B} = \{X\cap A:\ A\in\mathscr{A}\}.$$

By definition, for any subset $Y\subseteqq X$, one can find a function f: $Y\to\{+1, -1\}$ so that

$$\max_{B\in\mathscr{B}}\Big|\sum_{p\in Y\cap B} f(p)\Big| = \sup_{A\in\mathscr{A}}\Big|\sum_{p\in Y\cap A} f(p)\Big| \leqq T(Y,\mathscr{A}) \leqq T(|Y|,\mathscr{A}) \leqq T(N,\mathscr{A}).$$

Thus Lemma 3.2 yields the existence of a subset $Y_k\subset X$ such that $|Y_k|=\lfloor |X|2^{-k}\rfloor = \lfloor N2^{-k}\rfloor$ and

(6) $$\left||Y_k\cap A|-|X\cap A|2^{-k}\right| \leqq 2T(N,\mathscr{A})$$

for all $A\in\mathscr{A}$.

On the other hand, by the definition of $\Delta(Y_k,\mathscr{A})$

(7) $$\sup_{A\in\mathscr{A}}\left||Y_k\cap A|-|Y_k|\mu(A)\right| = \Delta(Y_k,\mathscr{A}).$$

By (7)

(8) $$\Delta(\lfloor N2^{-k}\rfloor,\mathscr{A}) = \Delta(|Y_k|,\mathscr{A}) \leqq \Delta(Y_k,\mathscr{A}) =$$

$$= \sup\left||Y_k\cap A|-|Y_k|\mu(A)\right| \leqq \sup\left||Y_k\cap A|-|X\cap A|2^{-k}\right|+$$

$$+\sup\left||X\cap A|2^{-k}-|X|2^{-k}\mu(A)\right|+$$

$$+\sup\left||X|2^{-k}\mu(A)-|Y_k|\mu(A)\right|,$$

where the supremum is extended over all $A\in\mathscr{A}$.

Since, by definition

$$\big||X\cap A|-|X|\mu(A)\big| \leqq D(A) \leqq \Delta(X,\mathscr{A})$$

and $|Y_k|=\lfloor |X|2^{-k}\rfloor$, then by (8) and (6) we obtain

$$\Delta(\lfloor N2^{-k}\rfloor,\mathscr{A}) \leqq 2T(N,\mathscr{A})+\Delta(X,\mathscr{A})2^{-k}+1,$$

or equivalently

$$\Delta(\lfloor N2^{-k}\rfloor,\mathscr{A})-\Delta(X,\mathscr{A})2^{-k} \leqq 2T(N,\mathscr{A})+1.$$

Choosing X so that $\Delta(X,\mathscr{A})\leqq\Delta(N,\mathscr{A})+\varepsilon$,

$$\Delta(\lfloor N2^{-k}\rfloor,\mathscr{A})-\Delta(N,\mathscr{A})2^{-k} \leqq 2T(N,\mathscr{A})+1+\varepsilon,$$

which was to be proved. ∎

Proof of the lower bound in Theorem 1.1. Let $\mathscr{A}$ be the class of two-dimensional boxes with sides parallel to the coordinate axes. A fundamental result of the classical discrepancy theory states that

$$\log N \gg \Delta(N,\mathscr{A}) \gg \log N. \tag{9}$$

Here the upper bound is due to van der Corput [11] (cf. [10]) and the lower bound is due to W. M. Schmidt [9].

Choosing k so that $2^k\leqq\log N<2^{k+1}$, by Theorem 3.1 we obtain

$$2T(N)+1 = 2T(N,\mathscr{A})+1 \geqq \Delta(\lfloor N2^{-k}\rfloor,\mathscr{A})-\Delta(N,\mathscr{A})2^{-k} \gg$$
$$\gg \log(N/\log N)-\Delta(N,\mathscr{A})/\log N \gg \log N.$$

The theorem follows. ∎

4. Proof of Theorem 1.2

In this section let $\mathscr{A}$ denote the class of those plane regions obtained by intersecting the unit square with rectangles in arbitrary position. A surprising result of W. M. Schmidt [8] says that

$$\Delta(N,\mathscr{A}) \gg N^{1/4-\varepsilon}. \tag{10}$$

First we prove that this estimate is nearly sharp. In fact, we shall prove a bit more. Let $\mathscr{B}$ denote the class of (not necessarily rectangular) *convex quadrilaterals*.

Theorem 4.1. $\Delta(N,\mathscr{B})\ll N^{1/4}(\log N)^{1/2}$.

