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HYPERGRAPHS DO NOT JUMP

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The number α, $0 \leqq \alpha \leqq 1$, is a jump for r if for any positive ε and any integer m, $m \geqq r$, any r-uniform hypergraph with $n > n_0(\varepsilon, m)$ vertices and at least $(\alpha+\varepsilon)\binom{n}{r}$ edges contains a subgraph with m vertices and at least $(\alpha+c)\binom{m}{r}$ edges, where $c = c(\alpha)$ does not depend on ε and m. It follows from a theorem of Erdős, Stone and Simonovits that for $r=2$ every α is a jump. Erdős asked whether the same is true for $r \geqq 3$. He offered \$ 1000 for answering this question. In this paper we give a negative answer by showing that $1-\frac{1}{l^{r-1}}$ is not a jump if $r \geqq 3$, $l > 2r$.

1. Introduction

For a finite set V and a positive integer r we denote by $\binom{V}{r}$ the family of all r-element subsets of V. We call $G=(V, E)$ an r-uniform hypergraph or shortly an r-graph if $E \subseteq \binom{V}{r}$. The density $d(G)$ of G is defined by $d(G) = E \Big/ \left|\binom{V}{r}\right|$.

Definition 1. 1. A sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$, $G_n = (V_n, E_n)$ of r-graphs is *admissible* if the following two conditions are verified

(i) $|V_n| \geqq r$ and $|V_n| \to \infty$ as $n \to \infty$
(ii) the limit $\lim\limits_{n\to\infty} d(G_n) = d(\mathcal{G})$ exists.

We call $d(\mathcal{G})$ the *density* of the admissible sequence $\mathcal{G}$. For $k \geqq r$ we define

$$\sigma_k(\mathcal{G}) = \max_n \max_{V \in \binom{V_n}{k}} \frac{\left|E_n \cap \binom{V}{r}\right|}{\binom{k}{r}}.$$

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A simple averaging argument yields (cf. Katona, Nemetz, Simonovits [7]): $\sigma_r(\mathcal{G}) \geqq \sigma_{r+1}(\mathcal{G}) \geqq \dots$.

Therefore $\lim_{k\to\infty} \sigma_k(\mathcal{G}) = \bar{d}(\mathcal{G})$ exists, we call it the *upper density* of $\mathcal{G}$.

P. Erdős, A. H. Stone, M. Simonovits [3] proved that the only possible values of $\bar{d}(\mathcal{G})$ for $r=2$ (i.e. for admissible sequences of graphs) are $1-\frac{1}{l}$ $(l=1, 2, 3 \dots)$ and 1. This result easily follows from the following

Theorem (Erdős and Stone [4]). *Suppose $\varepsilon>0$, l, m are positive integers and G is a graph on n vertices with $d(G) \geqq 1-\frac{1}{l}+\varepsilon$. If $n>n_0(l, m, \varepsilon)$ then G contains a complete $(l+1)$-partite subgraph with partition classes of size m (i.e. there exist $l+1$ pairwise disjoint subsets $V_1, \dots, V_{l+1}$ such that (x_i, x_j) is an edge of G whenever $x_i \in V_i$, $x_j \in V_j$ and $i \neq j$ hold).* ∎

P. Erdős asked whether for $r \geqq 3$ the set of possible values of $\bar{d}(\mathcal{G})$ for admissible sequences of r-uniform hypergraphs forms a well-ordered sequence. We give a negative answer:

Theorem 1. 2. *Suppose $r \geqq 3$, $l>2r$ Then for an arbitrary positive ε there exists an admissible sequence of r-graphs, satisfying*

$$(1) \qquad 1-\frac{1}{l^{r-1}} < \bar{d}(\mathcal{G}) < 1-\frac{1}{l^{r-1}}+\varepsilon.$$

In the next section we introduce the Langrange function, $\lambda(G)$, which proves to be a helpful tool in calculating the upper density of certain admissible sequences. In Section 3 we prove a necessary and sufficient condition for α to be a jump. The proof of Theorem 1.2 is given in Section 4.

2. The Lagrange function of hypergraphs

For an r-graph G with vertex set $\{1, 2, \dots, n\}$, edge set $E(G)$ and a vector $\vec{x}=(x_1, \dots, x_n) \in \mathbf{R}^n$ denote by $\lambda(G, \vec{x})$ the Lagrange function of G defined as

$$\lambda(G, \vec{x}) = \sum_{\{i_1, i_2, \dots, i_r\} \in E(G)} x_{i_1} x_{i_2} \dots x_{i_r}.$$

Set $S=\left\{\vec{x}=(x_1, x_2, \dots, x_n)\colon \sum_{i=1}^{n} x_i = 1, x_i \geqq 0\right\}$,

$$\lambda(G) = \max\{\lambda(G, \vec{x})\colon \vec{x} \in S\}.$$

For $J \subset \{1, 2, \dots, n\}$, $J \neq \emptyset$, set $S_J = \{\vec{x}=(x_1, \dots, x_n)\colon \vec{x} \in S,\ x_i>0 \text{ iff } i \in J\}$. Then we write supp $\vec{x}=J$. Clearly, we have

$$\lambda(G) = \max_{\emptyset \neq J \subseteq \{1, 2, \dots, n\}} \{\lambda(G, \vec{x})\colon \vec{x} \in S_J\}.$$

The next theorem follows from the theory of Lagrange multipliers (cf. [4]).

