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A NOTE ON THE GIRTH OF DIGRAPHS

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Behzad, Chartrand and Wall conjectured that the girth of a diregular graph of order n and outdegree r is not greater than $\left[\frac{n}{r}\right]$. This conjecture has been proved for $r=2$ by Behzad and for $r=3$ by Bermond. We prove that a digraph of order n and halfdegree $\geqq 4$ has girth not exceeding $\left[\frac{n}{4}\right]$. We also obtain short proofs of the above results. Our method is an application of the theory of connectivity of digraphs.

1. Introduction

We use notations of Berge [3]. Let G be a digraph having at least one circuit. The *girth* of G is the minimal length of a circuit of G. It will be denoted by $g(G)$. The minimal outdegree (resp. indegree) of G will be denoted by $\delta^+(G)$ (resp. $\delta^-(G)$). No general result relating the girth of a digraph to its minimal degree is known. There are only the following conjectures.

Conjecture A (Behzad, Chartrand and Wall [1]). Let $G=(V,E)$ be a diregular digraph of outdegree r. Then $|V|\geqq r\left(g(G)-1\right)+1$.

A generalization of this conjecture to arbitrary digraphs would be the following.

Conjecture B. Let $G=(V,E)$ be a digraph such that $\min\left(\delta^+(G), \delta^-(G)\right)\geqq r$. Then $|V|\geqq r\left(g(G)-1\right)+1$.

These two conjectures are implied by the following.

Conjecture C (Caccetta and Häggkvist [5]). Let $G=(V,E)$ be a digraph. Then $|V|\geqq \delta^+(G)\left(g(G)-1\right)+1$.

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Conjecture A has been proved for $r=2$ by Behzad [2] and for $r=3$ by Bermond [4]. Conjecture C has been proved for $\delta^+(G)=2$ by Caccetta and Häggkvist [5].

In [8] we solved this problem completely in the case of a digraph having a transitive group of automorphisms, using a connectivity method. We use in this paper the theory of connectivity of a digraph to prove Conjecture B for $r\leqq 4$. Our proofs are easier than those obtained in [2, 4].

2. Connectivity of a digraph

Let $G=(V,E)$ be a digraph and A be a subset of V. We put $N^+(A)=$ $=\Gamma^+(A)-A$ and $R^-(A)=V-(A\cup N^+(A))$. Similarly we define $N^-(A)$ and $R^+(A)$. Two paths are called *openly* disjoint if their intersection is contained in the set of their extremities. Given a family of paths, we will say that these paths are disjoint (resp. openly disjoint) if any two of them are disjoint (resp. openly disjoint). All paths will be supposed elementary. Consider two vertices x and y of G. The maximal number of openly disjoint paths of G connecting x to y will be denoted by $\tau(x,y)$ (or $\tau_G(x,y)$ when confusion is possible). If (x,y) is not an arc of G, the minimal order of a subset of $V-\{x,y\}$ separating y from x (i.e. subset whose deletion leaves no path connecting x to y) will be denoted by $\varkappa(x,y)$ (or $\varkappa_G(x,y)$). The *diconnectivity* of a digraph $G=(V,E)$ which is not complete-symmetric is

$$\varkappa(G) = \min\{\varkappa(x,y)|(x,y)\notin E\}.$$

The diconnectivity of a complete-symmetric digraph $G=(V,E)$ is $\varkappa(G)=|V|-1$. We can easily see that for a digraph $G=(V,E)$ which is not complete-symmetric

$$\varkappa(G) = \min\{|N^+(A)|\big|A\neq\emptyset \text{ and } R^-(A)\neq\emptyset\},$$

$$\varkappa(G) = \min\{|N^-(A)|\big|A\neq\emptyset \text{ and } R^+(A)\neq\emptyset\}.$$

Let $G=(V,E)$ be a strongly connected digraph which is not complete-symmetric and F be a subset of V. We say that F is a *positive* (resp. *negative*) *fragment* if $|N^+(F)|=\varkappa(G)$ and $R^-(F)\neq\emptyset$ (resp. $|N^-(F)|=\varkappa(G)$ and $R^+(F)\neq\emptyset$). A fragment of minimal cardinality is called an *atom*. These notions have been studied [7].

