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QUASI-SYMMETRIC 2, 3, 4-DESIGNS

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Quasi-symmetric designs are block designs with two block intersection numbers x and y It is shown that with the exception of $(x, y)=(0, 1)$, for a fixed value of the block size k, there are finitely many such designs. Some finiteness results on block graphs are derived. For a quasi-symmetric 3-design with positive x and y, the intersection numbers are shown to be roots of a quadratic whose coefficients are polynomial functions of v, k and λ. Using this quadratic, various characterizations of the Witt—Lüneburg design on 23 points are obtained. It is shown that if $x=1$, then a fixed value of λ determines at most finitely many such designs.

1. Introduction

Block designs with two block intersection numbers have been objects of considerable interest in recent years (see the references). These designs are known as quasi-symmetric designs. If the definition is slightly generalized then the class of these designs can be made sufficiently broad to include all the symmetric designs.

In the literature, 2-designs with two block intersection numbers x and y, $x<y$ have been studied but almost invariably these objects have been studied with additional constraints. For example, under the assumption that the design has no three blocks that mutually intersect in x points. Such designs were studied in [1, 11, 14]. The case $x=0$ is particularly important since most of the known examples seem to belong to this type (or its complement). This was considered in [13] where it was shown that the number of such designs is finite if the block size k or the occurrence of a pair $\lambda \geqq 2$ is held fixed.

The preliminary aim of this paper is to study quasi-symmetric designs with any intersection number pair (x, y), where x is not necessarily assumed to be zero. The main aim, however, is to develop sufficient machinary toward classification of all the 3-designs with two block intersection numbers. This problem (and even a more difficult one concerning the classification of quasi-symmetric 2-designs) has been posed as an open problem in the t-design survey of Hedayat and Kageyama [6, 9].

The classification of 3-designs with $x=0$ was given by Cameron [2]. Unfortunately, there is no obvious way of extending Cameron's proof to cover $x\neq 0$, since

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his proof heavily relies on the fact that any contraction of such an object is a symmetric design, a structure which can be characterized by the design parameters alone. However, we show the existence of a quadratic polynomial $f(\alpha)$ with coefficients in v, k, and λ whose roots are the block intersection numbers. We would like to point out that this is similar to the Delsarte polynomial for tight t-designs used by researchers in the area of t-designs [5, 12, 16]. As pointed out earlier, this quadratic is a step in the direction of the classification of quasi-symmetric 3-designs.

Section 2 is concerned with quasi-symmetric 2-designs D. A useful tool in the study of such designs is its block graph Γ, where two vertices are joined if and only if the corresponding blocks intersect in y points. It is well known that the block graph is a strongly regular graph whose parameter set (n, a, c, d) is expressible in terms of the design parameters and the intersection numbers (see e.g. [15]). Many strongly regular graphs are, in fact, obtained in this manner.

Some other preliminary results required in Sections 3 and 4 are also proved in Section 2. In Proposition 2.2, we show that if $\lambda \neq 1$, then the number of points v of D is bounded by k^2/x, where x is the smaller intersection number. It then follows that barring the pair $(x, y)=(0, 1)$ (which corresponds to Steiner systems), if the block size k is fixed, then the number of such designs is finite (Theorem 2.6 and Remark 2.7). It is easy to see that quasi-symmetric t-designs do not exist for $t>4$ (e.g. [6]). For $t=4$, N. Ito [8] has shown that, up to complementation such an object is unique.

In Section 3 we study 3-designs with two block intersection numbers. We show that if the block intersection numbers are positive, then they are roots of a quadratic polynomial $f(\alpha)=A\alpha^2+B\alpha+C$, where A, B and C are polynomial functions in v, k, and $\lambda=\lambda_3$ (Theorem 3.2). To do this, we look at D and its point contraction as quasi-symmetric designs and obtain two simultaneous linear equations in $x+y$ and xy. It is shown that the Delsarte polynomial for 4-designs referred to in [12] can also be obtained from our quadratic (Corollary 3.5).

