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A TOPOLOGICAL APPROACH TO EVASIVENESS

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The complexity of a digraph property is the number of entries of the vertex adjacency matrix of a digraph which must be examined in worst case to determine whether the graph has the property. Rivest and Vuillemin proved the result (conjectured by Aanderaa and Rosenberg) that every graph property that is monotone (preserved by addition of edges) and nontrivial (holds for some but not all graphs) has complexity $\Omega(v^2)$ where v is the number of vertices. Karp conjectured that every such property is evasive, i.e., requires that every entry of the incidence matrix be examined. In this paper the truth of Karp's conjecture is shown to follow from another conjecture concerning group actions on topological spaces. A special case of the conjecture is proved which is applied to prove Karp's conjecture for the case of properties of graphs on a prime power number of vertices.

1. Introduction

Suppose we are given a digraph D on v vertices via an oracle which answers questions of the form "is (x, y) an edge of D?" (We may think of this oracle as the adjacency matrix of D.) Our objective is to determine whether D has a given (isomorphism invariant) property P while minimizing the worst case number of queries to the oracle. This minimum is called the *complexity* of P, denoted $c(P)$. When $c(P)$ is as large as possible, i.e., equal to $v(v-1)$, P is said to be *evasive*. Analogous definitions can be made for (isomorphism-invariant) properties of undirected graphs—henceforth *graph properties*—and in particular such a property is evasive if its complexity is $\binom{v}{2}$.

In 1973 S. Aanderaa and R. L. Rosenberg [14] proposed the

Aanderaa—Rosenberg conjecture. *There is a constant $\varepsilon>0$ such that any nontrivial monotone digraph property on v vertices has complexity at least εv^2.*

A *nontrivial* property is one which holds for some, but not all digraphs. A *monotone* property is one which is not destroyed by addition of edges. (Rosenberg's

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original version of the conjecture, which Aanderaa disproved, did not assume monotonicity.)

The Aanderaa—Rosenberg conjecture was proved by Rivest and Vuillemin [12]. Their value of $\varepsilon=1/16$ was subsequently improved by Kleitman and Kwiatkowski [8] to $\varepsilon=1/9$. (Note: These papers and most later work deal with graph, rather than digraph, properties. In fact, the digraph results follow from the (not entirely trivial) observation that if $m(v)$ and $M(v)$ are respectively the minimum complexities of monotone graph and digraph properties on v vertices, then $m(v)\leqq M(v)\leqq 2m(v)$.)

As we shall see, the main results of this paper easily imply the A—R conjecture "asymptotically" for any $\varepsilon<1/4$; precisely:

$$m(v) > v^2/4+o(v^2).$$

But such improvements are really of secondary interest, since, remarkably, no counterexample is known to the following conjecture, which Rosenberg attributes to Karp.

Conjecture 2. *Every nontrivial monotone digraph property is evasive.*

This "Karp conjecture" was the starting point for our investigations. It (or strictly speaking the analogue for undirected graphs) is known to hold for a number of specific properties (see [1], [2], [7], [10], and also [3, chap. 8] for a survey of results in this area). Here we will prove, inter alia,

Theorem 1. *If v is a prime power then every nontrivial monotone graph or digraph property on v vertices is evasive.*

The notion of evasiveness generalizes naturally as follows. We are given a a Boolean function f of variables $t_1, \ldots, t_n$ and wish to evaluate f (for fixed but unknown values of the t_i) by asking questions of the form "what is the value of t_i?" Alternately, we may associate with each vector $(t_1, \ldots, t_n)$ the subset $\{x_i: t_i=1\}$ of $X=\{x_1, \ldots, x_n\}$, and take F to be the collection of subsets of X whose associated vectors have f-value 1, to obtain the equivalent, but for us more convenient formulation: given a collection F of subsets of X and a fixed but unknown subset A of X, determine whether A is in F by asking questions of the form "is x_i an element of A?" As before, a function f or collection F is *evasive* if every questioning strategy requires n questions in worst case. The definitions of nontriviality and monotonicity are likewise extended in the obvious ways (for instance, F is monotone if $A\supseteq B\in F\Rightarrow A\in F$). We will usually refer to F as a *set property*, meaning that A has the property iff $A\in F$.

Of course we recover the original digraph problem by taking X to be the collection of ordered pairs of distinct elements of the set $V=\{1, \ldots, v\}$ (and equating a digraph D with its edge set). But observe that digraph properties are distinguished from arbitrary set properties by their symmetry: the group S_v acts transitively on the $v(v-1)$ elements of X, while preserving the set of digraphs having a given property. This suggests the following generalization of Conjecture 2.

