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MATRICES WITH THE EDMONDS—JOHNSON PROPERTY

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A matrix $A=(a_{ij})$ has the *Edmonds—Johnson property* if, for each choice of integral vectors d_1, d_2, b_1, b_2, the convex hull of the integral solutions of $d_1 \leqq x \leqq d_2$, $b_1 \leqq Ax \leqq b_2$ is obtained by adding the inequalities $cx \leqq \lfloor\delta\rfloor$, where c is an integral vector and $cx \leqq \delta$ holds for each solution of $d_1 \leqq x \leqq d_2$, $b_1 \leqq Ax \leqq b_2$. We characterize the Edmonds—Johnson property for integral matrices A which satisfy $\sum_j |a_{ij}| \leqq 2$ for each (row index) i. A corollary is that if G is an undirected graph which does not contain any homeomorph of K_4 in which all triangles of K_4 have become odd circuits, then G is t-perfect. This extends results of Boulala, Fonlupt, Sbihi and Uhry.

1. Introduction

Edmonds and Johnson [5, 6] derived from Edmonds' characterization of the matching polytope [4] that if $A=(a_{ij})$ is an integral $m\times n$-matrix such that

$$\sum_{i=1}^{m} |a_{ij}| \leqq 2 \quad (j = 1, \dots, n), \tag{1}$$

then A has the following *Edmonds—Johnson property*: if d_1, d_2, b_1, b_2 are integral vectors (of appropriate sizes), then the integer hull (= convex hull of the integral solutions) of

$$d_1 \leqq x \leqq d_2, \quad b_1 \leqq Ax \leqq b_2 \tag{2}$$

is obtained from (2) by adding the inequalities ("Gomory cuts")

$$cx \leqq \lfloor\delta\rfloor \tag{3}$$

($\lfloor\ \rfloor$ means rounding down), where $cx \leqq \delta$ is an inequality valid for *all* solutions of (2), and c is integral. (So it means that (2) has "rank 1" in the sense of [2], while rank 0 would mean A being totally unimodular.)

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The Edmonds—Johnson property is not maintained when passing to transposes: (1) may not be replaced by

(4) $$\sum_{j=1}^{n} |a_{ij}| \leqq 2 \quad (i = 1, \dots, m),$$

as the matrix

(5) $$M(K_4) := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{pmatrix}$$

(the incidence matrix of the undirected graph K_4) does not have the Edmonds—Johnson property (consider $0 \leqq x \leqq 1$, $0 \leqq M(K_4)x \leqq 1$). Our main result is that $M(K_4)$ is essentially the only counterexample among the matrices satisfying (4):

Theorem. *An integral matrix satisfying* (4) *has the Edmonds—Johnson property, if and only if it cannot be transformed to* $M(K_4)$ *by a series of the following operations:*

(6) (i) *deleting or permuting rows or columns, or multiplying them by* -1;

(ii) *replacing matrix* $\begin{pmatrix} 1 & g \\ f & D \end{pmatrix}$ *by the matrix* $D - fg$.

Operation (ii) is called *contraction.* f is a column vector and g is a row vector, so that fg is a matrix of the same order as D.

In fact, if a matrix satisfying (4) has the Edmonds—Johnson property, we can describe a smaller set of Gomory cuts which are sufficient to give the convex hull of the integral solutions. To this end, we use the terminology of graph theory. Any integral matrix A satisfying (4) can be considered as a *bidirected graph*: the columns of A correspond to the nodes of this graph, and the rows to the edges. A row containing two $+1$'s corresponds to a $++$ edge connecting the two nodes where the $+1$'s occur. Similarly, there are $+-$ edges and $--$ edges. Moreover, there are $++$ loops (if a 2 occurs) and $--$ loops (if a -2 occurs), (and $+$ loops and $-$ loops for rows with exactly one ± 1, but they will be irrelevant in our discussion). It will be convenient to *identify* the matrix with this bidirected graph, the columns with the nodes, and the rows with the edges. Generally, we denote the set of nodes (= columns) of a bidirected graph A by $V(A)$ or just V, and the set of edges (= rows) by $E(A)$ or E.

A *cycle* in a bidirected graph is a square submatrix C of form:

(7) $$\begin{pmatrix} \pm 1 & \pm 1 & 0 & \cdots & 0 \\ 0 & \pm 1 & \pm 1 & & 0 \\ \vdots & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \pm 1 & \pm 1 \\ \pm 1 & 0 & \cdots & 0 & \pm 1 \end{pmatrix} \quad \text{or} \quad (\pm 2)$$

(possibly with rows or columns permuted). A cycle is *odd* (*even*, respectively) if the number of *odd* edges (:= $++$ edges and $--$ edges) in it is odd (even). We call a bidirected graph *bipartite* if it does not contain any odd cycle. It is well-known and easy that a bidirected graph is bipartite if and only if it is totally unimodular.