Section 4 is mainly concerned with quasi-symmetric 3-designs in which the smaller intersection number $x=1$. As mentioned earlier, the case $x=0$ was considered by Cameron [2]. The only known example of a 3-design with $x=1$ is the Witt—Lüneburg 4-design on 23 points or its residual. We obtain various characterizations of these designs (Proposition 4.3, Theorem 4.5). We also conjecture (Conjecture 4.6 and Conjecture 4.7) that if D is a quasi-symmetric 3-design with $x\neq 0$ then D is the Witt—Lüneburg design on 23 points or its residual or D is the complement of one of these designs or complement of designs in Cameron's list. We believe that the quadratic in Theorem 3.2, (though quite complicated) and the use of sophisticated number theoretic techniques will help settle these conjectures. We end this paper by partially answering this question (Theorem 4.8) and show that for $x=1$ and for any fixed $\lambda=\lambda_3$, there are finitely many such 3-designs.

2. Quasi-symmetric 2-designs

At the outset, we wish to point out that while the results of this section are needed in Sections 3 and 4, they are also of independent interest (for example, Theorem 2.6).

Suppose D is a quasi-symmetric 2-design with standard parameter set $(v, b, r,$

$k, \lambda;\ x, y)$ where x, y are the two block intersection numbers with $0 \leqq x < y < k$. We also assume that D is proper quasi-symmetric i.e., both the intersection numbers occur. Let Γ (respectively $\bar{\Gamma}$) denote the block graph (the complement of Γ) of D where, as usual, two vertices are adjacent if and only if the corresponding blocks intersect in y points (x points). The following result is well-known (see, e.g. [15]).

Lemma 2.1. *Both Γ and $\bar{\Gamma}$ are strongly regular graphs. If Γ is connected then its parameter set (n, a, c, d) is given by:*

$$n = b,\ a = \frac{k(r-1)-x(b-1)}{z},$$

$$c = d + \frac{(r-\lambda)-(k-x)}{z} - \frac{(k-x)}{z},$$

$$d = a - \frac{(r-\lambda)(k-x)}{z^2} + \frac{(k-x)^2}{z^2}$$

where, $z=y-x$ divides $k-x$. Let $k-x=mz$. Also if $\bar{\Gamma}$ is connected then the parameters $\bar{a}, \bar{c}, \bar{d}$, of $\bar{\Gamma}$ are obtained by interchanging x and y in the formulas of a, c, d respectively. ▮

Proposition 2.2. *With everything as above, let $\lambda \neq 1$. Then the smaller intersection number $x < k^2/v < \lambda$.*

Proof. Since $a > 0$ in Lemma 2.1, $x < \dfrac{k(r-1)}{(b-1)} < \dfrac{kr}{b} = \dfrac{k^2}{v}$. Since $r \geqq k+1$ (recall that D is not symmetric), $\lambda(v-1) = r(k-1) \geqq k^2-1$. So $\lambda v - k^2 \geqq \lambda - 1 > 0$, because $\lambda \neq 1$. Hence $k^2/v < \lambda$ proving the assertion. ▮

Corollary 2.3. *Let α be a positive real number and suppose D satisfies $x > k/\alpha$. Then the number of points v of D is bounded above by $k\alpha$.*

Corollary 2.4. *Let $v \geqq 2k$. Then the smaller intersection number x is at most $k/2$.*

Proofs. In Proposition 2.2, use $v/k \leqq k/x$ to obtain Corollary 2.3; Corollary 2.4 follows with $\alpha=2$. ▮

Lemma 2.5 [14, Lemma 2.3] *The following relation holds:*

$$(1) \qquad k(r-1)(y+x-1)-xy(b-1) = k(k-1)(\lambda-1).$$

Proof. Fix a block B_0 and count in two ways the number of triples (s, t, B), where $s, t \in B \cap B_0$, $B \neq B_0$. This gives $ay(y-1)+(b-1-a)\ \ x(x-1) = k(k-1)(\lambda-1)$. Now, use lemma 2.1 and substitute the value of a. The desired conclusion follows upon simplification. ▮

Theorem 2.6. *For a fixed value of the block size k, there exist only finitely many quasi-symmetric designs with larger intersection number $\geqq 2$.*

Proof. By [4], the inequality $b \leqq v(v-1)/2$ holds for any quasi-symmetric design. Hence it suffices to show that for a fixed k, v is bounded above. For $x \neq 0$, this is a consequence of Proposition 2.2, while the case $x=0$ is shown in [13, Corollary 3.2]. ∎

Remark 2.7. Since there are infinitely many designs for $\lambda=1$, the following interpretation of Theorem 2.6 is possible: for a fixed block size k, the population of quasi-symmetric designs is thicker at $(x, y)=(0, 1)$.