Proof. Assume that $N=n^2$, and divide the unit square into N congruent, pairwise disjoint "small squares" by $(n-1)$ vertical and $(n-1)$ horizontal lines. Let us associate with the "small squares" *independent* "random points" so that each of them has *uniform distribution* in its "small square", i.e.

$$\text{Prob}(\xi\in A) = \frac{\mu(A)}{\mu(Q)} = N\mu(A),$$

where Q is a "small square", $\xi=\xi(Q)$ is the "random point" associated with Q, A is an arbitrary measurable subset of Q and $\mu(\cdot)$ denotes the Lebesgue measure.

In what follows, we shall prove that the "random point-system" defined above has discrepancy $\ll N^{1/4}(\log N)^{1/2}$ with respect to $\mathcal{B}$ (i.e. the class of convex quadrilaterals) with probability $\geqq 1/2$.

Now consider a convex quadrilateral B. It is easy to see that the sides of B intersect $\ll N^{1/2}$ "small squares". Therefore, B is representable as the disjoint union of "small squares" entirely covered by B and the union of $\ll N^{1/2}$ "pieces" which are the intersection of some small squares and B, i.e.

$$B = \bigcup_{i\in I} Q_i \cup \bigcup_{j\in J} (Q_j \cap B),$$

where the index-set J has cardinality $\ll N^{1/2}$. Since every small square has Lebesgue measure $1/N$ and contains exactly one element of the "random point-system" above, the discrepancy of $\bigcup_{i\in I} Q_i$ is zero. Thus, it remains to investigate the discrepancy of $\bigcup_{j\in J} (Q_j \cap B)$.

For notational convenience let

$$R = \bigcup_{j\in J} (Q_j \cap B) \quad \text{and} \quad \xi_j = \xi(Q_j), \quad j\in J.$$

Define the random variables χ_j, $j\in J$ as follows: Let $\chi_j=1$ if $\xi_j\in Q_j\cap B$; otherwise let $\chi_j=0$. Then

$$D(R) = \Big|\sum_{j\in J} \chi_j - N\Big(\sum_{j\in J} \mu(Q_j\cap B)\Big)\Big|. \tag{11}$$

Since ξ_j is uniformly distributed in Q_j,

$$\text{Prob}\,(\chi_j = 1) = \mu(Q_j\cap B)/\mu(Q_j) = N\mu(Q_j\cap B).$$

Thus

$$\mathrm{E}\chi_j = N\mu(Q_j\cap B), \tag{12}$$

where $\mathrm{E}\chi_j$ denotes, as usual, the expected value of the random variable χ_j. By (12) and (11) the discrepancy $D(R)$ can be written in the form

$$D(R) = \Big|\sum_{j\in J} (\chi_j - \mathrm{E}\chi_j)\Big|. \tag{13}$$

Since the random variables χ_j, $j\in J$ are *independent*, in order to estimate the order of magnitude of the sum $D(R)$ we are able to apply the classical Bernstein—Chernoff inequality of large deviation type.

Lemma 4.2 (Bernstein—Chernoff). *Let $\eta_1, \dots, \eta_m$ be independent random variables with $|\eta_i|\leqq 1$, $1\leqq i\leqq m$. Denote by σ_i^2 the variance of η_i, i.e. $\sigma_i^2=\mathrm{E}(\eta_i-\mathrm{E}\eta_i)^2$. Put $S_m=\sum_{i=1}^{m}(\eta_i-\mathrm{E}\eta_i)$, $\beta=\left(\sum_{i=1}^{m}\sigma_i^2\right)^{1/2}$. Then*

$$\text{Prob}\,(|S_m| > \lambda) \leqq \begin{cases} 2e^{-\lambda/4} & \text{if } \lambda \geqq \beta^2 \\ 2e^{-\lambda^2/4\beta^2} & \text{if } \lambda \leqq \beta^2. \end{cases}$$

For a proof see e.g. Petrov [6].

Now let us return to (13). Let σ_j^2 denote $E(\chi_j - E\chi_j)^2$, $j \in J$, and set $\beta = \left(\sum_{j\in J} \sigma_j^2\right)^{1/2}$. Choosing $\lambda = cN^{1/4}(\log N)^{1/2}$ with a large enough constant c, by the application of Lemma 4.2 we obtain, after some easy calculation, that

$$\text{(14)} \qquad \operatorname{Prob}(D(R) > \lambda) \leqq \begin{cases} 2e^{-\lambda/4} \ll N^{-9} & \text{if } \lambda \geqq \beta^2 \\ 2e^{-\lambda^2/4\beta^2} \ll N^{-9} & \text{if } \lambda \leqq \beta^2. \end{cases}$$

since $\beta^2 \leqq |J| \ll N^{1/2}$ and $t^2/4\beta^2 \geqq t^2/4|J| \gg t^2/N^{1/2} \gg \log N$.

