COMBINATORICA 7 (1) (1987) 145—150

ON DIGRAPHS WITH NO TWO DISJOINT DIRECTED CYCLES

C. THOMASSEN

Received 11 February 1986

We obtain a result on configurations in 2-connected digraphs with no two disjoint dicycles. We derive various consequences, for example a short proof of the characterization of the minimal digraphs having no vertex meeting all dicycles and a polynomially bounded algorithm for finding a dicycle through any pair of prescribed arcs in a digraph with no two disjoint dicycles, a problem which is NP-complete for digraphs in general.

1. Introduction

Dirac [3] characterized the 3-connected graphs having no two disjoint cycles and Lovász [9] extended Dirac's result to all graphs. The corresponding problem for digraphs (directed graphs) with no two disjoint dicycles (directed cycles) is unsolved. A well-known conjecture of Gallai [6] (extended by Younger, see [2]) asserts that there exists a natural number m such that every digraph D with no two disjoint dicycles contains a set S of at most m vertices such that $D-S$ is acyclic. Combined with the solution of Fortune, Hopcroft and Wyllie [5] of the k-dipath problem for acyclic digraphs this would provide a so-called good characterization of the digraphs having no two disjoint dicycles although it might be difficult to apply since m might be large and since the characterization by Fortune et al. is (though elegant and significant) not very precise. A precise characterization of the acyclic digraphs having no k disjoint dipaths with prescribed ends is known only for $k=1, 2$ [11] and a natural consequence of that result is the characterization of the minimal digraphs having no vertex meeting all dicycles. A short independent proof of that was also given in [11] and it was pointed out how this implies results of Allender [1] and Kosaraju [7]. In this note we obtain another short proof based on a result on configurations in 2-connected digraphs with no two disjoint dicycles. As further consequences of that result we obtain a result of Ésik on dicycles of different lengths, a result of Kostocka [8] on dicycles through a given vertex and arc, and finally a result on dicycles through prescribed arcs e_1, e_2 in a digraph D with no two disjoint dicycles: Such a digraph D contains a dicycle through e_1 and e_2 if and only if, for each vertex v of D, $D-v$ has a dipath from the head of e_i to the tail of e_{3-i} for $i=1$ or 2. The problem of finding a dicycle through two prescribed arcs in a general digraph is NP-complete [5].

AMS subject classification (1980): 05 C 20, 05 C 38

2. Terminology

We use the same terminology as in [2]. *Splitting* a vertex v of a digraph D means that we delete v and add two new vertices v_1 and v_2 such that v_1 is dominated by all those vertices of $D-v$ which dominate v and v_2 dominates all those vertices of $D-v$ which are dominated by v and furthermore v_1 dominates v_2. A *split* of a digraph D is any digraph obtained from D by successively splitting vertices of indegree and outdegree at least 2. The three splits of the complete symmetric digraph K_3^* in Figure 1 (a) are shown in Figure 1 (b), (c), (d). The digraph T_5 of Figure 1 (e) is the unique tournament in which each vertex has indegree 2 and outdegree 2.

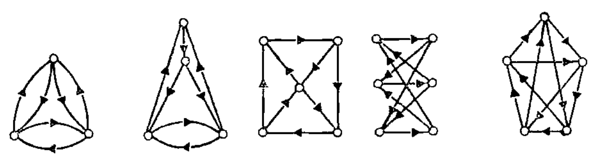

Fig. 1

A *subdivision* of a digraph D is any digraph obtained from D by successively inserting vertices of indegree and outdegree 1 on arcs. If S is a dipath or dicycle and u and v are vertices of S, then $S[u, v]$ denotes the dipath in S from u to v. If P is a dipath, then $t(P)$ and $h(P)$ denote the first vertex (tail) and the last vertex (head) of P, respectively.