If A is a bidirected graph, $x\in\mathbf{R}^V$, $b\in\mathbf{R}^E$, $e\in E$ and C is a submatrix of A, we denote:

(8) $x(e) :=$ entry in position e of Ax (so $x(e)=\pm x_v \pm x_w$ if e connects v and w);

$$x(C) := \frac{1}{2}\sum_{e\in E(C)} x(e),$$

$$b(C) := \sum_{e\in E(C)} b_e.$$

So $Ax\leqq b$ is the same as: $x(e)\leqq b_e$ for $e\in E$. If C is an odd cycle, the corresponding *odd cycle inequality* is:

(9) $$x(C) \leqq \left\lfloor \frac{1}{2} b(C)\right\rfloor.$$

So it is a special type of Gomory cut. In fact, for bidirected graphs, the odd cycle inequalities imply all other Gomory cuts:

Proposition. *Let A be a bidirected graph, with node set V and edge set E, and let $b\in\mathbf{Z}^E$. Then the system*

(10)
$$\begin{aligned} &Ax \leqq b,\\ &cx \leqq \lfloor\delta\rfloor \quad (\textit{if } Ax\leqq b \textit{ implies } cx\leqq\delta, \textit{ where } c \textit{ is integral}), \end{aligned}$$

has the same solution set as the system

(11)
$$\begin{aligned} &Ax \leqq b,\\ &x(C) \leqq \left\lfloor \frac{1}{2} b(C)\right\rfloor \quad (C \textit{ odd cycle}). \end{aligned}$$

Proof. It suffices to show that each solution of (11) satisfies each $cx\leqq\lfloor\delta\rfloor$ in (10). Choose c integral such that $Ax\leqq b$ implies $cx\leqq\delta$. By Farkas' lemma, $yA=c$, $yb\leqq\delta$ for some vector $y\geqq 0$. By Carathéodory's theorem, we may assume that the positive components of y correspond to linearly independent rows of A. As each nonsingular submatrix of A has half-integral inverse (as is easily checked) it follows that y is half-integral. Let A' be the submatrix of A consisting of those rows which have positive component in y.

If A' contains an odd cycle C (say), let y' be half of the characteristic vector of $E(C)$, and let $y'':=y-y'\geqq 0$. If $y''=0$, we know that $cx=x(C)\leqq\lfloor b(C)/2\rfloor = \lfloor yb\rfloor\leqq\lfloor\delta\rfloor$. If $y''\neq 0$, applying induction on $|y|$, we know that $(y''A)x\leqq\lfloor y''b\rfloor$ follows from (11). Hence:

(12) $$cx = (yA)x = (y'A)x+(y''A)x \leqq \lfloor y'b\rfloor+\lfloor y''b\rfloor \leqq \lfloor yb\rfloor \leqq \lfloor\delta\rfloor.$$

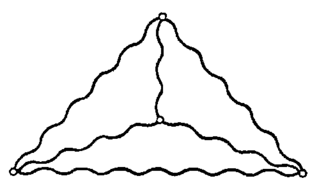
If A' contains no odd cycle, then A' is totally unimodular, and hence $Ax \leqq b$ implies $cx = yAx \leqq \lfloor yb \rfloor \leqq \lfloor \delta \rfloor$. ▮

Some further graph theory. Among the further graph terminology we will use is: an edge *contains* or *connects* the nodes where it has nonzeros; two nodes are *adjacent* if there is an edge connecting them; a bidirected graph is *connected* if we cannot split the node set into two nonempty classes such that no two nodes in different classes are adjacent; a *forest* is a bidirected graph without cycles; a *tree* is a connected forest.

What means contraction (operation (6) (ii))? If we apply operation (6) (ii), and the first row of the initial matrix is a $+-$ edge, we get the ordinary graph contraction: deleting the edge and identifying the two nodes contained in the edge. If the first row is a $++$ edge, contracting means deleting the edge, reversing the signs in node (= column) 1, and identifying the two nodes contained in edge 1. Thus we obtain the following equivalent form of our Theorem.

Corollary 1. *A bidirected graph has the Edmonds—Johnson property if and only if it does not have a subgraph of the form*

(13)



where the wriggled lines stand for (pairwise openly disjoint) paths, such that each of the four cycles in (13) *which have exactly three nodes of degree three, is odd.* ▮

For short, we call a graph (13) as forbidden an *odd-K_4*.

A consequence of Corollary 1 is the following. Chvátal [3] defined an undirected graph $G=(V, E)$ to be *t-perfect* if the convex hull of the characteristic vectors of cocliques (= stable sets) in G is given by:

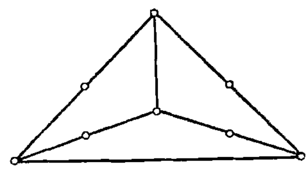
(14)
$$x_v \geqq 0 \qquad (v \in V),$$
$$x_v + x_w \leqq 1 \qquad (vw \in E),$$
$$\sum_{v \in V(C)} x_v \leqq \left\lfloor \frac{1}{2} |V(C)| \right\rfloor \qquad (C \text{ circuit with } |V(C)| \text{ odd}).$$

Then Corollary 1, together with the Proposition, directly give:

Corollary 2. *If* G *satisfies the condition described in Corollary* 1, *then* G *is t-perfect.* ▮