Proposition 2.8. *Let S be the set of all quasi-symmetric designs D in which the block intersection numbers x, y are fixed with $y\geqq 2$. Suppose any one of the following additional properties* (i) *through* (iv) *holds. Then S is finite.*

(i) *Γ is connected and $a-c$ is fixed.*
(ii) *$\bar{\Gamma}$ is connected and $a-d$ is fixed.*
(iii) *$\bar{\Gamma}$ is connected and $\bar{a}-\bar{c}$ is fixed.*
(iv) *Γ is connected and $\bar{a}-\bar{d}$ is fixed.*

Proof. (i): From lemma 2.1, $(c-a)(y-x)=\theta-2mz-\theta m+m^2z$, where $\theta=r-\lambda$ and z and m are explained in lemma 2.1. Writing this equation modulo $(m-1)$, gives $z(c-a+1)\equiv 0 \pmod{m-1}$. Observe that $m\neq 1$ since we have no repeated blocks. Also if $c=a-1$ then it is easy to see that Γ is either disconnected or complete. This is a contradiction since Γ is connected and D is proper quasi-symmetric. Since $z=y-x$ is fixed, $m-1$ divides a fixed positive integer and consequently m has finitely many possibilities. This implies that $k=mz+x$ also has finitely many possibilities. Use of Theorem 2.6 completes the proof of (i).

(ii) As in (i), m divides $(a-d)z^2\neq 0$ since $\bar{\Gamma}$ is connected. Now argue as in (i).

(iii) and (iv) are proved similarly and are left to the reader. ∎

Remarks 2.10. We would like to point out that Theorem 2.6 is a generalization of the results in [1, 13, and 14]. The special case $(x, \bar{c})=(0, 0)$ was considered in [1, also 11]. This was generalized in one direction to allow arbitrary x in [14] while in [13], x was restricted to 0 but $\bar{c}$ was allowed to be an arbitrary fixed integer. With a fairly lengthy and tedious argument (which involves obtaining a quadratic equation in λ with coefficients in k, x, y and $\bar{c}$) it may be possible to show a very generalized form of the main results in [1, 13, 14]: Let $x\geqq 0$, $y\geqq 2$ and $\bar{c}\geqq 0$ be fixed arbitrary integers. Then there are finitely many quasi-symmetric designs with given values of x, y and $\bar{c}$.

We would also like to mention that if $x=0$, then it is easy to see that $y<\lambda$ for a proper quasi-symmetric design. However, if $x\neq 0$, then Proposition 2.2 gives $x<\lambda$. This shows the importance of studying the pair $(x, y)=(x, \lambda)$. This has been considered by Holliday in [7].

3. A polynomial associated with a quasi-symmetric 3-design

Throughout this section and section 4, D will denote a quasi-symmetric 3-design with two block intersection numbers x and y, $0\leqq x<y$. For $x=0$, such designs were classified by P. J. Cameron [2, 4] in the following theorem.

Theorem 3.1. *D is a quasi-symmetric 3-design with an intersection number 0 if and* *only if D is the extension of a symmetric 2-design. In that case the parameters of D are one of the following four types:*

(i) *D is Hadamard 3-design.*

(ii) $v=(\lambda+2)(\lambda^2+4\lambda+2)+1$, $k=\lambda^2+3\lambda+2$, and $\lambda=1, 2, \ldots$.

(iii) *D is the extension of a projective plane of order* 10.

(iv) $v=496$, $k=40$ *and* $\lambda=3$. ∎

We add that (iii) has recently been ruled out by Lam et. al [10]. A classification analogous to Theorem 3.1 for $x\neq 0$ is not known (see, e.g. [6, 9]). While this question seems to be fairly difficult, we show the existence of a Delsarte-type polynomial see [5, 12] for a 3-design with *two positive block intersection numbers.* Using this polynomial in Section 4, we make an attempt to classify 3-designs with $x=1$.