Theorem 2.1. *There exists a non-empty set* $J \subset \{1, 2, \ldots, n\}$ *and a vector* $\vec{y} = (y_1, y_2, \ldots, y_n) \in S_J$ *so that the following three conditions are satisfied:*

(i) $\lambda(G) = \lambda(G, \vec{y})$

(ii) $\frac{\partial}{\partial x_j} \lambda(G, \vec{y}) = \sum_{j \in \{i_1, i_2, \ldots, i_r\} \in E(G)} \frac{y_{i_1} y_{i_2} \ldots y_{i_r}}{y_j} = r\lambda(G), \quad j \in J.$

(iii) *The Jacobian matrix of second derivatives* $\left(\frac{\partial^2}{\partial x_i \partial x_j} (G, \vec{y})\right)_{ij}$ *is negative semidefinite on the hyperplane* $\sum_{1 \leqq i \leqq n} y_i = 0$. ∎

This theorem gives necessary conditions for a vector $\vec{y} \in S$ to fulfill (i). Unfortunately we do not know any satisfactory (i.e. easily verifiable) sufficient condition. In the case of graphs, i.e. if $r=2$, the situation is much simpler. The next theorem can be deduced from Turán's theorem [7]. We give an alternative proof later, using Theorem 2.3.

Theorem 2.2. *Let G be a graph in which the largest clique has size t. Then we have*

$$\lambda(G) = \frac{1}{2}\left(1 - \frac{1}{t}\right). \quad ∎ \tag{2}$$

Theorem 2.3. *Suppose* $\lambda(G) = \lambda(G, \vec{y})$ *for* $\vec{y} \in S_J$ *and* $|J|$ *is minimal subject to this condition. Then for any* $a, b \in J$ *there exists* $e \in E(G)$ *with* $\{a, b\} \subseteqq e \subseteqq J$.

Proof. Suppose the contrary. Then we have

$$\frac{\partial^2}{\partial x_a \partial x_b} \lambda(G, \vec{x}) = 0. \tag{3}$$

Suppose by symmetry $\frac{\partial}{\partial x_a} \lambda(G, \vec{y}) \leqq \frac{\partial}{\partial x_b} \lambda(G, \vec{y})$ holds and define $\delta = \min \{y_a, 1 - y_b\}$. Set $\vec{z} = (z_1, \ldots, z_n)$ with $z_a = y_a - \delta$, $z_b = y_b + \delta$ and $z_i = y_i$ otherwise. Then (3) and the fact that $\lambda(G, \vec{x})$ is linear in each variable imply

$$\lambda(G, \vec{z}) = \lambda(G, \vec{y}) + \delta\left(\frac{\partial}{\partial x_b} \lambda(G, \vec{y}) - \frac{\partial}{\partial x_a} \lambda(G, \vec{y})\right) \geqq \lambda(G, \vec{y}).$$

Howerever, $\vec{z} \in S$, $z_i = 0$ if $i \notin J$ and either $z_a = 0$ or $z_b = 0$ holds in addition, a contradiction. ∎

Proof of Theorem 2.2. Let $\lambda(G) = \lambda(G, \vec{y})$, $\vec{y} \in S_J$ and $|J|$ minimal subject to this condition. Then Theorem 2.3 implies that J is a clique in G, consequently $|J| = j \leqq t$.

Suppose for simplicity $J = \{1, 2, \ldots, j\}$. Then (ii) of Theorem 2.1 implies

$$\sum_{\substack{\nu \neq i \\ 1 \leqq \nu \leqq j}} y_\nu = 2\lambda(G) \quad \text{for} \quad i = 1, \ldots, j.$$

Taking the difference of the first and i'th equation we obtain $y_i - y_1 = 0$, i.e., $y_1 = y_i$. This means $y_1 = y_2 = \ldots = y_j$. Now $y_1 + y_2 + \ldots + y_j = 1$ implies $y_i = \frac{1}{j}$ and consequently $\frac{j-1}{j}=2\lambda(G)$ i.e., $\lambda(G)=\frac{1}{2}\left(1-\frac{1}{j}\right)$. Since $\lambda(G)$ is a maximum, we must have $j=t$ and thus $\lambda(G)=\frac{1}{2}\left(1-\frac{1}{t}\right)$. ∎

For $r\geqq 3$ the situation is much more complicated, the equations in (ii) of Theorem 2.1 are not linear any more. However, for our purposes it will be sufficient to investigate the Lagrange function of a special class of r-graphs.