This extends results of Boulala und Uhry [1] (each series-parallel graph is t-perfect), Sbihi and Uhry [9] (each series-parallel graph with some edges substituted by bipartite graphs is t-perfect), and Fonlupt and Uhry [7] (if all odd circuits in a graph contain one fixed node, the graph is t-perfect). There exist however t-perfect graphs which do not have the Edmonds—Johnson property, like

(15)



Remark 1. It follows with the ellipsoid method that if A is a bidirected graph with the Edmonds—Johnson property, and $b \in \mathbf{R}^E$ and $w \in \mathbf{R}^V$, we can solve the integer linear programming problem

(16) $$\max \{wx \mid Ax \leqq b,\ x \text{ integral}\}$$

in polynomial time. Indeed, we may suppose that b is integral. By the results described by Grötschel, Lovász and Schrijver [8] to show polynomial solvability of (16) it suffices to show that we can check in polynomial time whether a vector z belongs to the convex hull of the solution set of (16), and find a separating hyperplane if z is not in this convex hull. To this end, we first check $Az \leqq b$. If one of the constraints is violated, we find a separating hyperplane. Otherwise, we must check the odd cycle inequalities: $z(C) \leqq (b(C)-1)/2$ (C odd cycle, $b(C)$ odd). However, we may as well check: $z(C) \leqq (b(C)-1)/2$ (C cycle (odd or even), $b(C)$ odd), since $Az \leqq b$ implies $z(C) \leqq \lfloor b(C)/2 \rfloor$ for each even cycle C. This last checking can be done as follows. Define a length function l on the edges of A by $l := b - Az \geqq 0$. We must find a cycle C for which $b(C)$ is odd and

(17) $$z(C) > \frac{1}{2} b(C) - \frac{1}{2}, \quad \text{i.e.,} \quad l(C) < 1.$$

To this end, split each node v in V into two nodes v_+ and v_-, and make edges as:

(18) if edge e of A connects v and w and b_e is even, make edges v_+w_+ and v_-w_-, each with length l_e;
if edge e of A connects v and w and b_e is odd, make edges v_+w_- and v_-w_+, each with length l_e.

Then cycles C in A with $b(C)$ odd correspond to paths from v_+ to v_- for some v. So finding a cycle C with $b(C)$ odd and satisfying (17) is equivalent to finding a path from v_+ to v_-, of length less than 1, for some v. This can be done in polynomial time, with a shortest path algorithm.

Remark 2. If A is a bidirected graph, the collection

(19) $$\{E(C) \mid C \text{ odd cycle in } A\}$$

forms a so-called *binary hypergraph* (i.e., if E_1, E_2, E_3 belong to (19), the symmetric difference $E_1 \triangle E_2 \triangle E_3$ contains a set in (19) as a subset). Seymour [10] showed that "a binary hypergraph has the $\mathbf{Z}_+$-max-flow min-cut property, if and only if it does not contain Q_6 as a minor". For bidirected graphs (applying Seymour's result to

(19)) this can be seen to be equivalent to: a bidirected graph A does not contain an odd-K_4 as a subgraph if and only if the system

$$(20)\qquad \begin{aligned} x_e &\geqq 0 \quad (e\in E),\\ \sum_{e\in E(C)} x_e &\geqq 1 \quad (C \text{ odd cycle}) \end{aligned}$$

is totally dual integral, i.e., any linear program over (20) with integral objective function has integral optimum primal and dual solutions. In particular, if A has no odd-K_4, each vertex of (20) is integral. So there are three equivalent properties for a bidirected graph A:

(21)
(i) A has the Edmonds—Johnson property;
(ii) the system (20) is totally dual integral;
(iii) A does not contain an odd-K_4 as a subgraph.

Properties (i) and (ii) are very much related, but we could not find a direct way of deriving one from the other.

In fact, if the list of "minor-minimal counterexamples" for the "weak MFMC-property", given by Seymour [10] p. 200, can be proved to be complete — which is not known —, our Theorem would follow as a corollary.

A recent result of Truemper [11] shows that binary hypergraphs can be tested for having a Q_6 minor, in polynomial time. This implies that a bidirected graph can be tested for having the Edmonds—Johnson property, in polynomial time.

It can be derived from the results of Tseng and Truemper [12] that for every bidirected graph G without odd-K_4 we have one of the following:

(i) G has a node v_0 which is contained in each odd cycle;
or (ii) G is planar, with at most two odd facets;
or (iii) G has at most three nodes;
or (iv) G is "3-separable".

(iv) implies that G can be decomposed into smaller bidirected graphs without odd-K_4. Thus each bidirected graph without odd-K_4 can be composed from bidirected graphs of types (i), (ii) and (iii). This is elaborated in a forth-coming paper of Lovász, Schrijver, Seymour and Truemper.

Remark 3. We leave it to the reader to show that if A has the Edmonds—Johnson property, then in (2) we can also allow some of the components of d_1, d_2, b_1, b_2 to be $\pm\infty$.

2. Proof of the theorem

I.

To show necessity, it suffices to show that the Edmonds—Johnson property is maintained under the transformations (6), and that $M(K_4)$ does not have the Edmonds—Johnson property.

(i) Permuting rows or columns, or multiplying them by -1: trivially maintains the Edmonds—Johnson property.