Conjecture 3. *If F is any nontrivial monotone set property for which there exists a transitive group Γ of permutations of X preserving F, then F is evasive.*

(As usual, Γ acts on subsets of X by $\{x, y, \ldots, z\}^{\gamma}=\{x^{\gamma}, y^{\gamma}, \ldots, z^{\gamma}\}$, and is said to *preserve* F if $A\in F$, $\gamma\in\Gamma\Rightarrow A^{\gamma}\in F$.) This was shown by Rivest and Vuillemin [12] in case $|X|$ is a prime power, a fact which forms the basis for their proof of the Aanderaa—Rosenberg Conjecture. (In fact, they proposed a somewhat stronger version of Conjecture 3 in which monotonicity was replaced by the weaker condition: F contains exactly one of $\emptyset$, X; a counterexample to this was provided by Illies [6].)

The approach of the present paper is topological. We regard the collection of sets *not* belonging to F as an (abstract) simplicial complex, and after suitable preliminaries this allows us to invoke known results concerning finite group actions on topological spaces to prove Conjecture 3 under an additional hypothesis on Γ (see Theorem 2). Theorem 1 above is an easy consequence of this general result.

These developments are described in section 3, section 2 being devoted to a brief review of some topological background. Finally in section 4 we apply the ideas of section 3 to prove Conjecture 2 for undirected graphs when $v=6$. (The reader may wish to convince himself of the difficulty of a barehanded approach to even this seemingly small problem.)

2. Topological prerequisites

Recall that an (abstract simplicial) *complex* on a set X (always assumed to be finite) is just a collection Δ of subsets of X with the property that $A\subseteq B\in\Delta$ implies $A\in\Delta$. The members of Δ are called *faces*. The *dimension* of a face A is $|A|-1$. (Thus the empty set has dimension -1.) Faces of dimension 0 are called *vertices*. If f_i is the number of faces of Δ of dimension i, then the *Euler characteristic* of Δ is

$$\chi(\Delta) = \sum_{i\geqq 0}(-1)^i f_i.$$

It is often helpful to think of Δ in terms of its associated *geometric realization* $|\Delta|$: if $x=\{x_1, \ldots, x_n\}$, we think of each x_i as coinciding with the standard basis vector e_i in R^n, and take $|\Delta|$ to be the union of all the convex hulls $|A|=\operatorname{conv}(\{e_i\colon x_i\in A\})$ with A a face of Δ. (See e.g. [16], § 3—1.) The *automorphism group* of Δ, denoted $\operatorname{Aut}(\Delta)$, is the collection of permutations of X which preserve Δ. $\operatorname{Aut}(\Delta)$ also acts on $|\Delta|$ as a group of (piecewise linear) homeomorphisms, namely by regarding $\gamma\in\operatorname{Aut}(\Delta)$ as the map which for each $A\in\Delta$ sends the convex combination $\sum_{x_i\in A}\lambda_i e_i$ to $\sum_{x_i\in A}\lambda_i e_{\gamma(i)}$, where we write $x_{\gamma(i)}=x_i^{\gamma}$.

With each $x\in X$ there are associated two complexes on $X\setminus\{x\}$ which arise naturally in the context of evasiveness. These are the *link* and *contrastar* of x, given respectively by

$$\mathrm{LINK}(x) = \mathrm{LINK}(x, \Delta) = \{A\subseteq X\setminus\{x\}\colon A\cup\{x\}\in\Delta\}$$

and

$$\mathrm{COST}(x) = \mathrm{COST}(x, \Delta) = \{A\subseteq X\setminus\{x\}\colon A\in\Delta\}.$$

A *free face* of Δ is a non-maximal face which is contained in a unique maximal face. An *elementary* (simplicial) *collapse* of Δ is the process of removing from Δ some free face A together with all faces containing A. (This definition makes more sense when thought of in terms of $|\Delta|$.) We say that Δ *collapses* to a complex Δ' if Δ' can be obtained from Δ by a sequence of elementary collapses, and that Δ is *collapsible* if it collapses to the empty complex. For further information on collapsibility, see [4].