3. Dicycles and subdivisions in digraphs with no two disjoint dicycles

Lemma 1. *Let v and e be a vertex and an arc, resepctively, of a digraph D which has no two disjoint dicycles. If $D-v$ has a dicycle C through e and D has two dicycles C_1, C_2 having precisely v in common, then D contains a subdivision D' of a split of K_3^* or T_5 (of Figure 1 (a) and (e), respectively) such that D' contains e and v and such that v has indegree 2 and outdegree 2 in D'.*

Proof. Without loss of generality we can assume that number of segments in $(C_1 \cup C_2) \cap C$ is minimum. If e is in $C_1 \cup C_2$, then we let P denote any shortest dipath in C from C_1 to C_2. If e is not in $C_1 \cup C_2$, we denote by P the shortest dipath in C such that P contains e and $\{t(P), h(P)\} \subseteq C_1 \cup C_2$, say $t(P) \in C_1$. The minimality property of C, C_1, C_2 and the assumption that D has no two disjoint dicycles then imply that $h(P) \in C_2$. Following C from $h(P)$ we denote by P_1 the first (shortest) dipath with no arc in $C_1 \cup C_2$ and with $t(P_1) \in C_2$ and $h(P_1) \in C_1 \cup C_2$. Again, the minimality property of C, C_1, C_2 implies that $h(P_1) \in C_1$. Going backwards along C from $t(P)$ we define analogously P_2 as the first (shortest) dipath with no arc in $C_1 \cup C_2$ such that $h(P_2) \in C_1$ and $t(P_2) \in C_2$. If $h(P_1) \in C_1[v, t(P)]$, then $C_1 \cup C_2 \cup P \cup P_1$ is a subdivision of one of the digraphs of Figure 1 (a), (b), (c), so assume $h(P_1) \in$

$\in C_1[t(P), v]-t(P)$. Similarly, we can assume that $t(P_2)\in C_2[v, h(P)]-h(P)$. Following C from $h(P_1)$ we obtain P_3 in the same way as we obtained P_1 and following C backwards from $t(P_2)$ we obtain P_4 in the same way as we obtained P_2. The minimality property of C, C_1, C_2 implies that $h(P_3)\in C_2[v, h(P)]$ (for otherwise we can modify C and obtain a contradiction). If $h(P_3)\in C_2[v, t(P_2)]$, then $C_1\cup C_2\cup P\cup$ $\cup P_1\cup P_2\cup P_3$ is a subdivision of a split of T_5 (of Figure 1 (e)). So we can assume that $h(P_3)\in C_2[t(P_2), h(P)]-t(P_2)$. By a similar argument $t(P_4)\in C_1[t(P), h(P_1)]-$ $-h(P_1)$. But then $C_1\cup C_2\cup P_1\cup P_2\cup P_3\cup P_4$ contains two disjoint dicycles. This contradiction completes the proof. ∎

Theorem 1. *If D is a strongly 2-connected digraph with no two disjoint dicycles and v and e are a vertex and arc of D, then D has a subdivision D' of a split of one of the digraphs of Figure 1 (a), (e) such that D' contains v and e and v has indegree and outdegree 2 in D'. Also, D has a subdivision D'' of a split of K_3^* containing both v and e such that v has outdegree 2 or indegree 2 in D''.*

Proof. If e is not incident with v we apply Lemma 1 and if e is incident with v we apply the proof of Lemma 1 by letting e be in one of C_1, C_2. The last assertion follows from the observation that deleting any two consecutive arcs of the "outer" dicycle of Figure 1 (e) leaves a subdivision of the digraph of Figure 1 (b). ∎

Theorem 2. ([11]). *A digraph D contains a vertex meeting all dicycles of D if and only if D has no two disjoint dicycles and no subdivision of any digraph in Figure 1 (a), (b), (c), (d).*

Proof. The "only if" part is trivial and we prove the "if" part by contradiction assumng that D is a counterexample of minimum order. Clearly D is strong and has at least four vertices. By Theorem 1 D cannot be strongly 2-connected. Hence D has a vertex x such that $D-x$ is not strong. Let D_1 and D_2 be strong components of $D-x$ such that no arc enters D_1 and no arc leaves D_2 in $D-x$. Since D has no two disjoint dicycles, one of D_1, D_2 (say D_1) is a single vertex y of indegree 1 in D. Let D_3 be obtained from D by contracting the arc xy. Since D has no vertex meeting all dicycles, D_3 has no vertex meeting all dicycles and, by the minimality of D, D_3 has either two disjoint dicycles or a subdivision of a digraph in Figure 1 (a), (b), (c), (d). But then D has the same property. This contradiction proves the theorem. ∎

If we prove the weakened version of Lemma 1 where D' does not contain e, we get an even shorter proof of Theorem 1 (see [12]). We now recall two consequences of Theorem 2 given in [11].