(ii) Deleting a column, say corresponding to variable x_j: follows trivially by taking $(d_1)_j=(d_2)_j=0$.

(iii) Deleting a row, say the i-th row: follows trivially by taking $(b_1)_i=-\infty$, $(b_2)_i=+\infty$.

(iv) Replacing $\begin{pmatrix}1 & g\\ f & D\end{pmatrix}$ by $D-fg$: Suppose the first matrix has the Edmonds—Johnson property. Let d_1, d_2, b_1, b_2 be integral vectors (of appropriate order), and consider the systems

(22) $$d_1 \leqq x \leqq d_2, \quad b_1 \leqq (D-fg)x \leqq b_2$$

and

(23) $$\begin{pmatrix}-\infty\\ d_1\end{pmatrix} \leqq \begin{pmatrix}\lambda\\ x\end{pmatrix} \leqq \begin{pmatrix}+\infty\\ d_2\end{pmatrix}, \quad \begin{pmatrix}0\\ b_1\end{pmatrix} \leqq \begin{pmatrix}1 & g\\ f & D\end{pmatrix}\begin{pmatrix}\lambda\\ x\end{pmatrix} \leqq \begin{pmatrix}0\\ b_2\end{pmatrix}.$$

Let z be not in the integer hull of (22). It suffices to show that there exists a Gomory cut (3) violated by z. To this end, define

(24) $$\begin{pmatrix}\mu\\ z\end{pmatrix} := \begin{pmatrix}-gz\\ z\end{pmatrix}.$$

It is easily checked that this vector is not in the integer hull of (23). Hence, by assumption, there exists an inequality $(\alpha, c)\begin{pmatrix}\lambda\\ x\end{pmatrix} \leqq \delta$, valid for (23), such that $(\alpha, c)\begin{pmatrix}\mu\\ z\end{pmatrix} > \lfloor\delta\rfloor$ and α, c integral. Then $(c-\alpha g)x \leqq \delta$ is a valid inequality for (22), as if x satisfies (22), then $\begin{pmatrix}-gx\\ x\end{pmatrix}$ satisfies (23), and hence

(25) $$(c-\alpha g)x = (\alpha, c)\begin{pmatrix}-gx\\ x\end{pmatrix} \leqq \delta.$$

Similarly, $(c-\alpha g)z=(\alpha, c)\begin{pmatrix}\mu\\ z\end{pmatrix} > \lfloor\delta\rfloor$, so z is cut off from (22) by a Gomory cut.

(v) $M(K_4)$ has not the Edmonds—Johnson property: Consider the system

(26) $$0 \leqq x \leqq 1, \quad 0 \leqq M(K_4)x \leqq 1.$$

The integral solutions are $(0, 0, 0, 0)^T$, $(1, 0, 0, 0)^T$, $(0, 1, 0, 0)^T$, $(0, 0, 1, 0)^T$, $(0, 0, 0, 1)^T$. So $x_1+x_2+x_3+x_4 \leqq 1$ is a facet of the integer hull. However, this inequality is not a Gomory cut, as $\delta=2$ is the smallest δ for which $x_1+x_2+x_3+x_4 \leqq \delta$ is valid for (26) (since $(1/2, 1/2, 1/2, 1/2)^T$ belongs to (26)).

II.

The remainder of the paper is devoted to showing sufficiency in the Theorem. Suppose the condition is not sufficient. Then there exists a bidirected graph A without an odd-K_4, and an integral vector b, such that

(27) $$Ax \leqq b$$

together with the odd cycle inequalities

(28) $$x(C) \leqq \lfloor b(C)/2 \rfloor \quad (C \text{ odd cycle in } A)$$

is not enough for determining the integer hull of (27) (since joining A with unit basis row vectors, or with the opposite of any row of A, cannot make an odd-K_4). Let A be a smallest such matrix (i.e., with number of rows and columns as small as possible), and let P be the polyhedron defined by (27) and (28). Clearly, A is connected, as otherwise we can decompose A and get a smaller counterexample. We may assume that in each row the sum of the absolute values of the entries is *exactly* 2: all-zero rows trivially do not occur, while a row with one ± 1 can be replaced by the same row multiplied by 2.

Claim 1. *If z belongs to P and z has an integral component, z is a convex combination of integral solutions of* (27).

Proof. Suppose z_1 (say) is an integer. Let $z=\binom{z_1}{z'}$ and $A=[a_1\, B]$, where a_1 is the first column of A. Then z' satisfies

(29) $$Bx' \leqq b - a_1 z_1.$$

We show that z' cannot be cut off from (29) by an odd cycle inequality derived from (29). For suppose $(yB)x' \leqq \lfloor y(b-a_1z_1) \rfloor$ is such an inequality, cutting off z', where y is 0, 1/2-valued, with its 1/2's in positions corresponding to an odd cycle in B. This implies $ya_1=0$. Then

(30) $$(yA)z = (ya_1, yB)\binom{z_1}{z'} = (yB)z' > \lfloor y(b-a_1z_1) \rfloor = \lfloor yb \rfloor.$$

But this is an odd cycle inequality for (27) cutting of z, contradicting the fact that z is in P.

