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Two New Examples of Sets without Medians and Centers

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Abstract

Many problems in continuous location theory, reduce to finding a best location, in the sense that a facility must be located at a point minimizing the sum of distances to the points of a given finite set (median) or the largest distances to all points (center). The setting is often assumed to be a Banach space.
To have a better understanding concerning the structure of location problems, it is nice to see how, if the space is infinite-dimensional, the lack of optimal solutions may occur also in rather simple cases.
In this paper we indicate two simple examples of four-point sets such that one of the two problems indicated has a solution, while the other one has no solution. Also, we list papers containing examples previously given, dealing with this lack of optimal solutions.

Key Words: Location theory, Fermat point, Chebyshev center.

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1 Introduction

Let X be a normed real space. Given a set A, we set:

$$r(A,x) = sup\{\|x-a\|; a \in A\}; \quad r(A) = \inf_{x \in X} r(A,x).$$

We say that c is a *center* of A if $r(A,c) = r(A)$.

Let $A = \{a_1, a_2, \ldots, a_n\}$ be a finite set; we set

$$\mu(A,x) = \frac{1}{n}\sum_{i=1}^{n} \|x-a_i\|; \quad \mu(A) = \inf_{x \in X} \mu(A,x);$$

for a finite set, we say that m is a *median* of A if $\mu(A,m) = \mu(A)$.

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Many problems in continuous location theory reduce to finding a best location, in the sense that we must find a center or a median.

These two problems are often considered in two-dimensional spaces: in this case, from a theoretical point of view, things concerning existence of solutions go smoothly. Already for spaces of three or more dimensions, things behave in a different way: compactness helps in proving existence theorems, but when the setting is not Euclidean, solutions can be outside the convex hull of the set.

In general the setting is assumed to be a Banach space. If the underlying space has some "good" properties, but also in some "bad" spaces, it is known that both problems always admit (at least) one solution; but in general, if the space is infinite-dimensional, the two above problems do not necessarily have solutions.

Examples are known of sets where these problems have no solution. The general feeling on these optimization problems seems to be the following: at least for very simple sets, the above problems have solutions. Also, from the theoretical point of view, both problems are convex: it is rather unexpected to find simple examples where there are solutions for one of the two problems but not for the other one.

An example of a three-point set without a center was given already in Garkavi (1964), Theorem 1; we do not know if such a set has a median.

An example of a three-point set without a median was given in Veselý (1993), Remark; we do not know if such a set has a center.

An example of a three-point set without a median and without a center was given in Baronti et al. (1993), Example 5.2.

In this paper we want to indicate two examples of four-point sets: the first one, is a set having a center but no median. The second one, is a set having a median but no center.

This shows that having a median or a center are independent properties. As known, in many Banach spaces all closed bounded sets have centers, all finite sets have medians (see e.g. Veselý (1997) for general results of this type).

The following questions can be raised:

Q1 Are there spaces such that every finite set has a center but not every finite set has a median?

Q2 Are there spaces such that every finite set has a median but not every finite set has a center?

Recall that there are spaces such that every finite set (also, every compact set) has a center but there exist closed bounded and convex sets without a center (see Veselý (2002)). With respect to some properties, considering three-point sets is enough: for example, among normed spaces of dimension at least three, Hilbert spaces can be characterized by the property that for any three-point set, there exists a center which belongs to its convex hull; a similar characterization holds with existence of medians, see Benítez et al. (2002). Also, a characterization of reflexivity in terms of existence of centers and/or medians for three-point sets was given in Veselý (1993).

2 Examples

Our examples are given in the following space, already considered in Baronti et al. (1993). Let c_o be the space of all real sequences converging to 0, with the max norm; denote by e_i, $i \in N$, the elements of the natural basis. Consider the following functional on c_o: $f = (f_n)$ where $f_n = 1$ for $n = 1, 2, 3, 4$; $f_n = \frac{1}{2^{n-4}}$ for $n \geq 5$.