We do not define here the more familiar notions of homology and contractibility. (The standard text is [16], but see [9] or [17] for a less forbidding introduction to the subject. For the moment, the untopologized reader should still be able to follow our line of reasoning (taking certain results on faith), but of course the picture will be rather incomplete.) For an abelian group G (here either Z or Z_p) we will say Δ is *G-acyclic* if the (nonreduced) homology groups of Δ are:

$$H_0(\Delta, G) = G,$$
$$H_i(\Delta, G) = 0 \qquad i > 0.$$

It is easy to see ([4], p. 49) that if a nonempty complex Δ is collapsible, then it is also contractible, and we recall that we always have the additional implications

$$\text{contractible} \Rightarrow Z\text{-acyclic} \Rightarrow Z_p\text{-acyclic}.$$

Finally we define a complex Δ^*, called the *dual* of Δ, by

$$\Delta^* = \{A \subseteq X\colon X \setminus A \notin \Delta\},$$

and remark that the dual of Δ^* is Δ.

3. Proofs

For a family F of subsets of X it is clear that the problem of testing membership in F (as in the introduction) is equivalent to testing membership in the complementary family Δ of subsets of X not belonging to F. This trivial modification is in fact quite helpful; for saying that F is monotone is the same as saying that Δ is a simplicial complex, so that shifting our attention to Δ allows us to think topologically. So we are now faced with the problem: given a complex Δ on the set X and an (unknown) subset A of X, determine whether A is a face of Δ by asking questions of the form "is x in A?" As usual we say Δ is evasive if there is no strategy which always decides membership in Δ in fewer than n questions, and trivial if it is empty or a simplex (i.e. consists of *all* subsets of X). In this setting we find it convenient to reformulate Conjecture 3 as

Conjecture 3'. *If Δ is a nonempty, nonevasive simplicial complex on X, and* Aut (Δ) *is transitive on X, then Δ is a simplex.*

The topological connection is provided by

Proposition 1. *A nonevasive complex is collapsible.*

Proof. If Δ is not trivial then there is some $x \in X$ for which "is x in A?" is a *good* first question, i.e. the first question in a strategy which always decides membership in Δ in fewer than n questions. Having asked this question we have two possibilities: if the answer is "$x \notin A$", then A belongs to Δ iff it belongs to COST(Δ); if the answer is "$x \in A$", then A belongs to Δ iff $A \setminus \{x\}$ belongs to LINK(Δ). In either case we have (by our choice of x) a strategy which decides membership in the new complex in fewer than $n-1$ questions. In other words, *if Δ is nonevasive and nontrivial, then there is some $x \in X$ for which both* COST(x) *and* LINK(x) *are nonevasive.*

The proof of Proposition 1 is now an easy induction on $|X|$. If X is trivial (notice this includes the basis step), then it is also collapsible ([4], p. 49). Otherwise we may by induction and the above remark choose $x \in X$ with both LINK(x) and COST(x) collapsible. Now if $A_1, \ldots, A_k$ is a sequence of free faces used to collapse LINK(x), then $A_1 \cup \{x\}, \ldots, A_k \cup \{x\})$ $(=\{x\})$ is a sequence of free faces which collapses Δ to COST(x), so the collapsibility of Δ follows from that of COST(x). ∎

Although we cannot give a specific example, the converse of this proposition is false because the dual (see section 2) of a nonevasive complex is again nonevasive, whereas the dual of a collapsible complex may fail to be collapsible. (The first of these assertions is an easy exercise. The second is seen as follows. It is known (see [4], p. 69) that a collapsible Δ may be collapsed to a noncollapsible Σ. On the other hand, it is an easy consequence of the definitions that if Δ collapses to Σ, then Σ^* collapses to Δ^* (* is a contravariant functor), so in our case either Σ^* is collapsible or Δ^* is not, and this proves the second assertion).

We do not make use of the full power of Proposition 1: as often happens we are unable to exploit directly either the collapsibility or consequent contractibility of Δ, and our conclusions are limited to what we can glean from

Corollary 1. *Any nonevasive complex is acyclic over the integers.* ∎

A direct proof of this can be given along the lines of Proposition 1; but in view of the similarity between nonevasiveness and collapsibility it seems worthwhile pointing out how they are related.

At this point one can hardly avoid posing

Conjectures 4, 5, 6. *If $\Delta \neq \emptyset$ is a collapsible resp. contractible resp. Z-acyclic complex on X and* Aut (Δ) *is transitive on X then Δ is a simplex.*

Surprisingly, nothing beyond what we say here appears to be known about these very natural questions. (Incidentally, the authors believe that Conjectures 5 and 6 are probably false.)