So z' cannot be cut off from (29) by an odd cycle inequality. Hence, as B is smaller than A, z' is in the integer hull of (29), i.e., it is a convex combination of integral solutions of (29), say $z_1', \ldots, z_k'$. Then z is a convex combination of the integral solutions

(31) $$\binom{z_1}{z_1'}, \ldots, \binom{z_1}{z_k'}$$

of (27), proving our claim. ∎

Claim 2. *P has a vertex z with all components non-integral.*

Proof. It suffices to show that there exists a minimal face F of P such that all components of all vectors in F are non-integral (since this implies that F has dimension 0, i.e., is a vertex). In order to show this, observe that P has a minimal face F containing no integral vectors. If F would contain a vector z with at least one component integral, by Claim 1, this vector z is a convex combination of integral vectors in P, hence in F. Contradiction. ∎

From now on, fix a vertex z as described in Claim 2.

Claim 3. *$Az<b$, i.e., z satisfies each inequality in $Ax \leqq b$ strictly.*

Proof. Suppose to the contrary that the first inequality $a_1x \leqq b_1$ (say) has equality for z (where a_1 is the first row of A). Then a_1 contains two ± 1's: if it would contain a ± 2, and b_1 is even, Claim 2 is contradicted, while if b_1 is odd, z is cut off by the odd cycle inequality obtained from a_1.

Without loss of generality, $a_{11} = \varepsilon = \pm 1$. Let

$$z = \begin{pmatrix} z_1 \\ z' \end{pmatrix}, \quad A = \begin{pmatrix} \varepsilon & g \\ f & D \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b' \end{pmatrix}. \tag{32}$$

Then z' satisfies

$$(D - f\varepsilon g)x' \leqq b' - f\varepsilon b_1. \tag{33}$$

Moreover, z' cannot be cut off from (33) by an odd cycle inequality derived from (33). For suppose $y(D - f\varepsilon g)x' \leqq \lfloor y(b' - f\varepsilon b_1)\rfloor$ is such an inequality cutting off z', with $y \geqq 0$. Then

$$\begin{aligned}(\lceil yf\varepsilon\rceil - yf\varepsilon, y)\begin{pmatrix} \varepsilon & g \\ f & D \end{pmatrix}\begin{pmatrix} z_1 \\ z' \end{pmatrix} &= (\varepsilon\lceil yf\varepsilon\rceil, \lceil yf\varepsilon\rceil g + y(D - f\varepsilon g))\begin{pmatrix} z_1 \\ z' \end{pmatrix} \\ &= \varepsilon\lceil yf\varepsilon\rceil z_1 + \lceil yf\varepsilon\rceil gz' + y(D - f\varepsilon g)z' \\ &= \lceil yf\varepsilon\rceil b_1 + y(D - f\varepsilon g)z' > \lceil yf\varepsilon\rceil b_1 + \lfloor y(b' - f\varepsilon b_1)\rfloor \\ &= \left\lfloor (\lceil yf\varepsilon\rceil - yf\varepsilon, y)\begin{pmatrix} b_1 \\ b' \end{pmatrix}\right\rfloor,\end{aligned} \tag{34}$$

(using $\varepsilon z_1 + gz' = a_1 z = b_1$). So z would be cut off from $Ax \leqq b$ by a Gomory cut, contradicting the fact that z is in P.

So z' cannot be cut off from (33) by a Gomory cut. Hence, as $D - f\varepsilon g$ is smaller than A, z' is a convex combination of integral solutions of (33), say $z_1', \ldots, z_k'$. Then

$$\begin{pmatrix} \varepsilon b_1 - \varepsilon g z_1' \\ z_1' \end{pmatrix}, \ldots, \begin{pmatrix} \varepsilon b_1 - \varepsilon g z_k' \\ z_k' \end{pmatrix} \tag{35}$$

are integral solutions of $Ax \leqq b$, having z as a convex combination, contradicting our assumption. ▮

We call an odd cycle C *tight* if the corresponding odd cycle inequality is satisfied by z with equality. As z is a vertex, Claim 3 implies that z is uniquely determined by setting the tight odd cycle inequalities to equality. Moreover we have:

Claim 4. *Each edge of A is contained in at least one tight odd cycle.*

Proof. If not, deleting the edge gives a smaller counterexample. ▮

Without loss of generality, we assume

$$0 < z_v < 1 \quad (v \in V). \tag{36}$$

This is allowed, as replacing z by $z - \lfloor z\rfloor$, and b by $b - A\lfloor z\rfloor$ (where $\lfloor z\rfloor := (\lfloor z_v\rfloor | v \in V)$) gives again a counterexample. Having made assumption (36) we can prove:

Claim 5.

$b_e = +1$ if e is a $++$ edge;
$b_e = \ \ 0$ if e is a $+-$ edge;
$b_e = -1$ if e is a $--$ edge.