We denote by X the subspace of c_o containing all sequences $x = (x_1, x_2, \dots, x_n, \dots)$ such that $\sum_{n=1}^{\infty} f_n x_n = 0$.

Example 2.1. Let $A = \{a, b, c, d\}$ where $a = e_1 - e_4$; $b = e_2 - \frac{1}{4}e_4 - e_5$; $c = e_3 - \frac{1}{4}e_4 - e_5$; $d = e_2 + e_3 - e_4 - e_5 - \sum_{i=6}^{n+5} 2\alpha e_i$ with $\alpha = \frac{2^{n-1}}{2^n - 1}$ $(\in (\frac{1}{2}, 1))$.

Note that $A \subset X$; in particular $d \in X$ since $\frac{1}{2} - 2\alpha(\frac{1}{4} + \cdots + \frac{1}{2^{n+1}}) = \frac{1}{2} - \alpha(1 - \frac{1}{2^n}) = 0$.

We have: $\|c - d\| = 2\alpha$, so $r(A) \geq \alpha$; if we take:

$$x = \frac{a + d}{2} = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, -\frac{1}{2}, -\alpha, -\alpha, \dots, -\alpha, 0, 0, \dots\right)$$

we have: $\|x - a\| = \|x - b\| = \|x - c\| = \|x - d\| = \alpha$; therefore $r(A, x) = \alpha$, so $r(A) = \alpha$ and x is a center of A.

Now we want to prove that $\mu(A) = \frac{\alpha+1}{2}$ and that A has no median.

Let $m = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, -1, -\frac{1}{2}, -\frac{1}{2}, \ldots, -\frac{1}{2}, -\beta, -\beta, \ldots, -\beta, 0, 0, \ldots\right)$: $-\frac{1}{2}$ appears from the 5-th to the $(n+5)$-th position, and $-\beta$ in the next k positions with $k > n$.

We have: $m \in X$ if

$$\frac{1}{2} - \frac{1}{4} - \left(\frac{1}{2.2^2} + \frac{1}{2.2^3} + \cdots + \frac{1}{2}\frac{1}{2^{n+1}}\right) - \beta\left(\frac{1}{2^{n+2}} + \cdots + \frac{1}{2^{n+k+1}}\right) = 0$$

$$\Leftrightarrow \frac{1}{4} - \frac{1}{8}\left(1 + \cdots + \frac{1}{2^{n-1}}\right) = \beta\frac{1}{2^{n+2}}\left(1 + \cdots + \frac{1}{2^{k-1}}\right) = 0$$

$$\Leftrightarrow \frac{1}{4} - \frac{1}{8}\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right) = \frac{\beta}{2^{n+2}}\left(\frac{1-\frac{1}{2^k}}{1-\frac{1}{2}}\right) \Leftrightarrow \frac{1}{4\cdot 2^n} = \frac{\beta}{2^{n+1}}\left(1 - \frac{1}{2^k}\right)$$

$$\Leftrightarrow \beta = \frac{2^{k-1}}{2^k - 1}\left(\in\left(\frac{1}{2}, \alpha\right)\right).$$

Therefore: $\|m-a\| = \|m-b\| = \|m-c\| = \beta$; $\|m-d\| = 2\alpha - \frac{1}{2}$, so $4\mu(A,m) = 2\alpha - \frac{1}{2} + 3\beta$.

For $k \to \infty$, the value of β goes to $\frac{1}{2}$, so $\mu(A) = \inf_{x\in X}\mu(A,x) \leq \frac{2\alpha+1}{4}$; but also, for any $x \in X$ we have:

$$\mu(A,x) = \frac{1}{4}\Big(\|x-a\| + \|x-b\| + \|x-c\| + \|x-d\|\Big) \geq \frac{1}{4}\Big(\|a-b\| + \|c-d\|\Big) = \frac{2\alpha+1}{4},$$

so we conclude that $\mu(\mathrm{A}) = \frac{2\alpha+1}{4}$.