Corollary 1. (Allender [1]). *If D is a digraph with p vertices such that no vertex of D meets all dicycles, then D has a dicycle of length at most $2p/3$.*

Proof. By Theorem 2, D contains a subdivision of one of the digraphs of Figure 1 (a), (b), (c), (d) or two disjoint dicycles. In any case D has the desired dicycle. ∎

Corollary 1 is best possible and it is also easy to characterize the extremal digraphs (see [11]).

Corollary 2. (Kosaraju [7]). *If any three dicycles in the digraph D have nonempty intersection, then all dicycles of D have nonempty intersection.*

Proof. If no vertex of D meets all dicycles, then D has either two disjoint dicycles or a subdivision of one of the digraphs of Figure 1 (a), (b), (c), (d). In any case D has three dicycles with empty intersection. ∎

It would be interesting to establish a connection between Corollary 2 and Helly's theorem for compact convex sets in $\mathbf{R}^2$. One possibility would be to consider plane digraphs in which all cycles are convex polygons. But such digraphs are rare. For example, if a digraph of minimum degree at least 3 has a Hamiltonian cycle, then that cannot be convex in any plane representation. In order to avoid this obstacle one might consider only a subclass of all dicycles. But then Corollary 2 fails as shown by the cycles of length $n-1$ in the complete symmetric digraph of order n.

The following result of Ésik was drawn to the author's attention by A. Ádám and P. Erdős.

Corollary 3. (Ésik). *If D is a digraph with no two disjoint dicycles and with no vertex meeting all dicycles, then D has two dicycles of different lengths.*

Proof. It is easy to see that each digraph of Figure 1 (a), (b), (c), (d) has two dicycles of different lengths. Hence Corollary 3 follows from Theorem 2. ∎

Corollary 4. (Kostochka [8]). *If D is a strong digraph with no two disjoint dicycles, then D has a vertex v such that, for each* arc *e of D, there is a dicycle through v and e.*

Proof. By induction on the order of D. If D is 2-connected, then any vertex can play the role of v by Theorem 1. If D is not 2-connected, then we conclude as in the proof of Theorem 2 that D has an arc xy such that x has outdegree 1 or y has indegree 1. Then we contract xy and apply the induction hypothesis. ∎

Theorem 3. *If e_1 and e_2 are arcs in a digraph D with no two disjoint dicycles, then D has a dicycle through e_1 and e_2 if and only if, for each vertex v in D, $D—v$ has a dipath from $h(e_i)$ to $t(e_{3-i})$ for $i=1$ or 2.*

Proof. The "only if" part is trivial and we prove the if part by induction on the order of D. Since D has a dipath from $h(e_i)$ to $t(e_{3-i})$ for $i=1, 2$, e_1 and e_2 are in the same strong component of D and we can assume that D equals that component. If v is a vertex incident with both e_1 and e_2, then a dipath in $D-v$ from $h(e_i)$ to $t(e_{3-i})$ together with v, e_1 and e_2 form the desired dicycle. So we can assume that e_1 and e_2 have no end in common.

We consider first the case where D has no two internally disjoint dipaths from $h(e_1)$ to $t(e_1)$. Then, by Menger's theorem, D has a vertex u such that the vertex set of $D-u$ can be partitioned into sets A, B such that e_1 goes from A to B and there is no arc from B to A. Since D has no two disjoint dicycles we can assume that the subdigraph induced by A or B (say B) has no dicycle. Let P be a maximal dipath in B such that $t(P)=h(e_1)$. Then $h(P)$ dominates u only. If $e_2=h(P)u$, then a dipath from u to $t(e_1)$ together with P and e_1, e_2 is the desired dicycle. So we can assume that $e_2 \neq h(P)u$. Contracting $h(P)u$ into a single vertex yields a digraph satisfying the condition of the theorem and we complete the proof by induction.