Definition 2.4. Denote by $K(r, l, t)$ the complete l-partite r-graph with each color class of cardinality t. More explicitly $V(K(r, l, t))=V_1\cup\ldots\cup V_l$, $|V_i|=t$, $V_i\cap V_j=\emptyset$ for $1\leqq i\neq j\leqq l$, $E(K(r, l, t))=\binom{V}{r}-\bigcup_{1\leqq i\leqq l}\binom{V_i}{r}$.

Theorem 2.5.

$$\lambda(K(r, l, t)) = \frac{1}{(lt)^r}\left(\binom{lt}{r}-l\binom{t}{r}\right). \tag{4}$$

We need a lemma. We call two vertices i, j of G *equivalent* if for all $e\in\binom{V-\{i, j\}}{r-1}$, $e\cup\{j\}\in E(G)$ if and only if $e\cup\{i\}\in E(G)$.

Lemma 2.6. *Suppose G is an r-graph on $\{1, 2, \ldots, n\}$ in which i and j are equivalent. Then there exists $\vec{y}\in S_n$ with $\lambda(G)=\lambda(G, \vec{y})$ and $y_i=y_j$. Moreover, $y_i=y_j$ holds for any vector $\vec{y}$ satisfying $\lambda(G)=\lambda(G, \vec{y})$ and such that for some $e\in E(G)$, $\{i, j\}\subseteqq e \subseteqq \operatorname{supp}(\vec{y})\cup\{i\}$ holds.*

Proof. Suppose $y_i\neq y_j$. Define $\vec{z}$ by $z_i=z_j=\frac{y_i+y_j}{2}$, $z_v=y_v$ otherwise. It is easy to see that $\lambda(G, \vec{y})\leqq\lambda(G, \vec{z})$. Moreover, for an edge of the form $\{i, j, i_1, \ldots, i_{r-2}\}$, we replaced $y_i y_j \prod_{v=1}^{r-2} y_{i_v}$ by $\left(\frac{y_i+y_j}{2}\right)^2 \prod_{v=1}^{r-2} y_{i_v}$ which is strictly bigger if $y_{i_1}\ldots y_{i_{r-2}}\neq 0$. ∎

Proof of Theorem 2.5. We may assume without loss of generality that $V_i=\{(i-1)t+1, \ldots, it\}$ for $i=1, 2, \ldots, l$. Let $\vec{y}$ be an optimum vector, i.e. so that $\lambda(G)=\lambda(G, \vec{y})$ holds. Suppose $y_a, y_b\in V_i$ for some $1\leqq i\leqq l$. We want to show that $y_a=y_b$ holds. Suppose for contradiction $y_a>y_b$. Since $\vec{y}$ is an optimum vector, there is an $e'\in E(G)$ satisfying $y_a\in e'\subseteqq\operatorname{supp}(\vec{y})$. Define $e=e'$ if $b\in e$. If not, set $e=(e'-\{b'\})\cup\{b\}$ where $b'\in e'$, $b'\neq a$ and such that $e\subsetneqq V_i$. This is always possible because $e'\subsetneqq V_i$ and $r\geqq 3$. Now the second statement of Lemma 2.6 implies $y_a=y_b$.

Define ϱ_i as the common value of y_j for $j\in V_i$. With this notation

$$\lambda(G, \vec{y}) = \lambda(K(r, 1, lt), \vec{y})-\sum_{i=1}^{l}\binom{t}{r}\varrho_i^r.$$

Using Lemma 2.6 we infer

$$\lambda(K(r, 1, lt), \vec{y}) \leqq \lambda(K(r, 1, lt)) = \frac{1}{(lt)^r}\binom{lt}{r}.$$

On the other hand $\sum\varrho_i=1/t$, $\varrho_i\geqq 0$ and the Jensen inequality yield

$$\sum_{i=1}^{l}\binom{t}{r}\varrho_i^r \geqq \binom{t}{r}l\left(\frac{\sum\varrho_i}{l}\right)^r = \binom{t}{r}\frac{l}{(lt)^r}.$$

Combining these we obtain

$$\lambda(G, \vec{y}) \geqq \frac{1}{(lt)^r}\left(\binom{lt}{r} - l\binom{t}{r}\right)$$

and thus

$$\lambda(G) = \lambda\left(G, \left(\frac{1}{lt}, \frac{1}{lt}, \ldots, \frac{1}{lt}\right)\right) = \frac{1}{(lt)^r}\left(\binom{lt}{r} - l\binom{t}{r}\right) \quad \text{holds.} \quad \blacksquare$$

3. Jumps and thresholds

Definition 3.1. Let $\mathcal{G}=(G_n)_{n=1}^{\infty}$ be an admissible sequence of r-graphs. We say that $\mathcal{H}=(H_n)_{n=1}^{\infty}$ is a *subsequence* of $\mathcal{G}$ if H_n is a subgraph of G_n for all $n \geqq 1$ and $|V(H_n)| \to \infty$ as $n \to \infty$.