Now suppose that x is a median of A; clearly we must have $2\alpha + 1 = 4\mu(A,x) = \|x-a\| + \|x-b\| + \|x-c\| + \|x-d\| \geq \|a-b\| + \|c-d\| = 1 + 2\alpha$, so $\|x-a\| + \|x-b\| = 1$, and $\|x-c\| + \|x-d\| = 2\alpha$; also, since $\|a-c\| = 1$ and $\|b-d\| = 2\alpha, \|x-a\| + \|x-c\| = 1$ and $\|x-b\| + \|x-d\| = 2\alpha$; similarly, $\|b-c\| = 1$ and $\|a-d\| = 2\alpha$, imply $\|x-b\| + \|x-c\| = 1$ and $\|x-a\| + \|x-d\| = 2\alpha$. These inequalities together imply $\|x-a\| = \|x-b\| = \|x-c\| = \frac{1}{2}$; $\|x-d\| = 2\alpha - \frac{1}{2}$. But the first three conditions are not compatible since, as shown in Example 5.2 in Baronti et al. (1993), the set $A' = \{a,b,c\}$ has no median ($r(A') = \mu(A') = \frac{1}{2}$); so A has no median.

Example 2.2. Consider again the space X defined above. Let $B = \{a,b,c,e\}$ where a, b, c are the same points as in Example 2.1, while $e = \left(\frac{4}{5}, \frac{1}{5}, 0, -\frac{9}{10}, -\frac{1}{5}, 0, 0, \ldots\right)$.

If $x \in X$, then $\|x-a\|+\|x-b\|+\|x-c\|+\|x-e\| \geq \|a-b\|+\|c-e\| = 2$; also, $\|e-a\| + \|e-b\| + \|e-c\| = \frac{1}{5} + \frac{4}{5} + 1 = 2$: this shows that e is a median of B.

Now let $x = \frac{a+d}{2}$, with d the point in Example 2.1; we have: $sup\{\|x-a\|, \|x-b\|, \|x-c\|, \|x-e\|\} = \alpha = \frac{2^{n-1}}{2^n-1} \geq r(B)$; since we can take n as large as we wish, we obtain $r(B) \leq \frac{1}{2}$. Also, if $C = \{a,b,c\} \subset B$, then $1 = diameter(C) \leq 2r(C) \leq 2r(B) \leq 1$, so $r(C) = r(B) = \frac{1}{2}$. But it was shown in Baronti et al. (1993) that there exists no $y \in X$ such that $\|y-a\| \leq \frac{1}{2}$, $\|y-b\| \leq \frac{1}{2}$, $\|y-c\| \leq \frac{1}{2}$ (so C has no center); this implies that neither B can have a center.

Remark 2.1. In example 2.2, we have $\mu(B) = r(B) = \frac{1}{2}$: note that when a set satisfies a similar equality, centers (if they exist) are also medians.

Remark 2.2. Let A be a finite set and $co(A)$ its convex hull; consider the numbers:

$$r_s(A) = \inf_{x \in co(A)} r(A,x); \quad \mu_s(A) = \inf_{x \in co(A)} \mu(A,x).$$

Clearly, these numbers are also minima.

In Example 2.1, we have: $r_s(A) = r(A)$; also, it is not difficult to see that $\mu_s(A) = \alpha$ (which is achieved by $x = \frac{a+d}{2}$).

In Example 2.2, we have $\mu_s(B) = \mu(B)$; also, it is not difficult to see that $r_s(A) = \frac{2}{3}$ $\big($which is achieved by $x = \frac{1}{3}(a+b+c)\big)$.

Remark 2.3. Example 2.1 shows a four point set which has a center, but there are three-point subsets (like $\{a,b,c\}$) without a center; Example 2.2 shows a similar fact for medians.

It could be of some interest to indicate a four-point set without a median, such that all three-point subsets have a median; similarly for centers.

Also, it could be interesting to produce similar examples in strictly convex spaces. We note in passing that an example of an infinite set without a center, in a strictly convex space, was given in Amir et al. (1982) (3.2. Example).

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