Theorem 3.2. *Let D be a quasi-symmetric 3-design with standard parameters v, k, $\lambda(=\lambda_3)$ and with two positive block intersection numbers. Then the block intersection numbers are roots of a quadratic $f(\alpha)=A\alpha^2+B\alpha+C$, where A, B, and C are (polynomial) functions of v, k and λ as given by*

$$A = (v-2)[\lambda(v-1)(v-2)-k(k-1)(k-2)],$$

$$-B = [(v-2)+2(k-1)^2](v-1)(v-2)\lambda-[k(v-2)+(k-1)^2]k(k-1)(k-2)$$

and

$$C = (k-1)^2k^2[\lambda(v-2)-(k-1)(k-2)].$$

Proof. Let x and y be the two block intersection numbers with $1\leqq x<y$. Suppose $\alpha_1=x+y$ and $\alpha_2=xy$. Using equation (1) of the previous section (notice that the λ of equation (1) is actually λ_2 of D), we obtain:

$$k(r-1)\alpha_1-(b-1)\alpha_2 = k(k-1)(\lambda_2-1)+k(r-1). \tag{2}$$

Let E be a point-contraction of D. Since $x\geqq 1$, E is a proper quasi-symmetric design with parameters (in terms of the original parameters of D): $k'=k-1$, $r'=\lambda_2$, $b'=\lambda_1=r$, $\lambda'=\lambda_3$, $x'=x-1$ and $y'=y-1$ where the two last symbols are the block intersection numbers of E. Applying (1) again to E and suitably rearranging terms, we obtain:

$$\begin{aligned}[(k-1)(\lambda_2-1)+(r-1)]\alpha_1-(r-1)\alpha_2 &= \\ =(k-1)(k-2)(\lambda_3-1)+3(k-1)(\lambda_2-1)+(r-1).\end{aligned} \tag{3}$$

Our proof involves solving (2) and (3) as simultaneous linear equations in unknowns α_1 and α_2. Let F, G and H denote the following determinants:

$$F = \begin{vmatrix} k(r-1) & -(b-1) \\ (k-1)(\lambda_2-1)+(r-1) & -(r-1) \end{vmatrix}$$

$$G = \begin{vmatrix} k(k-1)(\lambda_2-1)+k(r-1) & -(b-1) \\ (k-1)(k-2)(\lambda_3-1) & \\ +3(k-1)(\lambda_2-1)+(r-1) & -(r-1) \end{vmatrix}$$

and

$$H = \begin{vmatrix} k(r-1) & k(k-1)(\lambda_2-1)+k(r-1) \\ (k-1)(\lambda_2-1)+(r-1) & (k-1)(k-2)(\lambda_3-1) \\ & +3(k-1)(\lambda_2-1)+(r-1) \end{vmatrix}.$$

It is then clear by Cramer's rule that if $F \neq 0$ then $\alpha_1 = G/F$ and $\alpha_2 = H/F$. Hence the theorem is proved if F, G and H are simplified to a suitable form. To that end, note the following recursion relation which will be used throughout this paper.

Lemma 3.3. *In a $t-(v, k, \lambda)$ design, let λ_i be the number of blocks containing any given i-tuple, $i=0, 1, \ldots, t$ with $\lambda_t=\lambda$, $\lambda_0=b$ and $\lambda_1=r$. Then:*

$$\lambda_i = \frac{(v-i)}{(k-i)}\lambda_{i+1}, \quad i = 0, 1, \ldots, t-1. \tag{4}$$ ∎

Returning to our proof, the use of (4) and straightforward but tedious calculations simplify F, G and H to $F=\theta A$, $G=-\theta B$ and $H=\theta C$, where:

$$\theta = \frac{\lambda(v-k)^2(v-2)}{k(k-1)^2(k-2)^2}$$

and A, B, C are as given in the statement of Theorem 3.2. Observe that $F=0$ implies $A=0$, which on simplification gives $\lambda_3(v-1)(v-2)=k(k-1)(k-2)$. Using (4) two times, gives $r=k$, a contradiction since D is *not* symmetric. Hence $F\neq 0$, $\alpha_1=x+y=G/F=-B/A$ and $\alpha_2=xy=H/F=C/A$. Thus x, y are roots of $A\alpha^2+B\alpha+C=0$ as desired. ∎

The following theorem is a special case of a theorem due to Ray—Chaudhuri and Wilson [12].