Definition 3.2. For $0 \leqq \alpha < 1$ define

$$\Delta_r(\alpha) = \sup\{\delta\colon\ d(\mathcal{G}) > \alpha \text{ implies } \bar{d}(\mathcal{G}) \geqq \alpha + \delta \text{ for all admissible sequences of } r\text{-graphs}\}.$$

We call α a *jump* if $\Delta_r(\alpha) > 0$.

It is not hard to see that one may replace sup by max in Definition 3.2. Moreover, the definition is equivalent to: any admissible sequence $\mathcal{G}$ with $d(\mathcal{G}) > \alpha$ contains a subsequence with $\limsup_n d(H_n) \geqq \alpha + \Delta_r(\alpha)$.

Erdős' problem stated in the Introduction can be reformulated as follows: is it true that $\Delta_r(\alpha) > 0$ holds for all $r \geqq 2$ and $0 \leqq \alpha < 1$?

Definition 3.3. For $0 \leqq \alpha < 1$ and a family $\mathcal{F}$ of r-graphs, we say that α is a *threshold* for $\mathcal{F}$ if in any admissible sequence $\mathcal{G} = \{G_n\}_{n=1}^{\infty}$ with $d(\mathcal{G}) > \alpha$ all but finitely many G_n contain some member of $\mathcal{F}$ as a subgraph. We denote this fact by $\alpha \to \mathcal{F}$.

Note that $\alpha \to \mathcal{F}$ is equivalent to the following: for every ε there exists an $n_0 = n_0(\varepsilon, \alpha, r, \mathcal{F})$ such that $d(G) \geqq \alpha + \varepsilon$ and $|V(G)| > n_0$ imply that G has some member of $\mathcal{F}$ as a subgraph.

Theorem 3.4. (Erdős [1]) *Let $\mathcal{G}$ be an admissible sequence of r-graphs with $d(\mathcal{G}) > 0$. Then $\mathcal{G}$ contains an admissible subsequence $\mathcal{H}$ of complete r-partite graphs with each color class of the same cardinality, $d(\mathcal{H}) = \dfrac{r!}{r^r}$.* $\blacksquare$

In our terminology we have:

Corollary 3.5. *For any $r \geqq 2$, 0 is a jump and $\Delta_r(0) = \dfrac{r!}{r^r}$.* $\blacksquare$

Now we will define the blow-up of an r-graph which will play a central role in the rest of this section.

Definition 3.6. Let G be an r-graph with $V(G) = \{1, 2, \ldots, n\}$ and $\vec{p} = (p_1, \ldots, p_n)$ a non-negative integer vector. Define the $\vec{p}$ blow-up of G, $\vec{p} \otimes G$ as an n-partite r-graph with vertex set $V_1 \cup \ldots \cup V_n$, $|V_i| = p_i$, $1 \leqq i \leqq n$, and edge set

$$E(\vec{p} \otimes G) = \left\{\{v_{i_1}, v_{i_2}, \ldots, v_{i_r}\}\colon\ v_i \in V_i,\ \{i_1, i_2, \ldots, i_r\} \in E(G)\right\}.$$

Note that

(5) $\vec{p}\otimes(\vec{q}\otimes G) = (p_1q_1, \ldots, p_nq_n)\otimes G$ where $\vec{p} = (p_1, \ldots, p_n)$, $\vec{q} = (q_1, \ldots, q_n)$.

We omit the proof of the following easy but important proposition.

Proposition 3.7. *Set* $\bar{i}=(i, \ldots, i)$, $i\geqq 1$. *Then*

(6) $$\bar{d}(\{\bar{i}\otimes G\}_{i=1}^{\infty}) = r!\,\lambda(G)$$

holds for any r-graph G. ∎

The main purpose of this section is to prove the following.

Theorem 3.8. *The following are equivalent.*

(i) α *is a jump for r.*

(ii) $\alpha\to\mathcal{F}$ *for some finite family* $\mathcal{F}$ *of r-graphs satisfying* $\lambda(F)>\frac{\alpha}{r!}$ *for all* $F\in\mathcal{F}$.

Proof. (i)→(ii): Let us set $\Delta=\Delta_r(\alpha)$ (cf. Definition 3.2). Then $\Delta>0$. Let us fix $k=k_0(\Delta, \alpha)$ so that

(7) $$\frac{1}{k^r}\binom{k}{r}\left(\alpha+\frac{\Delta}{2}\right) > \frac{\alpha}{r!}.$$

Let $\mathcal{F}$ be the family of all r-graphs on k vertices and at least $\left(\alpha+\frac{\Delta}{2}\right)\binom{k}{r}$ edges. Then (7) implies $\lambda(F)>\frac{\alpha}{r!}$ for all $F\in\mathcal{F}$.