Let us take $\mathcal{G}$ to be the collection of all groups Γ in which there exists subgroups Γ_1, Γ_2 satisfying

(i) $\Gamma_1 \lhd \Gamma_2 \lhd \Gamma$ ('$\lhd$' means 'is a normal subgroup of'), and
(ii) Γ_1 is a p-group, Γ_2/Γ_1 is cyclic, and Γ/Γ_2 is a q-group, with p and q (not necessarily distinct) primes,

and $\mathcal{G}_0$ the collection of those Γ in $\mathcal{G}$ for which Γ_2 (as above) is equal to Γ. What we can show regarding Conjecture 3 and the variants suggested above is

Theorem 2. *Let Δ be a nonempty Z-acyclic complex on X and Γ a vertex-transitive subgroup of* Aut (Δ). *Then if Γ is a member of $\mathcal{G}$, Δ is a simplex.*

Of course this implies the corresponding results for contractible, collapsible and nonevasive Δ, so in particular we have

Corollary. *Conjectures 3—6 are true whenever Γ is a member of $\mathcal{G}$.*

Before proving Theorem 2, we observe that Theorem 1 is indeed a special case. For let P be a digraph property on the vertex set V, with $|V|=p^\alpha$ a prime power. Identifying V with $\mathrm{GF}(p^\alpha)$, we find that the "affine group"

$$\Gamma = \{x \to ax+b\colon\ a, b\in \mathrm{GF}(p^\alpha),\quad a\neq 0\}.$$

is doubly transitive on V, i.e. is transitive on our set E of ordered pairs of distinct elements of V. Moreover, the translation subgroup

$$\Gamma = \{x \to x+b\colon\ b\in \mathrm{GF}(p^\alpha)\}$$

is a normal p-subgroup of Γ, and $\Gamma/\Gamma_1 \cong \mathrm{GF}(p^\alpha)^x$ is cyclic; that is, $\Gamma\in\mathscr{G}$. Thus Theorem 2 contains Theorem 1. ▮

Now to prove Theorem 2, observe that we need only show that there is some nonempty $A\in\Delta$ satisfying $A^\gamma=A$ for all $\gamma\in\Gamma$; for the transitivity of Γ implies that such an A must be equal to X.

On the other hand, we will have $A^\gamma=A$ if (and only if) γ acting on $|\Delta|$ (section 2) fixes some $x\in|A|^0$ (the relative interior of $|A|$). So writing $|\Delta|_\Gamma$ for the space of fixed points of Γ, i.e.

$$|\Delta|_\Gamma = \{x\in|\Delta|\colon\ x^\gamma = x, \forall\gamma\in\Gamma\},$$

we see that Theorem 2 is true provided $|\Delta|_\Gamma\neq\emptyset$.

We pause here to describe an abstract version of $|\Delta|_\Gamma$ which will be of use especially in section 4. For Δ, Γ as above, define a complex Δ_Γ by:

(i) the vertices of Δ_Γ are the minimal nonempty Γ-invariant faces of Δ (or equivalently, those Γ-orbits on X which are faces of Δ).

(ii) If $A_1, \dots, A_k$ are vertices of Δ_Γ, then $\{A_1, \dots, A_k\}$ is a face of Δ if $A_1\cup\dots\cup A_k$ is a face of Δ.

Thus the faces of Δ_Γ are just the Γ-invariant faces of Δ, regarded in the natural way as subsets of the minimal nonempty Γ-invariant faces. The reader will easily see that if we identify each vertex A_i of Δ_Γ with the barycenter of $|A_i|$ in $|\Delta|$, then the geometric realization of $|\Delta_\Gamma|$, constructed as in section 2, is just $|\Delta|_\Gamma$.

We now return to complete the proof of Theorem 2, which we have seen requires only that $|\Delta|_\Gamma\neq\emptyset$. In fact, this is known to be true, being a consequence of a theory of fixed points of finite group actions on topological spaces whose foundation is the following result of P. A. Smith [15].

Theorem 3. *If Γ is a p-group acting on a Z_p-acyclic complex Δ, then $|\Delta|_\Gamma$ is again Z_p-acyclic.* ▮

As observed in [11], this leads quickly, via the Hopf-Lefschetz trace formula [16, p. 195], to

Theorem 4. *If the group Γ acts on the Z_p-acyclic complex Δ, and if Γ contains a normal p-subgroup Γ_1 such that Γ/Γ_1 is cyclic, then $\chi(|\Delta|_\Gamma)=1$.* ▮

It is this result which we shall find most useful in the arguments of section 4. Observe also that it provides sufficient power for the proof of Theorem 1 given above.