Theorem 3.4. *A 4-design in quasi-symmetric if and only if $b=\binom{v}{2}$.* ∎

Application of Theorem 3.2 yields the following corollary given in Ray—Chaudhuri and Wilson [12, Theorem 5].

Corollary 3.5. *Let D be a quasi-symmetric 4-design. Then x and y are roots of the (Delsarte) quadratic*

$$f(\alpha) = \alpha^2 - \left[\frac{2(k-1)(k-2)}{v-3}+1\right]\alpha + \lambda\left(2+\frac{4}{k-3}\right).$$

Proof. First note that if $x=0$, then by (Cameron's) Theorem 3.1, the design is an extension of a symmetric design E which must be simultaneously a 3-design, a contradiction. So $x\neq 0$. By Theorem 3.4, $b=\binom{v}{2}$. Using lemma 3.3,

$$\lambda_3 = \frac{k(k-1)(k-2)}{2(v-2)}.$$

Substitution of this value in Theorem 3.2 yields the assertion. ∎

4. The Case $x=1$

We begin this section by noting the following simple consequences of Theorem 3.2.

Proposition. 4.1 *Let D be a $3-(v, k, \lambda)$ quasi-symmetric design. Then the following assertions hold.*

(i) *$v-2$ divides $k(k-1)^3(k-2)$.*

(ii) *Let $x \neq 0$ and suppose α is x or y. Then the following equation holds:*

$$\lambda_3[(k-1)^2(v-2)\{k^2-2\alpha(v-1)\}+\alpha(\alpha-1)(v-1)(v-2)^2] = \tag{5}$$
$$= k(k-1)(k-2)(k-\alpha)[(k-1)^2-\alpha(v-2)].$$

Proof. (i): If $x=0$, then use Theorem 3.1 and verify the assertion. If $x \neq 0$, then by Theorem 3.2, $x+y=-B/A$, where A, B are as in the statement of Theorem 3.2. Since $v-2$ divides A, it must also divide B. This yields the assertion.

(ii) By Theorem 3.2, $x+y=-B/A$ and $xy=C/A$. Use the values of A, B, C and eliminate one of the variables to get equation (5) in the other variable. Since $f(\alpha)=A\alpha^2+B\alpha+C$ is symmetric in x or y the proof is obvious. ▮

We also need the following result whose proof can be found in [13].

Theorem 4.2. *For a fixed value of $\lambda \geqq 2$, the number of proper quasi-symmetric designs with $x=0$ (and arbitrary y) is finite.* ▮

Proposition 4.3. *Let D be a quasi-symmetric 3-design with $x=1$. Then the following equations are satisfied:*

$$\frac{\lambda(v-2)}{k(k-2)} = \frac{(k-1)^2-(v-2)}{k^2-2(v-1)} \tag{6}$$

$$(k-2)(\lambda-1) = (\lambda_2-1)(y-2). \tag{7}$$

$$\lambda = \lambda_3 = \frac{(k-2)(k-y)}{(k-2)^2-(v-2)(y-2)}. \tag{8}$$

$$yv^2-y(k^2-k+3)v+k(k-1)^2(k-2)+2y(k^2-k+1) = 0. \tag{9}$$

Proof. Substitute $\alpha=x=1$ in (5) to get $\lambda(v-2)[k^2-2(v-1)]=k(k-2)[(k-1)^2-(v-2)]$. In this equation, $k^2-2(v-1)=0$ if and only if $(k-1)^2-(v-2)=0$. This is impossible since $k \geqq 3$ and the proof of (6) is clear. (7) follows by simple two-way counting (see e.g. [13]).

For (8), obtain the values of λ_2 from (7) and Lemma 3.3 and equate them. Finally for (9), substitute the value of λ_3 from (8) into (6) to get the required quadratic in v. ▮

The following result appears in [13, Theorem 4.1].