A fundamental property of the connectivity of a digraph is the following.

Theorem 2.a (Menger—Dirac). *Let $G=(V,E)$ be a strongly connected digraph, x and y two vertices of G such that $(x,y)\notin E$. Then $\tau(x,y)=\varkappa(x,y)$.* ▮

We will use the following property of atoms.

Theorem 2.b [7]. *Let $G=(V,E)$ be a diconnected digraph, A be a positive atom of G and F be a positive fragment of G. Then $A\subset F$ or $A\cap F=\emptyset$.*

We note that Theorem 2.b generalizes a result proved in the undirected case by Mader [9].

3. A relation between girth and connectivity

We proved in [8] that a digraph of order n and diconnectivity h has girth not exceeding $\left[\frac{n}{h}\right]$. We relate now the girth of a digraph to its local connectivities.

Proposition 3.1. *Let $G=(V,E)$ be a digraph of girth $g\geqq 3$ and (x,y) be an arc of G. Then*

$$|V| \geqq (g-3)\tau(y,x)+d^+(y)+d^-(y)+1.$$

Proof. Put $t=\tau(y,x)$. The proposition is true for $g=3$ (in this case we have $N^+(y)\cap N^-(y)=\emptyset$), suppose $g>3$. There are t openly disjoint paths connecting y to x. Take on each path a vertex not joined by an arc to y whose successor on the path considered is joined to y by an arc (observe that $(x;y)$ is an arc of G and $g>3$). This construction induces t disjoint paths connecting $N^+(y)$ to to $N^-(N^-(y))$. Let $\{L_i;\ 1\leqq i\leqq t\}$ be a family of paths joining $N^+(y)$ to $N^-(N^-(y))$ such that $\Sigma|L_i|$ is minimal $(|L|=l(L)+1)$. This choice implies that $|N^+(y)\cap L_i|=1$ and $N^-(y)\cap L_i=\emptyset)$; $1\leqq i\leqq t$. We see easily that $|L_i|\geqq g-2$. It follows that

$$|V| \geqq |\cup L_i|+|N^-(y)|+|N^+(y)-\cup L_i|+1.$$

Therefore $|V|\geqq t(g-2)+d^-(y)+d^+(y)-t+1=t(g-3)+d^+(y)+d^-(y)+1$. ∎

Corollary 3.2. *Let $G=(V,E)$ be an anti-symmetric digraph having at least one circuit and (x,y) be an arc of G. Then*

$$|V| \geqq \big(g(G)-3\big)\varkappa(y,x)+d^+(y)+d^-(y)+1 \geqq \big(g(G)-1\big)\varkappa(G)+1.$$

Proof. The first part of this corollary is an application of Menger—Dirac's Theorem 2.a and Proposition 3.1. The second part is a consequence of the well known relation

$$\varkappa(G) \leqq \min\big(d^+(y), d^-(y)\big). \quad ∎$$

Convention. If a digraph G contains no circuit, we take $g(G)=0$.

Remark. A digraph which is not anti-symmetric satisfies the conjectures given in the introduction. Thus we have to consider the case of an anti-symmetric digraph only.

