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THRESHOLD FUNCTIONS

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It is shown that every non-trivial monotone increasing property of subsets of a set has a threshold function. This generalises a number of classical results in the theory of random graphs.

Let $X=X_n=\{1, 2, ..., n\}$ and $\mathcal{P}=\mathcal{P}(X)$. A *property* $Q=Q_n$ of the subsets of X is identified with the set of subsets of X having Q, i.e. we consider Q as a subset of $\mathcal{P}$. We call Q *monotone increasing* or simply *monotone* if $A\in Q$ and $A\subset B\subset X$ imply $B\in Q$. Similarly, Q is *monotone decreasing* or an *ideal* if $A\in Q$ and $B\subset A$ imply $B\in Q$. Note that Q is monotone increasing iff $Q'=\neg Q$ is monotone decreasing. If $\emptyset\neq Q\neq\mathcal{P}$ then Q is a *non-trivial* property. Thus a monotone increasing property Q is non-trivial iff $\emptyset\notin Q$ and $X\in Q$.

For $0\leqq k\leqq n$ let $X^{(k)}$ be the set of k-subsets of X and for $Q\subset\mathcal{P}$ define $Q_k=Q\cap X^{(k)}$. The *probability that a random k-set of X has Q* is defined to be $P_k(Q)=|Q_k|/|X^{(k)}|=|Q_k|\Big/\binom{n}{k}$.

If $k=k(n)$ is such that $\lim_{n\to\infty} P_k(Q)=1$ then we say that *almost every* (a.e.) k-subset of X has Q or that a k-subset of X *has Q almost surely* (a.s). Similarly, we say that *almost no k*-subset of X has Q or that a k-subset of X *fails to have Q almost surely* if $\lim_{n\to\infty} P_k(Q)=0$. If Q is a non-trivial monotone increasing property then $P_0(Q)=0$, $P_n(Q)=1$ and $P_k(Q)$ is a monotone increasing function of k. A function $m^*=m^*(n)$ is said to be a *threshold function* for a monotone increasing property Q if for $m/m^*\to 0$ almost no m-subset has Q and for $m/m^*\to\infty$ almost every m-subset has Q.

Erdős and Rényi were the first to prove that many a graph property has a threshold function in the sense above, if we identify a graph with the set of its edges. (For many examples of threshold functions of graphs and for an extensive account of the theory of random graphs, see [1].) Our main aim in this note is to point out that, in fact, every non-trivial monotone increasing property has a threshold function.

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Given natural numbers k and m, there are unique natural numbers $n_0>n_1>\dots$ $\dots>n_l$, $n_i\geqq k-i$, $l\leqq k-1$, such that

$$m=\sum_{i=0}^{l}\binom{n_i}{k-i}.$$

Define

$$f_k(m)=\sum_{i=0}^{l}\binom{n_i}{k-1-i}.$$

It is clear that we obtain the same function f_k if instead of requiring $n_i>n_{i+1}$ for every i, we relax this inequality to $n_i\geqq n_{i+1}$ for one value of i. Our results are based on the following theorem of Kruskal [3], discovered independently by Katona [2].

Theorem 1. *Let X be a finite set and let $Q\subset\mathcal{P}(X)$ be an ideal. Then $|Q_{k-1}|\geqq$ $\geqq f_k(|Q_k|)$.* ▮

Let $A_n=\binom{n}{k}$ and $B_n=\binom{n}{k-1}$. Then $f_k(A_n)=B_n$. In fact, f_k is at least as large as the piecewise linear function determined by the points (A_n, B_n):

$$f_k(A_{n-1}+x)\geqq B_{n-1}+\frac{B_n-B_{n-1}}{A_n-A_{n-1}}x=B_{n-1}+\frac{k-1}{n-k+1}x \tag{1}$$

for all x, $0\leqq x\leqq A_n-A_{n-1}=B_{n-1}$, provided $n\geqq k+1$.

To prove (1), we have to show that

$$\sum_{i=1}^{l}\binom{n_i}{k-1-i}\geqq\frac{k-1}{n-k+1}\sum_{i=1}^{l}\binom{n_i}{k-i} \tag{2}$$

for $n-1>n_1>n_2>\dots>n_l$. The difference of the terms in (2) depending on n_l is

$$\binom{n_l}{k-1-l}\left\{1-\frac{(n_l+l-k+1)(k-1)}{(n-k+1)(k-l)}\right\}.$$

If this is positive then, in proving (2), we may as well replace n_l by 0. If this is negative then it suffices to prove (2) in the case when n_l is as large as possible. In fact, we may take $n_l=n_{l-1}$, enabling us to omit n_l and replace n_{l-1} by $n_{l-1}+1$; if $n_{l-1}+1=n_{l-2}$ then we may omit $n_{l-1}+1$ as well and replace n_{l-2} by $n_{l-2}+1$, etc. In other words, there are natural numbers $n_1'>n_2'>\dots>n_t'$, $t<l$, such that

$$\sum_{i=1}^{l}\binom{n_i}{k-j-i}=\sum_{i=1}^{t}\binom{n_i'}{k-j-i}.$$

In the sequence $(n_i')_1^t$ either $n_1'<n-1$ or $t=1$. By repeating this reduction we see that (2) holds if it holds for $l=1$ and $n_1=n-1$. As in that case (2) is an equality, the proof of (1) is complete.