Let $\mathcal{G}$ be an admissible sequence with $d(\mathcal{G})>\alpha$. Then by definition $\bar{d}(\mathcal{G})\geqq\geqq\alpha+\Delta$ holds. Therefore we may choose $\mathcal{H}=\{H_n\}_{n=1}^{\infty}$, a sequence of subhypergraphs of members of $\mathcal{G}$ so that $\limsup_n d(H_n)\geqq\alpha+\Delta$ and $|H_n|\geqq k$ for $n>n_0(k)$. Thus we may choose H_n to satisfy $d(H_n)\geqq\alpha+\frac{\Delta}{2}$, $|V(H_n)|\geqq k$. Thus $\sigma_k(H_n)\geqq\alpha+\frac{\Delta}{2}$ therefore H_n has a subgraph F with $|V(F)|=k$ $d(F)\geqq\alpha+\frac{\Delta}{2}$, i.e. $F\in\mathcal{F}$, as desired.

(ii)→(i): Let $\mathcal{F}=\{F_1, \ldots, F_m)$ be the finite family with $\lambda(F_i)>\frac{\alpha}{r!}$ satisfying $\alpha\to\mathcal{F}$. Let us set $|V(F_i)|=v_i$.

Let $\mathcal{G}=\{G_n\}_{n=1}^{\infty}$ be an admissible sequence with $d(\mathcal{G})=\alpha+\varepsilon$, $\varepsilon>0$. Let us choose n_1 so that for $n\geqq n_1$, $r!\binom{n}{r}\Big/n^r>\frac{\alpha+\varepsilon/2}{\alpha+2\varepsilon/3}$ holds. Take n so that $|G_n|\geqq n_2=\max\left\{n_1, n_0\left(\frac{\varepsilon}{4}, \alpha, r, \mathcal{F}\right)\right\}$ (c.f. Definition 3.3) and $d(G_n)>\alpha+\frac{2\varepsilon}{3}$ hold. Then the average density of the n_1-vertex spanned subhypergraphs of G_n is at least $\alpha+\frac{\varepsilon}{2}$, therefore there are at least $\frac{\varepsilon}{4}\binom{|V(G_n)|}{n_2}$ n_2-subsets, $W\subset V(G_n)$ which span a subhypergraph with density at least $\alpha+\frac{\varepsilon}{4}$. As $n_2 \geqq n_0\left(\frac{\varepsilon}{4}, \alpha, r, \mathscr{F}\right)$, all these W contain a copy of some member of $\mathscr{F}$. However, a given copy of F_i is contained in at most $\binom{|V(G_n)|-v_i}{n_2-v_i}$ of the n_2-subsets of $V(G_n)$. Therefore, there exists a positive constant c such that, if $|V(G_n)| \geqq n_3 = n_3(n_2, c)$, then for some i, $1 \leqq i \leqq m$, G_n contains at least $\frac{c}{m}\binom{n}{v_i}$ copies of F_i. Let the vertices of F_i be $x_1, \ldots, x_{v_i}$ in an arbitrary but fixed order. For a random partition $V(G)=X_1 \cup \ldots \cup X_{v_i}$ and a given copy of F_i, the probability $x_i \in X_i$ for $i=1, \ldots, v_i$ is $\frac{1}{v_i^{v_i}}$. Therefore, there exists a partition, say $V(G_n) = V_1 \cup \ldots \cup V_{v_i}$, so that the corresponding v_i-partite v_i-graph contains at least $\frac{1}{v_i^{v_i}} \frac{c}{m}\binom{n}{v_i} > c'\binom{n}{v_i}$ copies of F_i, all in the same position.

Thus, for any positive integer t and $n > n_0(c, v_i)$, we may apply Theorem 3.4 and obtain a v_i-partite complete v_i-graph of copies of F_i, i.e. $\vec{t} \otimes F_i$ is a subgraph of $G_{n'}$ whenever $|V(G_{n'})| > n_0(c, v_i)$.

Consequently, for some i, $1 \leqq i \leqq m$, we can find a sequence $1 \leqq n_1 \leqq n_2 \leqq \ldots$ so that G_{n_j} has $\vec{j} \otimes F_i$ as a subgraph. Hence

$$\bar{d}(\mathscr{G}) \geqq \bar{d}\left(\{G_{n_j}\}_{j=1}^{\infty}\right) \geqq \bar{d}\left(\{\vec{j} \otimes F_i\}_{j=1}^{\infty}\right) = r!\,\lambda(F_i).$$

Thus, setting $\Delta = \min\limits_{1 \leqq i \leqq m} (r!\,\lambda(F_i)-\alpha)$, we have $\bar{d}(\mathscr{G}) \geqq \alpha+\Delta$, i.e. α is a jump. ∎

4. The proof of Theorem 1.2

Consider first the sequence $\mathscr{G}(t)=\{G_n\}_{n=1}^{\infty}$ where $G_n=(n, n, \ldots, n) \otimes K(r, l, t)$. Theorem 2.5 and Proposition 3.7 imply

Proposition 4.1.