Lemma 4.4. *Let E be a proper quasi-symmetric 2-design with $x=0$. Let $k=my$. Then the integer m divides $y\{\lambda^2-(y+1)\lambda+y\}$.* ▮

Theorem 4.5. *Let D be a quasi-symmetric* $3-(v, k, \lambda)$ *design with* $x=1$. *Then*

(i) $v-2$ *divides* $k(k-1)^2(k-2)$,

(ii) *The larger intersection number* $y \geqq 3$ *with equality if and only if D is the (unique) Witt—Lüneburg* 4—(23, 7, 1) *design or its residual a* 3—(22, 7, 4) *design.*

(iii) $\lambda \geqq y$.

(iv) $\lambda_3 \geqq 4$ *with equality if and only if D is the* 3—(22, 7, 4) *design.*

(v) $\lambda_3=5$ *if and only if D is the Witt—Lüneburg design on* 23 *points.*

(vi) $\lambda_3 \leqq k-2$ *with equality if and only if D is the Witt—Lüneburg design on* 23 *points.*

(vii) $\lambda_2 \leqq v-2$ *with equality if and only if D is the Witt—Lüneburg design on 23 points.*

(viii) v *is bounded below and above as follows:*

$$\frac{(k-2)^2(y-1)}{y(y-2)}+\frac{(k-2)}{y}-1 \leqq v-3 \leqq \frac{(k-2)(k-3)}{(y-2)}. \tag{10}$$

Further, in (10) *the upper bound is sharp with equality if and only if D is the Witt—Lüneburg 4-design on* 23 *points.*

Proof. For (i) consider a cross multiplication in (6).

(ii) The discriminant $\Delta(y)$ of the quadratic (9) is given by

$$\Delta(y) = y^2(k^2-k+3)^2-8y^2(k^2-k+1)-4yk(k-1)^2(k-2).$$

It is easy to see that $\Delta(2)$ is negative for all $k \geqq 2$. Hence $y \neq 2$. Let $y=3$. Then $\Delta(3)=-3k^4+30k^3-69k^2+42k+9$ which is negative for all $k \geqq 9$. Since (by Lemma 2.1) $y-1$ divides $k-1$, $k=5$ or 7 and for $k=5$, $\Delta(3)$ is not a perfect square. So $k=7$, substitute this value in (9) with $y=3$ to get $v=22$ or 23. Substitute these values in (6) to get $\lambda=4$ or 5. Hence D is either a $3-(22, 7, 4)$ or a $3-(23, 7, 5)$ design. In the latter case one can exactly calculate the occurrences of 4-tuples on blocks to arrive at the required conclusion. The former parameter set is clearly the residual of the latter 4-design. Hence (ii) is proved.

(iii) Contract D to get a 2-design E in which any point is on λ_2 blocks and the block size is $k-1$. Also E is not symmetric, for otherwise, $x=0$ by Cameron's theorem. Hence by Fisher's inequality, $k \leqq \lambda_2$. Using this in (7) we get $\lambda \geqq y$. This proves (iii).

Consider (iv). If $\lambda \leqq 3$, then by (iii) and (ii), $y=3$. This implies $\lambda_3=4$ or 5, a contradiction. So $\lambda_3 \geqq 4$. If $\lambda_3=4$, then by (iii), $y \leqq 4$. If $y=3$, then use (ii) to complete the proof. Otherwise $y=4$. Let E be a point-contraction of D. Then E has $y'=3$, $\lambda'=4$ and $k'=k-1$. Using Lemma 4.4, m divides 9. So $m=3$ or $m=9$. If $m=3$, then $k-1=9$ i.e., $k=10$ and using (7), $\lambda_2=13$, which, by Lemma 3.3 implies that $v-2=26$. This contradicts (i). Let $m=9$. Then $k-1=27$ i.e., $k=28$ and using (7), $\lambda_2=40$, which by Lemma 3.3 implies $v-2=260$, again a contradiction to Proposition 4.1 (i). This proves (iv).