Although the class $\mathscr{B}$ of convex quadrilaterals is uncountable, in fact it suffices to consider a subclass of cardinality $\ll N^8$. Actually, we shall restrict ourselves to the class $\mathscr{B}_N$ of convex quadrilaterals such that each of their four corner-points can be written in the form $p = (i/N, j/N)$, where i and j, $0 \leqq i, j \leqq N$, are integers. Simple calculation shows that the cardinality of $\mathscr{B}_N$ is $\ll N^8$.

Consider a convex quadrilateral B. It is easy to see that there exist $B_1, B_2 \in \mathscr{B}_N$ such that $B_1 \subseteqq B \subseteqq B_2$ and $N(\mu(B_2) - \mu(B_1)) \ll 1$. From this follows that

$$\text{(15)} \qquad D(B) \ll \max\{D(B_1), D(B_2), 1\},$$

that is, the class $\mathscr{B}_N$ is sufficiently "rich".

Now we are able to complete the proof of Theorem 4.1. The random "point-system" defined above has the property that fixing any convex quadrilateral B, we have (see (14)):

$$\operatorname{Prob}(D(B) \gg N^{1/4}(\log N)^{1/2}) \ll N^{-9}.$$

Since $|\mathscr{B}_N| \ll N^8$, we obtain

$$\operatorname{Prob}(D(B) \gg N^{1/4}(\log N)^{1/2} \text{ for some } B \in \mathscr{B}_N) \ll |\mathscr{B}_N| N^{-9} \ll N^{-1},$$

that is, there must exist an N-element point-set having discrepancy $\ll N^{1/4}(\log N)^{1/2}$ with respect to $\mathscr{B}_N$, and hence with respect to $\mathscr{B}$ (see (15)). ▮

Proof of Theorem 1.2. We recall that in this section $\mathscr{A}$ denotes the class of those plane regions obtained by intersecting the unit square with arbitrary position. By (10) and Theorem 4.1

$$N^{(1/4)+\varepsilon} \gg \Delta(N, \mathscr{A}) \gg N^{(1/4)-\varepsilon}.$$

Choosing k so that $2^k \leqq N^{4\varepsilon} < 2^{k+1}$, by Theorem 3.1 we obtain

$$2T^*(N)+1 = 2T(N, \mathscr{A})+1 \geqq \Delta(\lfloor N2^{-k} \rfloor, \mathscr{A}) - \Delta(N, \mathscr{A})2^{-k} \gg$$

$$\gg (N^{1-4\varepsilon})^{(1/4)-\varepsilon} - N^{(1/4)+\varepsilon}/N^{4\varepsilon} \gg N^{(1/4)-2\varepsilon} - N^{(1/4)-3\varepsilon} \gg N^{(1/4)-2\varepsilon},$$

which was to be proved. ▮

5. The case of dimension $\geqq 3$

Let $T_r(N)$ be the least integer so that given any N points in the r-dimensional unit cube $[0, 1]^r$ there exists an assignment of ± 1 to the points so that all r-dimensional boxes (Cartesian products of intervals) with sides parallel to the coordinate axes have sums with absolute values at most $T_r(N)$. That is, $T_2(N)=T(N)$.

Repeating the argument in Section 2 without any essential modification one can easily obtain $T_r(N) \ll (\log N)^{2r}$. The following theorem is an improvement on it for $r \geqq 4$.

Theorem 5.1. $(\log N)^{(r-1)/2} \ll T_r(N) \ll (\log N)^{r-1}$.

The proof of the upper bound goes along the lines of the proof in Section 2, but instead of Theorem 2.1 we need the following general result (see Theorem 2.1 in [1]).

Theorem 5.2. *Let us be given an s-element set S and m subsets* $S_1, \dots, S_m$ *of S such that each element is in at most k sets. Then it is possible to assign* $+1$ *and* -1 *to the elements of S so that all sets* S_i *have sums with absolute values at most* $ck^{1/2}(\log m)^{1/2} \log s$, *where c is a universal constant.*

Finally, consider the lower bound in Theorem 5.1. Its proof is the same as the proof for the case $r=2$ (see the end of Section 3), but instead of (9) we have to apply the current estimates for the measure theoretic discrepancy of N-element sets with respect to the class $\mathscr{C}$ of d-dimensional boxes with sides parallel to the coordinate axes. The best estimates (for $r \geqq 3$) at present are

$$(\log N)^{(r-1)/2} \ll \Delta(N, \mathscr{C}) \ll (\log N)^{r-1}.$$

The upper bound is due to J. H. Halton [4], cf. [10], the lower bound to K. F. Roth [7]. Details are left to the reader.

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