$$d(\mathscr{G}(t)) = \bar{d}(\mathscr{G}(t)) = r!\,\lambda(K(r, l, t)) = \frac{r!}{(lt)^r}\left(\binom{lt}{r} - l\binom{t}{r}\right). \quad ∎$$

Thus we have

$$\bar{d}(\mathscr{G}(t)) = 1-\frac{1}{l^{r-1}}-\frac{1}{t}\binom{r}{2}\left(\frac{1}{l}-\frac{1}{l^{r-1}}\right)+O\left(\frac{1}{t^2}\right), \tag{8}$$

i.e., the $\mathscr{G}(t)$, $t=1, 2, \ldots$ form a sequence of admissible sequences with monotone increasing upper density tending to $1-\frac{1}{l^{r-1}}$.

We shall prove Theorem 1.2 by showing that one can add ct^{r-1} edges to one of the color classes of $K(r, l, t)$—in order to obtain a new graph $K^*(t)=K^*(r, l, t)$ such that $\lambda(K^*(t))=\bar{d}(\vec{n} \otimes K^*(t))$ is slightly greater than $1-\frac{1}{l^{r-1}}$. Doing this for every t we obtain a decreasing infinite sequence of densities—tending actually to $1-\frac{1}{l^{r-1}}$.

First we need a lemma.

Lemma 4.2. *Suppose r, k and c are fixed. Then for every $t>t_0(r,k,c)$ there exists an r-graph H satisfying:*

(i) $|V(H)|=t$
(ii) $|E(H)|\geqq ct^{r-1}$
(iii) *For all* $V_0\subset V(H)$ $r\leqq|V_0|\leqq k$ *we have*

$$\left|E(H)\cap\binom{V_0}{r}\right|\leqq|V_0|-r+1.$$

Proof. Consider a random r-graph, H^* on $V=V(H^*)=\{1,2,\dots,t\}$ whose edges are chosen independently with probability $\frac{2c}{t}r\,!$. Then the expected number of edges, $M(|E(H^*)|)=2c(t-1)\dots(t-r+1)>\frac{3}{2}ct^{r-1}$ if $t>t_0(r)$.

For $2\leqq l\leqq k$ call $V_0\subset V(H^*)$, $|V_0|=l$, *bad* if (iii) fails for V_0, i.e., if V_0 contains at least $l-r+2$ edges. There are $\binom{\binom{l}{r}}{l-r+2}$ choices for $l-r+2$ edges, and the probability that these edges are in H^* is $\left(r\,!\frac{2c}{t}\right)^{l-r+2}$, thus the expected number of bad V_0's satisfies

$$M(\text{bad }V_0)\leqq\sum_{r\leqq l\leqq k}\binom{t}{l}\binom{\binom{l}{r}}{l-r+2}\left(\frac{r\,!\,2c}{t}\right)^{l-r+2}<\frac{c}{2\binom{k}{r}}t^{r-1}\quad\text{if}\quad t>t_0(r,k,c).$$

Thus, $M(|E(H^*)|)-\binom{k}{r}M(\text{bad }V_0)>ct^{r-1}$. Consequently, for some value of H^*, the number of edges remaining after removing all edges contained in a bad V_0 is still at least ct^{r-1}, and the r-graph formed by these remaining edges show the validity of the lemma. ∎

Set now $c=2+\binom{r}{2}\left(\frac{1}{l}-\frac{1}{l^{r-1}}\right)$. Let $K(r,l,t)$ have color classes $V_1,\dots,V_l$ and let H be an r-graph on V_1 satisfying the statement of Lemma 4.2. Define $K^*(r,l,t)$ as $K(r,l,t)$ together with the edges of H. In view of (8) we have for $t>t_0$

$$r\,!\frac{|E(K^*(r,l,t))|}{(lt)^r}\geqq 1-\frac{1}{l^{r-1}}+\frac{1}{t},$$

thus,

$$\lambda(K^*(r,l,t))\geqq\frac{1}{r!}\left(1-\frac{1}{l^{r-1}}+\frac{1}{t}\right).\tag{9}$$

Suppose now that $1-\frac{1}{l^{r-1}}$ is a jump. In view of Theorem 3.8 there exists a finite collection $\mathcal{F}$ of r-graphs with $\lambda(F)\geqq\frac{1}{r!}\left(1-\frac{1}{l^{r-1}}+\delta\right)$, $\delta>0$ for all $F\in\mathcal{F}$, such that $1-\frac{1}{l^{r-1}}$ is a threshold for $\mathscr{F}$ $\left(1-\frac{1}{l^{r-1}}\to\mathscr{F}\right)$. Let us set $k=\max_{F\in\mathscr{F}}|V(F)|$, and consider the sequence $\{\vec{n}\otimes K^*(r,l,t)\}_{n=1}^{\infty}$. Then (9) implies that for $n>n_0(k,\delta)$ some member of $\mathscr{F}$ is a subhypergraph of $\vec{n}\otimes K^*(r,l,t)$.