(v) Let $\lambda_3=5$. Then $y \leqq 5$ and by (ii), $y \geqq 3$. If $y=3$, then we are done by (ii). Let, for the sake of contradiction, $y \geqq 4$. Then $y=4$ or $y=5$. If $y=4$, then application of Lemma 4.4 to a contraction of D implies that m divides 24. So $m=2, 3, 4, 6, 8, 12$ or 24. This gives $k-1=m(y-1)=3m$ and hence k. From this we calculate λ_2-1 and hence λ_2 using (7). Finally using Lemma 3.3 (standard recursion), we get $v-2$. These values are tabulated in the table below:

m	2	3	4	6	8	12	24
$k-1$	6	9	12	18	24	36	72
$k-2$	5	8	11	17	23	35	71
λ_2	11	17	23	35	47	71	143
$v-2$	11	f	f	119	f	497	f.

In this table, f denotes a non-integer and the corresponding case is ruled out. Finally using (i), $v-2$ divides $k(k-1)^2(k-2)$. This rules out all the remaining cases.

Now let $y=5$. Again using Lemma 4.4, m divides 16 and hence $m=2, 4, 8$ or 16. Following the same approach as before (use (7), and then Lemma 3.3) to get the following table:

m	2	4	8	16
$k-1$	8	16	32	64
$k-2$	7	15	31	63
λ_2	f	20	f	84
$v-2$	–	60	–	f.

The second column is finally ruled out $\lambda_2(v-1)=r(k-1)$ which forces r to be a non-integer, a contradiction.

(vi) and (vii). Contract D two times to get a 1-design E with block intersection number $y-2$, which is positive by (ii). Hence E is the dual of a 2-design. Application of Fisher's inequality yields $\lambda_2 \leqq v-2$ or equivalently $\lambda_3 \leqq k-2$. Also, in the case of equality, E is a symmetric design i.e. D is a double extension of a symmetric design. Using standard rationality conditions or using results in Cameron—Van Lint [4], E is the Witt—Lüneburg 4-design on 23 points.

Consider (viii). Use (8) and the bound $\lambda_3 \leqq k-2$ proved in (vi) to yield the desired upper bound in (10). Also in case of equality, $\lambda_3=k-2$ which can be handled in (vi) again. Now use (8) again and the inequality $\lambda_3 \geqq y$ obtained in (iii) to get the lower bound. This proves (viii) and the proof of Theorem 4.5 is now complete. ∎

One can make use of various equations proved in this paper (this is already indicated in the proof of Theorem 4.5 (iv), (vi) to rule out many values of k and $\lambda=\lambda_3$. This leads us to make the following two conjectures:

Conjecture 4.6. Let D be a quasi-symmetric 3-design with $x=1$. Then D is either the Witt—Lüneburg 4-design on 23 points or its residual a $3-(22, 7, 4)$ design.

Conjecture 4.7. Let D be a quasi-symmetric 3-design. Then one of the following cases occurs:

(i) $x=0$ and D is a design in Cameron's family (see Theorem 3.1).
(ii) $x=1$ and D is the Witt—Lüneburg design on 23 points or its residual.
(iii) D is the complement of some design in (i) or (ii) above.

Observe that Conjecture 4.7 implies the well-known result of N. Ito on tight 4-designs [8]. Also we note an open question of Hedayat and Kageyama [9, Question 5] in the light of our results. While it seems difficult to prove Conjecture 4.7 using our (quadratic) polynomial in Section 3, we believe that Conjecture 4.6 is provable using the various equations in Section 4 and some number theory. To that end, we offer the following observation:

Theorem 4.8. *Let s be a fixed positive integer and S be the class of all quasi-symmetric $3-(v, k, \lambda)$ design with $\lambda=s$ and smaller intersection number $x\leqq 1$. Then S is finite.*

Proof. The case $x=0$ is shown in Theorem 3.1. For $x=1$, contract D to a 2-design E with block intersection numbers 0 and $y-1$. Since $\lambda_3\neq 1$ (by Theorem 4.5, (iv)), using Theorem 4.2, there are finitely many such designs E. Hence $k-1$ has finitely many possibilities. For each such value of $k-1$ and hence k, v is bounded by k^2 (use Proposition 2.2). Therefore S is a finite union of finite sets. ∎

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