Proof. We only show the first line — the other are similar. Let e' be a $++$ edge. By Claim 3 and (36), $b_{e'}>z(e')>0$. So $b_{e'}\geqq 1$. To show the reverse inequality, let C be a tight odd cycle containing e' (exists by Claim 4). Let e' connect nodes v and w, say. Consider the system of linear inequalities

$$(37)\qquad \begin{aligned} x(e) &\leqq b_e \quad (e\in E(C),\ e\neq e'),\\ x_v &\leqq 1, \quad x_w \leqq 1. \end{aligned}$$

For each x satisfying (37) we have

$$(38)\qquad x(C)=\frac{1}{2}\sum_{e\in E(C)\setminus e'} x(e)+\frac{x_v}{2}+\frac{x_w}{2}\leqq 1+\frac{1}{2}\sum_{e\in E(C)\setminus e'} b_e.$$

Now the constraint matrix of (37) is totally unimodular. Hence for each x satisfying (37) we have

$$(39)\qquad x(C)\leqq 1+\left\lfloor \frac{1}{2}\sum_{e\in E(C)\setminus e'} b_e\right\rfloor.$$

Since z satisfies all inequalities in (37) strictly (Claim 3 and (36)), we have

$$(40)\qquad \left\lfloor \frac{1}{2}\sum_{e\in E(C)} b_e\right\rfloor=\left\lfloor \frac{1}{2} b(C)\right\rfloor = z(C) < 1+\left\lfloor \frac{1}{2}\sum_{e\in E(C)\setminus e'} b_e\right\rfloor.$$

Therefore, $b_{e'}<2$, and hence $b_{e'}=1$. ∎

We call a cycle C in a bidirected graph A *non-separating* if for each two edges e and f not contained in C, there exist nodes $v_1, \dots, v_k$ not on C such that v_1 is contained in e, v_k is contained in f, and v_j and v_{j+1} are adjacent $(j=1, \dots, k-1)$. So C is separating, if removing C from A (including the nodes of C) topologically disconnects A.

Claim 6. *There are no separating tight odd cycles.*

Proof. Suppose C is such a cycle. Then we can split the edges not in C into two nonempty classes E' and E'' such that if $e\in E'$ and $f\in E''$ intersect, then their common node(s) are contained in C. Let V' (V'') be the set of nodes which are not in C and are covered by at least one edge in E' (E''). Consider the submatrix A' (A'') of A induced by the rows $E(C)\cup E'$ and columns $V(C)\cup V'$ ($E(C)\cup E''$ and $V(C)\cup V''$). Let z' (z'') be the restriction of z to $V(C)\cup V'$ ($V(C)\cup V''$). Let b' (b'') be the restriction of b to $E(C)\cup E'$ ($E(C)\cup E''$).

Clearly, $A'z'\leqq b'$ and $A''z''\leqq b''$, and z' satisfies the odd cycle inequalities for $A'x'\leqq b'$, and z'' satisfies those for $A''x''\leqq b''$. Moreover, $z'(C)=\lfloor b'(C)/2\rfloor$ and $z''(C)=\lfloor b''(C)/2\rfloor$, as z' and z'' are the same as z on C, and b' and b'' are the same as b on C, and $z(C)=\lfloor b(C)/2\rfloor$.

Since A' is smaller than A, we know that A' has the Edmonds—Johnson property. Hence z' is a convex combination of integral solutions of $A'x'\leqq b'$. Similarly, z'' is a convex combination of integral solutions of $A''x''\leqq b''$. Therefore, there exists a natural number N such that

$$(41)\qquad Nz' = z_1'+\dots+z_N', \quad Nz'' = z_1''+\dots+z_N'',$$

for certain integral solutions $z'_1, \dots, z'_N$ of $A'x' \leqq b'$, and certain integral solutions $z''_1, \dots, z''_N$ of $A''x'' \leqq b''$. Moreover we know, since $x'(C) \leqq \lfloor b(C)/2 \rfloor$, is attained by z' with equality, the same holds for $z'_1, \dots, z'_N$. Similarly for $z''_1, \dots, z''_N$.

Let $e_1, \dots, e_k$ be the edges in C, and consider the corresponding inequalities (say)

$$(42) \qquad x'(e_1) \leqq b'_1, \dots, x'(e_k) \leqq b'_k.$$

As $z'_i(C) = \lfloor b(C)/2 \rfloor$, and $b(C)$ is odd by Claim 5, we know:

$$(43) \qquad z'_i(e_1) + \dots + z'_i(e_k) = b_1 + \dots + b_k - 1,$$

for $i = 1, \dots, N$. Hence each z'_i has equality in all constraints (42) except for one, where there is a rest of 1. Let λ'_j be the number of indices i for which z'_i has rest 1 in the j-th inequality in (42). Similarly, λ''_j is defined. Then trivially

$$(44) \qquad z(e_1) = b_1 - \frac{\lambda'_1}{N}, \dots, z(e_k) = b_k - \frac{\lambda'_k}{N}.$$

Similarly, for the λ''_j. Hence $\lambda'_j = \lambda''_j$ for each j. So we may assume that z'_i and z''_i have rest 1 at the same edge in (42). As $e_1, \dots, e_k$ are linearly independent rows of A, it follows that z'_i and z''_i are the same on $V(C)$. So we can combine z'_i and z''_i to one integral solution z_i of $Ax \leqq b$, so that z_i restricted to A' is z'_i, and z_i restricted to A'' is z''_i. But then $Nz = z_1 + \dots + z_N$, contradicting our assumption that z is a non-integral vertex of P. ∎

Claim 7. *Each tight odd cycle has at least three nodes of degree at least three.*

Proof. Suppose C is a tight odd cycle, with less than 3 nodes of degree at least 3. Assume C has more than 2 edges. Then C contains a node u of degree 2. If C is the only tight odd cycle containing u, we could delete u together with the two edges containing u. In the remaining bidirected graph, the remaining z_v ($v \in V \setminus u$) is uniquely determined by the remaining tight odd cycles (as only one tight odd cycle is deleted). Hence we obtain a smaller counterexample.