We shall establish the contradiction by proving

Lemma 4.3. *Suppose H is a subhypergraph of $K^*(r,l,t)$, $|V(H)|\leqq k$, $t>t_0(k)$. Then*

$$\lambda(H)\leqq\frac{1}{r!}\left(1-\frac{1}{l^{r-1}}\right).$$

Proof. Let us define $U_i=V(H)\cap V_i$. Let $\vec{\xi}=(\xi_1,\xi_2,\dots,\xi_m)$ be an optimal vector. In view of Lemma 2.6. $\vec{\xi}$ is a constant on U_i for $i=2,3,\dots,l$. Let ϱ_i be the corresponding value, set $\alpha_i=|U_i|\varrho_i$, $\alpha_1=1-\sum_{2\leqq i\leqq l}\alpha_i$. Define $H_0=(U_1, E(H)\cap[U_1]^r)$ i.e. the subhypergraph of H, induced on U_1. If $E(H_0)=\emptyset$ then H is a complete l-partite r-graph and Theorem 2.5 yields the statement. Thus we may assume $|V(H_0)|=r-1+d$ with d a positive integer. In view of Lemma 4.2, H_0 has at most d edges. Assume $V(H_0)=\{v_1,v_2,\dots,v_{r-1+d}\}$ and suppose $x_1,x_2,\dots,x_{r-1+d}$ are the corresponding values of $\vec{\xi}$ with $x_1\geqq x_2\geqq\dots\geqq x_{r-1+d}$.

Claim 4.4. *It is sufficient to prove Lemma 4.3 for the case*

$$E(H_0)=\big\{\{v_1,v_2,\dots,v_{r-1},v_i\};\ r\leqq i<r+d\big\}.$$

Proof of the claim. Let us order the edges of H_0 in a decreasing order $e_1,e_2,\dots,e_s$, i.e. $\prod_{v_i\in e_\nu}x_i\geqq\prod_{v_i\in e_\mu}x_i$ for $\nu<\mu$. In view of the construction, $s\leqq d$, moreover, $|e_1\cup e_2\cup\dots\cup e_\mu|\geqq r-1+\mu$ for $\mu=1,2,\dots,s$. Consequently, at least one of the edges out of $e_1,e_2,\dots,e_\mu$ contains some v_i with $i\geqq r-1+\mu$ and thus satisfies

$$\prod_{x_i\in e}x_i\leqq x_1x_2\dots x_{r-1}x_{r-1+\mu}.$$

Thus, by monotonicity the same holds for e_μ and

$$\sum_{1\leqq\mu\leqq s}\prod_{x_i\in e_\mu}x_i\leqq\sum_{1\leqq j\leqq d}x_1x_2\dots x_{r-1}x_{r-1+j}$$

follows. Consequently, replacing the "old" H_0 by the "new" one does not decrease $\lambda(H)$. ∎

We go on with the proof of Lemma 4.3. In view of Lemma 2.6, $x_1=x_2=\dots=x_{r-1}\overset{\text{def}}{=}\varrho_0$, $x_r=x_{r+1}=\dots=x_{r-1+d}\overset{\text{def}}{=}\varrho_1$ holds. Observing that each term in $\lambda(H)$ appears $r!$ times in the expansion of $(\xi_1+\xi_2+\dots+\xi_m)^r$ but this expansion contains lots of terms not appearing in $\lambda(H)$ as well, we infer

$$(10)\quad r!\,\lambda(H)\leqq 1-\sum_{i=1}^{l}\alpha_i^r+r!\sum_{1\leqq j\leqq d}x_1x_2\dots x_{r-1}x_{r-1+j}-\binom{r}{2}\sum_{1\leqq j\leqq r-1+d}x_j^2(1-\alpha_1)^{r-2}$$

It will be sufficient to show that the RHS of (10) is less than or equal to $1-\frac{1}{l^{r-1}}$.

Let us set $\alpha_1=\frac{1}{l}+(l-1)\varepsilon$. Then $1-\alpha_1=\frac{l-1}{l}-(l-1)\varepsilon=\alpha_2+\ldots+\alpha_l$ implies

(11) $$\alpha_1^r+\alpha_2^r+\ldots+\alpha_l^r \geqq \left(\frac{1}{l}+(l-1)\varepsilon\right)^r+(l-1)\left(\frac{1}{l}-\varepsilon\right)^r.$$

For $r=3$ we deduce

(12) $$\sum_{i=1}^{l}\alpha_i^3 \geqq \frac{1}{l^2}+3(l-1)\varepsilon^2+l(l-1)(l-2)\varepsilon^3.$$

Note that

(13) $$r!\sum_{1\leqq j\leqq d} x_1x_2\ldots x_{r-1}x_{r-1+j} = r!\,\varrho_0^{r-1}(\alpha_1-(r-1)\varrho_0),$$