Corollary 3.3 (Behzad [2]). *Let $G=(V,E)$ be a digraph such that $\delta^+(G)\geqq 2$ and $\delta^-(G)\geqq 2$. Then $|V|\geqq 2\big(g(G)-1\big)+1$.*

Proof. (By the above remark, we may assume G anti-symmetric.) By Corollary 3.2, it is sufficient to prove the corollary for $\varkappa(G)\leqq 1$. We can exclude easily the case $\varkappa(G)=0$. Suppose $\varkappa(G)=1$ and take a minimum cutset $\{t\}$. There are two distinct components C and C' of G_{V-t} such that $N^+(C)=\{t\}$ and $N^-(C')=\{t\}$. Let H (resp. H') be the subgraph of G induced by C (resp. C'). We have $N_H^+(x)\supset \supset N_G^+(x)-\{t\}$ for every $x\in C$. Hence $\delta^+(H)\geqq 1$. Therefore H contains a circuit. It follows that $|C|\geqq g(G)$. Similarly, we have $\delta^-(H')\geqq 1$ and $|C'|\geqq g(G)$. These relations imply that the corollary is verified in this case as well. ∎

4. Girth of a digraph with minimal halfdegree 4

In this section, we will use the following result due to Caccetta and Häggkvist [5].

Theorem 4.c (Caccetta and Häggkvist [5]). *Let $G=(V,E)$ be a digraph such that* $\delta^+(G)\geqq 2$ or $\delta^-(G)\geqq 2$. *Then*

$$|V| \geqq 2(g(G)-1)+1. \quad \blacksquare$$

We shall also use the following property of atoms.

Proposition 4.1. *Let $G=(V,E)$ be a digraph, A a positive atom of G, and x and y be two distinct vertices of A such that $(x,y)\notin E$. Then $\tau(x,y)\geqq\varkappa(G)+1$.*

Proof. Suppose on the contrary that $\tau(x,y)\leqq\varkappa(G)$. By the Menger—Dirac Theorem 2a, $\varkappa(x,y)\leqq\varkappa(G)$. Hence $\varkappa(x,y)=\varkappa(G)$. This means that there is a $T\subset V$ such that G_{V-T} contains no path joining x to y and $|T|=\varkappa(G)$. Let F be the set of vertices of $V-T$ which can be reached from x by a path contained in $V-T$. Clearly we have $x\in F$; $y\notin F$ and $N^+(F)\subset T$. These relations show that F is a positive fragment of G. But $F\cap A\neq 0$ and $y\in A-F$, contradicting Theorem 2.b. $\blacksquare$

Theorem 4.2. *Let $G=(V,E)$ be a digraph such that $\delta^+(G)\geqq 4$ and $\delta^-(G)\geqq 4$. Then $|V|\geqq 4\big(g(G)-1\big)+1$.*

Proof. We may suppose G strongly connected and anti-symmetric. Let A be an atom of G. We can take A positive, since a negative atom of G is a positive atom of the inverse digraph of G (cf. [7]). If $|A|=1$, then $\varkappa(G)=\min\big(\delta^+(G),\delta^-(G)\big)\geqq 4$ and the Theorem holds by Corollary 3.2. Suppose $|A|>1$. If $\varkappa(G)=3$, consider two vertices x and y of A such that (x,y) is an arc of G (observe that an atom is strongly connected cf. [7]). By Proposition 4.1, $\tau(y,x)\geqq 4$, and the Theorem holds by Proposition 3.1. We can thus suppose $\varkappa(G)\leqq 2$. We have $|N_H^+(x)|\geqq|N_G^+(x)|-2\geqq 2$, where $H=G_A$. By Theorem 4.c (Caccetta and Häggkvist), we have

$$|A| \geqq 2\big(g(H)-1\big)+1 \geqq 2\big(g(G)-1\big)+1.$$

But $|R^-(A)|\geqq|A|$ (observe that $R^-(A)$ is a negative fragment and that an atom is a fragment with minimal cardinality). These relations clearly imply the Theorem. $\blacksquare$

Corollary 4.3 (Bermond [3]). *Let $G=(V,E)$ be a digraph such that $\delta^+(G\geqq 3$ and $\delta^-(G)\geqq 3$. Then $|V|\geqq 3\big(g(G)-1\big)+1$.*

The proof can be obtained using the arguments of the proof of Theorem 4.2. $\blacksquare$

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