The extra refinement which yields Theorem 2 is contained in the following result of R. G. Oliver [11], which serves also to define the limits of the present approach.

Theorem 5. *If $k \in \mathcal{G}$ acts on the Z_p-acyclic complex Δ, then $|\Delta|_\Gamma \neq \emptyset$.* ▌

(Here the prime p is as in the definition of $\mathcal{G}$.)

We conclude this section with the asymptotic result claimed in the introduction.

Theorem 6. *Any nontrivial montone property of graphs (or digraphs) on v vertices has complexity at least $v^2/4 + o(v^2)$.*

Proof. We will prove this for (ordinary) graphs; the result for digraphs will then follow from the observation made earlier relating the complexities of graph and digraph properties.

The key result here is a lemma of Kleitman and Kwiatkowski [8] which states that $m(v) \geqq \min \{m(v-1), q(v-q)\}$ where q is the prime power nearest to $v/2$. Using this lemma and Theorem 1 we see that $m(v) \geqq q'(q+1-q')$ where q is the largest prime power less than v and q' is the prime power nearest to $(q+1)/2$. It is an easy consequence of the prime number theorem [cf. 5] that there is a real function $\delta(x) = o(x)$ with the property that for all x there is a prime between x and $x+\delta(x)$. We conclude that $m(v) \geqq v^2/4 + o(v^2)$. ▌

4. Graphs on six vertices

In this section we give some further illustration of the application of the ideas introduced above by sketching a proof of Conjecture 2 for undirected graphs in the case $v=6$. Incidentally, the truth of the Conjecture in this case was asserted in [12] but no proof was given there; indeed, we see no manageable approach which does not use the present machinery.

From now on we take Δ to be a nonevasive monotone decreasing graph property on 6 vertices, i.e. a nonevasive complex on the 15 two-element subsets of $\{1, \ldots, 6\}$ invariant under the natural action of S_6. If H is a graph we use "Δ contains H", "H belongs to Δ", etc. to mean that any (equivalently, some) graph on $\{1, \ldots, 6\}$ isomorphic to H is a face of Δ. We assume for a contradiction that H is neither empty nor a simplex. Observe that our situation at this point is self-dual: our assumptions on Δ are also true of Δ^*, and so anything we can say of one must also apply to the other.

Our proof proceeds by considering the actions of various subgroups Γ of S_6 and applying Theorems 4 and 5 to the complexes Δ_Γ. Verifications are for the most part left to the reader.

(1) *Perfect matchings belong to Δ.*

Proof. Let Γ be the stabilizer in S_6 of the set of pairs $M = \{\{12\}, \{34\}, \{56\}\}$ (e.g. $(14)(23)(56) \in \Gamma$). Then $\Gamma \in \mathcal{G}$ (we may take $\Gamma_1 \cong Z_2^3$ and $\Gamma/\Gamma_1 \cong S_3$), and has precisely two orbits, M and $\overline{M}$. At least one of these is in Δ, by Theorem 5; but if $\overline{M}$ is in Δ, then so is M, since it is (isomorphic to) a subgraph of $\overline{M}$. ▌

(2) *Exactly one of $2K_3$, K_{33} belongs to Δ.*

(Here $2K_3$ is the union of two disjoint triangles, and K_{33} is as usual the complete bipartite graph.)

Proof. Let $\Gamma=\langle(123), (456), (14)(25)(36)\rangle$. Then $\Gamma\in\mathcal{G}_0$ and the orbits of Γ are the copies of $2K_3$ and K_{33} shown in figure 1. So by Theorem 5, at least one of $2K_3$, K_{33} belongs to Δ. But if both did, then Theorem 4 would force their union to belong to Δ, and Δ would be a simplex. ▮

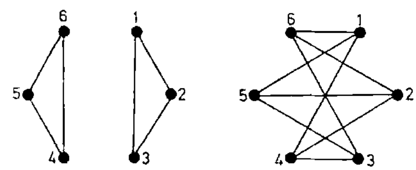

Fig. 1. The two orbits under the action of the group $\langle(123), (456), (14)(25)(36)\rangle$

Notice that (by the definition of Δ^*) whichever of the graphs in (2) belongs to Δ must also be the one belonging to Δ^*.