(14) $$\binom{r}{2}\sum_{1\leqq j\leqq r-1+d} x_j^2 \geqq \binom{r}{2}(r-1)\varrho_0^2.$$

Thus, for $r=3$, it is sufficient to show by (10), (11), (12), (13) and (14), that

$$3(l-1)\varepsilon^2+l(l-1)(l-2)\varepsilon^3-6\varrho_0^2\left(\frac{1}{l}+(l-1)\varepsilon-2\varrho_0\right)+6\varrho_0^2\left(\frac{l-1}{l}-(l-1)\varepsilon\right) \geqq 0,$$

or, equivalently,

(15) $$3(l-1)\varepsilon^2+l(l-1)(l-2)\varepsilon^3 \geqq 6\varrho_0^2\left(2(l-1)\varepsilon-2\varrho_0-\frac{l-2}{l}\right).$$

Using the inequality between the arithmetic and geometric means we get that the RHS can be bounded from above by

$$6\left(\frac{2(l-1)\varepsilon-\frac{l-2}{l}}{3}\right)^3 \leqq 6\left(\frac{l\varepsilon}{3}\right)^3,$$

using $\varepsilon\leqq 1/l$. Now (15) follows from

$$l(l-1)(l-2)\varepsilon^3 \geqq \frac{6}{27}l^3\varepsilon^3, \quad \text{i.e.} \quad (l-1)(l-2) \geqq \frac{2}{9}l^2 \quad \text{for} \quad l\geqq 3,$$

and from

$$3(l-1)\varepsilon^2 = 3\varepsilon^3 \geqq \frac{6\cdot 8}{27}\varepsilon^3 \quad \text{for} \quad l=2.$$

Consider now the case $r>3$, $l>2r$. By the inequality between arithmetic and geometric means for (13), we have

(16) $$r!\,\varrho_0^{r-1}(\alpha_1-(r-1)\varrho_0) \leqq \frac{r!}{r^r}\alpha_1^r = \frac{r!}{r^r}\left(\frac{1}{l}+(l-1)\varepsilon\right)^r.$$

Suppose first that $\varepsilon\geqq\frac{1}{lr}$. Then $(l-1)\varepsilon\geqq\frac{2}{l}$, thus, for (16) we obtain

$$\frac{r!}{r^r}\left(\frac{1}{l}+(l-1)\varepsilon\right)^r \leqq \frac{r!}{r^r}\left(\frac{3}{2}(l-1)\varepsilon\right)^r \leqq \frac{1}{2}(l-1)^r\varepsilon^r,$$

as $\frac{r!}{r^r}\left(\frac{3}{2}\right)^r \leqq \frac{1}{2}$ holds for $r\geqq 4$.

On the other hand from (11) we have

$$\sum_{i=1}^{l} \alpha_i^r \geqq \frac{1}{l^{r-1}} + (l-1)^r \varepsilon^r - (l-1)\varepsilon^r \geqq \frac{1}{l^{r-1}} + \frac{1}{2}(l-1)^r \varepsilon^r. \tag{17}$$

From (16) and (17) we infer adding (11)+(14)−(13) that

$$\sum_{i=1}^{l} \alpha_i^r + \binom{r}{2} \sum_{1 \leqq j \leqq r-1+d} x_j^2 - r! \sum_{1 \leqq j \leqq d} x_1 x_2 \dots x_{r-1} x_{r-1+j} \geqq$$

$$\geqq \frac{1}{l^{r-1}} + \frac{1}{2}(l-1)^r \varepsilon^r + \binom{r}{2}(r-1)\varrho_0^2 - \frac{1}{2}(l-1)^r \varepsilon^r \geqq \frac{1}{l^{r-1}},$$

as desired.

Suppose last $\varepsilon < \frac{1}{lr}$. We settle this case by showing (13)≦(14), equivalently,

$$r!\, \varrho_0^{r-3}(\alpha_1 - (r-1)\varrho_0) \leqq \binom{r}{2}(r-1)(1-\alpha_1)^{r-2}. \tag{18}$$

We increase the LHS by writing $r!\frac{r-1}{r-3}\left(\frac{\frac{r-3}{r-1}\alpha_1}{r-2}\right)^{r-2}$ substituting the value of α_1 and noticing that the LHS increases and RHS decreases with α_1, i.e. it is sufficient to check the inequality for $\varepsilon = \frac{1}{lr}$, i.e. $\alpha_1 = \frac{1}{l} + \frac{l-1}{lr}$. For $l \geqq r-1$ we have $\alpha_1 \leqq \frac{2}{r}$ and $1 - \alpha_1 \geqq 1 - \frac{2}{r}$. Thus, it is sufficient to have

$$r!\frac{r-1}{r-3}\left(\frac{2(r-3)}{r(r-1)(r-2)}\right)^{r-2} \leqq \binom{r}{2}(r-1)e^{-2},$$

which holds for $r \geqq 4$ as the LHS is <1 and the RHS is >1. ▮▮

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