So there exists another tight odd cycle C' containing u. As C' is non-separating, C and C' together form the whole bidirected graph. But then A has at least 3 vertices, and exactly two odd cycles, contradicting the fact that z is uniquely determined by the tight odd cycle inequalities.

Hence C has at most two edges. But then the odd cycle inequality is equivalent to $\pm x_v \leqq \lfloor b(C)/2 \rfloor$ for a node v on C, which is tight for z, contradicting Claim 2. ∎

We now prove a Lemma which can be understood independently of the present proof.

Lemma. *Let A be a bidirected graph not containing an odd-K_4. Let C be a non-separating odd cycle in A, containing at least 3 nodes of degree at least 3. Let the edges contained in $V \setminus V(C)$ form a bipartite bidirected graph. Then all odd cycles of A contain one fixed node of A.*

Proof. Clearly $V \setminus V(C) \neq \emptyset$, as if $V = V(C)$, there are at least two edges not in $E(C)$ connecting nodes of V, contradicting the fact that C is non-separating.

Let T be a tree spanning $V\setminus V(C)$ (which exists, as C is non-separating). Now delete all edges contained in $V\setminus V(C)$ which are not in T, and contract the edges in T. As the edges contained in $V\setminus V(C)$ form a bipartite graph, each odd cycle in the original bidirected graph contains an odd cycle in the contracted graph. So it suffices to show that the contracted graph has a node contained in each odd cycle. Hence we may assume that A is the contracted graph, i.e., $V=V(C)\cup\{w\}$ for some node w.

Let C' be an odd cycle in A which has a minimum number of edges in common with C. Choose $u\in V(C)\cap V(C')$ arbitrarily. We show that each odd cycle in A contains u. Suppose to the contrary that odd cycle C'' does not contain u. We consider three cases (cf. (45)).

Case 1. $|E(C'')|\geqq 3$, *and* C' *and* C'' *have a node on* C *in common.*

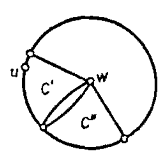
As C'' does not contain u, and as $|E(C')|$ is minimal, it follows that A contains an odd-K_4.

Case 2. $|E(C'')|\geqq 3$, *and* C' *and* C'' *have no node on* C *in common.*

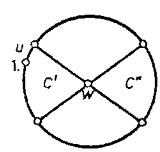
Then it follows directly that A contains an odd-K_4.

Case 3. $|E(C'')|=2$.

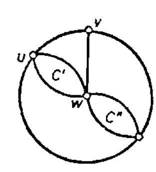
Then also $|E(C')|=2$ (by the minimality of $|E(C')|$). As C has at least 3 nodes of degree at least 3, there is a node v on C, which is connected to w, and which is not contained in C' or C''. Now again it follows that A contains an odd-K_4. ∎



Case 1



Case 2



Case 3

We now return to the main line of the proof. In the following Claim we use the Lemma twice.

Claim 8. *A has a node* u *which is contained in each odd cycle.*

Proof. By the Lemma, it suffices to show that if C is a tight odd cycle, then the edges contained in $V\setminus V(C)$ form a bipartite bidirected graph (using Claims 6 and 7). So it suffices to show that each two odd cycles have a node in common. Assume C' and C'' are odd cycles which do not have a node in common. As A is connected, and each edge is contained in a tight odd cycle, there exist tight odd cycles $C_1, \ldots, C_k$ such that

$$(46)\qquad V(C')\cap V(C_1)\neq\emptyset,\; V(C_1)\cap V(C_2)\neq\emptyset,\; V(C_2)\cap V(C_3)\neq\emptyset,\ldots$$
$$\ldots, V(C_{k-1})\cap V(C_k)\neq\emptyset,\; V(C_k)\cap V(C'')\neq\emptyset.$$

We may assume that k is as small as possible. Hence $V(C')\cap V(C_2)=\emptyset$. So without loss of generality, $C''=C_2$.

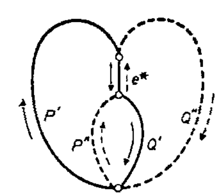
As C_1 is nonseparating, $V \setminus V(C_1)$ spans a connected graph. Let T be a tree spanning $V \setminus V(C_1)$ such that T contains all edges of $E(C')$ and $E(C'')$ which do not intersect $V(C_1)$. This is possible, as $V(C') \cap V(C'') = \emptyset$. Next delete all edges which are contained in $V \setminus V(C_1)$ and which do not occur in T. Let A' be the bidirected graph left. Since T is bipartite, we can apply the Lemma to A'. It follows that $V(C')$ and $V(C'')$ intersect, contradicting our assumption. ▮

We now define an orientation on the edges of A. This orientation can be such that a $+-$ edge is oriented from $+$ to $-$ or from $-$ to $+$.