Suppose first that $2K_3\in\Delta$ (and Δ^*), and let $\Gamma=\langle(153624)\rangle$. Then $\Gamma\in\mathcal{G}_0$, and the orbits of Γ are (the edge sets of) the graphs shown in figure 2. We know that $A, B\in\Delta$, and it follows that $C\in\Delta$; since otherwise Theorem 4 forces $A\cup B\in\Delta$, and then C, being isomorphic to a subgraph of $A\cup B$, must also be in Δ. But now we have a contradiction: Everything so far asserted for Δ is also valid for Δ^*; so we have $A, B, C\in\Delta^*$ and so $\bar{A}=B\cup C$, $\bar{B}=A\cup C$ and $\bar{C}=A\cup B\notin\Delta$, which contradicts Theorem 4. ▮

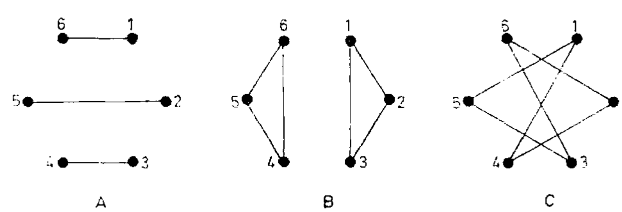

Fig. 2. The three orbits under the action of the group $\langle(153624)\rangle$

Now suppose that $K_{33}\in\Delta$ (and Δ^*). We first assert that

(3) *K_3 cannot belong to both of Δ, Δ^*.*

For let $\Gamma=\langle(123), (456)\rangle$. Then $\Gamma\in\mathcal{G}_0$ and the orbits of Γ are as shown in figure 3. If $K_3\in\Delta$, then we have $A, B, C\in\Delta$, so by Theorem 3 $A\cup B(\cong A\cup C)\in\Delta$. But then $\overline{A\cup B}=C$ cannot belong to Δ^*, and this proves (3). ▮

We may thus assume that $K_3\notin\Delta$. Now let $\Gamma=\langle(12), (3456), (35)\rangle$ (the direct product of $\langle(12)\rangle$ and a dihedral group on $\{3, 4, 5, 6\}$). Then $\Gamma\in\mathcal{G}_0$, and the orbits of Γ are as in figure 4.

(4) *The faces of Δ_Γ are* $\emptyset, A, B, C, A\cup B, A\cup C$.

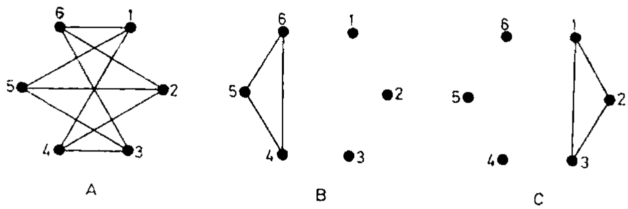

Fig. 3. The three orbits under the action of the group ⟨(123), (456)⟩

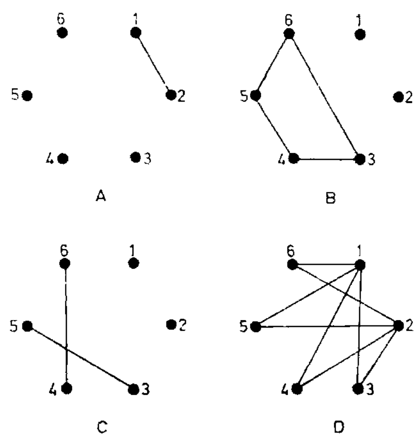

Fig. 4. The four orbits under the action of the group ⟨(12), (3456), (35)⟩

Proof. Each of the graphs listed is contained in K_{33}, so a face of Δ_Γ. On the other hand, $A \cup D$, $B \cup D$, $C \cup D$ and $B \cup C$ all contain K_3 so do not belong to Δ. Finally, Theorem 4 implies that $D \notin \Delta_\Gamma$. ▮

Now the vertex A of Δ is fixed by Γ, so that Γ acts on LINK (A, Δ). The fixed points of this action are given by

$$(\mathrm{LINK}\,(A, \Delta))_\Gamma = \mathrm{LINK}\,(A, \Delta_\Gamma),$$

the latter being a complex with two vertices and no edges. On the other hand, LINK (A, Δ) is nonevasive. (This is true of some vertex of Δ, as in the proof of Proposition 1 of section 3; but then it must be true of *every* vertex since Aut (Δ) is transitive.) So by Theorem 4, $(\mathrm{LINK}\,(A, \Delta))_\Gamma$ must have Euler characteristic 1, a contradiction which completes our proof. ▮

Note added in proof. R. G. Oliver (personal communication) has recently provided a counterexample to Conjectures 5 and 6 and a plausibility argument for the falsity of Conjecture 4.

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