We can therefore assume that D contains two internally disjoint dipaths $P_{i,1}$, $P_{i,2}$ from $h(e_i)$ to $t(e_i)$ for $i=1, 2$. If e_2 joins two vertices of $P_{1,1} \cup P_{1,2}$, then it is easy to find a dicycle through e_1 and e_2 using the facts that D has no two disjoint dicycles and e_1 and e_2 have no common end. So assume that $P_{1,1}$ contains no end of e_2. Now we delete all arcs which have an end in common with e_2 except those in $P_{2,1} \cup P_{2,2}$ and we then contract e_2 into a single vertex v. The resulting digraph satisfies the assumption of Lemma 1 with e_1, $P_{2,1}$, $P_{2,2}$, $P_{1,1} \cup \{e_1\}$ instead of e, C_1, C_2, C, respectively. The conclusion of Lemma 1 gives the desired dicycle. ∎

Corollary 5. *In a strongly 2-connected digraph with no two disjoint dicycles any two non-adjacent arcs are on a common dicycle.* ∎

Fortune et al [5] showed that the problem of finding a dicycle through two prescribed arcs in a digraph is NP-complete. Theorem 3 shows that this problem can be solved in polynomial time when it is restricted to the class of digraphs having no two disjoint dicycles. The conditions for such a dicycle through two arcs e_1, e_2 given in Theorem 3 and Corollary 5 are not sufficient in the class of digraphs containing two vertices whose deletion leaves an acyclic digraph. This is shown by the type of digraph in Figure 2. Note that such a digraph contains no three pairwise disjoint dicycles.

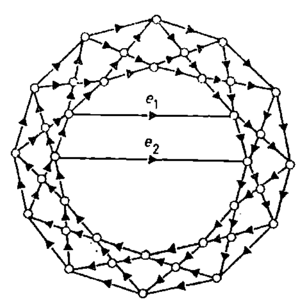

Fig. 2

We conclude with an arc version of Theorem 2.

Theorem 4. *A digraph D contains an arc meeting all dicycles of D if and only if D has no two dicycles with at most one vertex in common and no subdivision of the digraph of Figure 1 (d) and no subdivision of the directed multigraph obtained from a dicycle of length at least 2 by replacing each arc by two parallel arcs.*

Proof. The "only if" part is trivial so assume D has no arc meeting all dicycles. If D has no vertex meeting all dicycles we apply Theorem 2. On the other hand, if $D-x$ is acyclic, then Menger's theorem implies two arc-disjoint dicycles C_1, C_2 each containing x and now $C_1 \cup C_2$ is one of the configurations described in the theorem. ∎

Theorem 4 implies in particular that any planar digraph has either two arc-disjoint dicycles or an arc meeting all dicycles. This also follows as a special case of a theorem of Lucchesi and Younger [10].

References

[1] E. ALLENDER, On the number of cycles possible in digraphs with large girth, *Discrete Appl. Math.*, **10** (1985), 211—225.
[2] J.-C. BERMOND and C. THOMASSEN, Cycles in digraphs—a survey, *J. Graph Theory*, **5** (1981), 1—43.
[3] G. A. DIRAC, Some results concerning the structure of graphs, *Canad. Math. Bull.*, **6** (1963) 183—210.
[4] Z. ÉSIK, On cycles of directed and undirected graphs, *to appear.*
[5] S. FORTUNE, J. HOPCROFT and J. WYLLIE, The directed subgraph homeomorphism problem, *J. Theoret. Comput. Sci.*, **10** (1980), 111—121.
[6] T. GALLAI, Problem 6 *in: Theory of Graphs*, Proc. Colloq. Tihany 1966, Academic Press, New York (1968), 362.
[7] S. R. KOSARAJU, On independent circuits of a digraph. *J. Graph Theory*, **1** (1977) 379—382.
[8] A. V. KOSTOCHKA, A problem on directed graphs (in Russian with English summary), *Acta Cybernet.*, **6** (1983), 89—91.
[9] L. LOVÁSZ, On graphs not containing independent circuits (in Hungarian), *Mat. Lapok*, **16** (1965), 289—299.
[10] C. L. LUCCHESI and D. H. YOUNGER, A minimax theorem for directed graphs, *J. Lond. Math. Soc.*, **17** (1978), 369—374.
[11] C. THOMASSEN, The 2-linkage problem for acyclic digraphs, *Discrete Math.*, **55** (1985), 73—87.
[12] C. THOMASSEN, Paths, circuits and subdivisions, *in: Selected Topics in Graph Theory III* (L. W. Beineke and R. J. Wilson eds.) Academic Press, *to appear.*

C. Thomassen

Math. Inst.
The Technical University of Denmark
DK—2800 Lyngby
Denmark