Claim 9. *The edges of A can be oriented in such a way that each tight odd cycle becomes a directed cycle, and each directed cycle through u comes from an odd cycle in A.*

Proof. As after deleting u, A becomes bipartite, we can split the edges containing u into two classes E_1 and E_2 such that each odd cycle contains one edge in E_1 and one edge in E_2. Now for each tight odd cycle C, we orient the edges in C to a directed cycle such that the edge in E_1 is directed out of u, and the edge in E_2 is directed into u. We show that this gives a unique orientation to each edge. Suppose to the contrary that there exists an edge e^* which is passed by tight odd cycle C' in one direction, and by tight odd cycle C'' in the other direction.

(47)



Let P' be the set of edges of C' on the part from u to e^*, and let Q'' be the set of edges of C'' on the part from e^* to u. Then

(48)

$$\left(\frac{1}{2}b(C')-\frac{1}{2}\right)+\left(\frac{1}{2}b(C'')-\frac{1}{2}\right) = z(C')+z(C'') =$$

$$= \sum_{e\in C'\Delta C''} z(e)/2 + \sum_{e\in C'\cap C''} z(e) <$$

$$< (b(C'\Delta C'')/2-1) + \sum_{e\in C'\cap C''} b_e =$$

$$= (b(C')/2-1/2)+(b(C'')-1/2).$$

Here we use Claim 5 and that $C'\Delta C''$ contains two edge-disjoint odd cycles (since all degrees in $(V, C'\Delta C'')$ are even and since $C'\Delta C''$ contains an even number of odd edges, and it contain at least one odd walk (viz. $P'Q''$)). Moreover $z(e) < b_e$ for all $e \in C' \cap C''$ (Claim 3).

However, (48) includes a contradiction. ▮

Let $\tilde{A}$ be the incidence matrix of the directed graph D obtained by Claim 9. $\tilde{A}$ has one $+1$ (for a head) and one -1 (for a tail) in each row, and the support of $\tilde{A}$ (set of nonzero positions) is the same as that of A.

As z is not half-integral (by Claims 3 and 5 and assumption (36)), there exists an integral vector c such that $\max\{cx|x\in P\}$ is attained by z and the maximum value is not a half-integer.

Define

$$(49)\qquad \begin{aligned} b'_e &:= b_e - 1, && \text{if } e \text{ leaves } u \text{ (in the directed graph } D),\\ b'_e &:= b_e, && \text{otherwise.} \end{aligned}$$

Let $\mathscr{C}$ be the collection of all cycles C in A which form a directed cycle in D. Now $z(C)\leqq b'(C)/2$ for each C in $\mathscr{C}$: if C is odd, C passes u once, and $z(C)\leqq \leqq \lfloor b(C)/2\rfloor = b'(C)/2$; if C is even, C does not pass u, and $z(C)\leqq b(C)/2=b'(C)/2$. As $\mathscr{C}$ contains the tight odd cycles (Claim 9), z attains

$$(50)\qquad \max\{cx|x(C)\leqq b'(C)/2 \text{ for } C \text{ in } \mathscr{C}\}.$$

Now this value is equal to

$$(51)\qquad \max\{cx|Ax+\tilde{A}y\leqq b' \text{ for some vector } y\in\mathbf{R}^v\}.$$

Indeed, the feasible regions of (50) and (51) are the same. If $x(C)\leqq b'(C)/2$ for all C in $\mathscr{C}$, define a length function l by $l_e := b'_e - x(e)$. Then each C in $\mathscr{C}$ has length $\sum_{e\in E(C)} l_e$ nonnegative. So there exist y_v ($v\in V$) such that for each e in E we have: $y_{\text{head}(e)} - y_{\text{tail}(e)} \leqq l_e$ (head and tail with respect to D). As $l=b'-Ax$, this means $\tilde{A}y\leqq b'-Ax$, i.e., x is a feasible solution for (51). Conversely, if x is a feasible solution for (51), and C is in $\mathscr{C}$, let w be the incidence vector of the set of edges in C. Then $x(C)=wAx/2\leqq wb'/2-w\tilde{A}y/2=wb'/2=b'(C)/2$. This shows that (50) and (51) are the same.

Now (51) is equal to

$$(52)\qquad \frac{1}{2}\max\left\{c\tilde{x}+c\tilde{y}\,\middle|\,\frac{1}{2}(A+\tilde{A})\tilde{x}+\frac{1}{2}(A-\tilde{A})\tilde{y}\leqq b'\right\}$$

by the substitution $\tilde{x}=x+y$, $\tilde{y}=x-y$ (so $x=(\tilde{x}+\tilde{y})/2$, $y=(\tilde{x}-\tilde{y})/2$).

However, the constraint matrix $[(A+\tilde{A})/2, (A-\tilde{A})/2]$ in (52) is totally unimodular, as the matrix $[(\tilde{A}+A)/2, (\tilde{A}-A)/2]$ has one $+1$ and one -1 in each row. Therefore, (52) is half-integer, contradicting our assumption. ∎

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