Theorem 2. *Let Q be a monotone decreasing property of subsets of a set X. Then for $0\leqq j<k\leqq n=|X|$ we have $P_j(Q)^k\geqq P_k(Q)^j$.*

Proof. In proving the assertion we may and shall assume that $j=k-1$. Set $M_i=|Q_i|$ and $m_i=P_i(Q)=M_i\Big/\binom{n}{i}$. By Theorem 1, $M_{k-1}\geqq f_k(M_k)$, so it suffices to show that

$$f_k(M_k)\Big/\binom{n}{k-1} \geqq \left\{M_k\Big/\binom{n}{k}\right\}^{(k-1)/k}. \tag{3}$$

Since

$$\binom{n}{k-1}\binom{n}{k}^{(1-k)/k}$$

is a monotone decreasing function of n, we may assume that n is as small as possible, i.e., with the previous notation, $A_{n-1}<M_k=A_{n-1}+x\leqq A_n$. But then, by (1),

$$f_k(M_k)\Big/\binom{n}{k-1} = f_k(A_{n-1}+x)/B_n \geqq B_{n-1}/B_n+(k-1)x/\{(n-k+1)B_n\} \geqq$$

$$\geqq \big((A_{n-1}+x)/A_n\big)^{(k-1)/k} = \left\{M_k\Big/\binom{n}{k}\right\}^{(k-1)/k}.$$

Here the second inequality holds because it holds at $x=A_n-A_{n-1}=B_{n-1}$ and because the derivative of $\big((A_{n-1}+x)/A_n\big)^{(k-1)/k}$ at $x=B_{n-1}$ is precisely

$$(k-1)/\{k\cdot A_n\} = (k-1)/\{(n-k+1)B_n\}. \blacksquare$$

Corollary 3. *Let Q be a property of subsets of a set of order n and let $k_1<k<k_2$. If Q is monotone decreasing then*

$$P_{k_2}(Q)^{k/k_2} \leqq P_k(Q) \leqq P_{k_1}(Q)^{k/k_1}$$

and if Q is monotone increasing then

$$P_{k_1}(Q)^{(n-k)/(n-k_1)} \leqq P_k(Q) \leqq P_{k_2}(Q)^{(n-k)/(n-k_2)}.$$

Proof. To deduce the second relation, note that if Q is monotone increasing then $Q^*=\{A\in\mathscr{P}(X)\colon X\setminus A\in Q\}$ is monotone decreasing and $P_k(Q)=P_{n-k}(Q^*)$. $\blacksquare$

Our main result is an easy consequence of Corollary 3.

Theorem 4. *Let Q be a monotone increasing non-trivial property of subsets of a set X, $|X|=n$. Let $m^*(n)=\max\{l\colon P_l(Q)\leqq 1/2\}$ and $\omega(n)\geqq 1$. If $m\leqq m^*/\omega(n)$ then*

$$P_m(Q) \leqq 1-2^{-1/\omega}$$

and if $m\geqq\omega(n)\cdot(m^*+1)$ then

$$P_m(Q) \geqq 1-2^{-\omega}.$$

In particular, m^* is a threshold function of Q.

Proof. If $m\leqq m^*/\omega$ then

$$P_m(\neg Q) \geqq P_{m^*}(\neg Q)^{1/\omega} \geqq 2^{-1/\omega}$$

and if $m\geqq\omega(n)\cdot(m^*+1)$ then

$$P_m(\neg Q) \leqq P_{m^*+1}(\neg Q)^{\omega} \leqq 2^{-\omega}. \blacksquare$$

If $\underline{\lim}\, m^*(n)/n > 0$ then the assertion 'for $m/m^* \to \infty$ a.e. m-set has Q' is vacuous. In fact, in this case we can do better: the second relation in Corollary 3 implies that if $(n-m)/(n-m^*) \to 0$ then $P_m(Q) \to 1$. In particular, if $0 < \underline{\lim}\, m^*(n)/n \leqq \overline{\lim}\, m^*(n)/n < 1$ then $n/2$, say, is a threshold function in the following sense: if $m = o(n)$ then almost no m-set has Q and if $m = n - o(n)$ then almost every m-set has Q.

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