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Discretization and Resolution of the $(r|X_p)$-Medianoid Problem Involving Quality Criteria

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Abstract

In this paper an extension of the $(r|X_p)$-medianoid on networks introduced by Hakimi (1983) is studied. In this extension the customer considers not only the distance but some characteristics of the facilities such as store size, quality of service and parking space. A new firm wants to establish r new facilities which have to compete with the p facilities that already exist in the market. The entry firm wants to find their locations and characteristics to maximize profits. Three different customer choice rules (binary, partially binary and proportional preferences) are considered. Some discretization results are obtained and a resolution procedure is proposed. The problem is solved combining a global search algorithm based on a branch and bound procedure with some combinatorial heuristics (greedy, interchange, and tabu search). Some computational experiences are presented.

Key Words: Competitive location, medianoid, attractiveness.

AMS subject classification: 90B85.

1 Introduction

The competitive location analysis was initialized by Hotelling in his work "Stability in Competition" (1929). Later many papers in this field have appeared where different competitive location models have been studied (Achabal, Gorr and Mahajan (1982), Drezner (1994a, 1994b), Drezner and Drezner (1996), Eiselt and Laporte (1989b), Friesz, Miller and Tobin (1988), Goodchild (1984), Hakimi (1983, 1990), Huff (1964), Peeters and Plastria (1998), ReVelle (1986)) and many reviews can be found in

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the literature (Craig, Ghosh and McLafferty (1984), Eiselt and Laporte (1989a), Eiselt, Laporte and Thisse (1993), Plastria (2001), Serra and ReVelle (1995)).

Many competitive location models assume that customers choose the facilities taking the distance as the unique criterion. Nevertheless, this assumption makes sense only if differences between facilities do not exist or transport is difficult. Usually customers consider the distance and certain attributes of the facilities such as size, quality of products and services, parking space, etc. In this case the amount of demand in a zone which is captured by a facility depends on the attraction that the facility exerts towards the customer at that zone. This attraction can be formulated as a function with two components, the first component depends on the distance between facilities and demand points and the second depends on its attributes.

In the model proposed by Huff (1964), the attraction felt by a customer at zone i towards a facility j at x_j is directly proportional to the size of the facility and inversely proportional to a power of the distance between zone i and x_j, that is

$$a_{ij} = \frac{a_j}{d_{ij}^{\beta}} \tag{1.1}$$

where a_j is the *attractiveness* or *quality* (size) of the facility j and β is a parameter which represents the effect of travel distance (or travel time), d_{ij}, on the behaviour of consumers and must be estimated empirically.

Models which use the previous attraction function can be found in Drezner (1994b), Eiselt and Laporte (1988a, 1988b, 1989b), Eiselt, Laporte and Pederzoli (1989), and Plastria (1997).

A more general formulation of the attraction function is given by

$$a_{ij} = \frac{a_j}{f\left(d_{ij}\right)}$$

where f is a non-decreasing function. This attraction function was used by Peeters and Plastria (1998) in the analysis of the Huff and Pareto-Huff models on networks, for which they proved some discretization results.

Nakanishi and Cooper (1974) introduced the Multiplicative Competitive

Interaction (MCI) model defining the attraction function

$$a_{ij} = \prod_{k=1}^{s} x_{ijk}^{\beta_k}, \tag{1.2}$$

where x_{ijk} is the kth attribute describing a facility j by customers at i, and β_k is the weight of the kth attribute. Function (1.1) is a particular case of (1.2) when facility size and distance are the only attributes utilized. They estimated the parameters β_k by means of the ordinary least square method on the log-transformed centered form of the equation. This multiplicative function has been employed by multiple authors, including Achabal, Gorr and Mahajan (1982), Ghosh and McLafferty (1982) and Ghosh and Craig (1983). Colomé (2002) applied this methodology to solve a location-attractiveness problem in the supermarket sector. She estimated the supermarket key attributes through a factor analysis applied to the survey database and used the factors found in this analysis as variables to specify the MCI model.

Drezner (1994a) utilized an additive utility function of which the general expression is

$$U = \sum_{k=1}^{s} \beta_k f_k(x_k)$$

where s attributes x_k, $k = 1, 2, \ldots, s$ are considered, each with an associated weight β_k. In this problem, customers patronize the facility with the highest utility (binary model) and the best location is found using the break-even distance concept. At this distance the utilities of the existing and new facilities are equal. In Drezner and Drezner (1996), a linear utility function is used to define a stochastic location model where the utility function is assumed to be normally distributed.

Finally, some location models (Hodgson (1981)) incorporate an exponential attraction function given by

$$a_{ij} = a_j^{\alpha} e^{-\beta d_{ij}},$$

where a_j represents the quality (usually the facility size) of the facility and d_{ij} is the distance or travel time between demand point i and facility j. The parameters α and β are determined empirically. Comparing this with Huff's model, here the exponential formulation accelerates the distance decay.

Most of papers cited previously only deal with either the location problem or the attractiveness problem. To the authors' knowledge the problem where both location and attractiveness are decision variables has only been analysed on the plane (i.e. Plastria (1997), and Plastria and Carrizosa (2003)) and in the discrete problem (i.e. Achabal, Gorr and Mahajan (1982) or Eiselt and Laporte (1989b)), but not on networks. In this paper, the $(r|X_p)$-medianoid problem on networks is studied. The attractiveness of the facilities is incorporated into the model by using the attraction function $a_{ij} = \frac{a_j}{f_i(d_{ij})}$ where f_i is a positive, concave and non-decreasing function of distance. Three different customer choice rules (binary, partially binary and proportional preferences) are analysed. The entry firm wants to find the locations and the attractiveness of the new facilities in order to maximize its profits. For each customer choice rule, if the attractiveness of each new facility is given and its value is equal to 1, then the model studied by Hakimi (1990) is obtained.

The rest of the paper is organized as follows. In section 2 the model and the different scenarios are defined. In sections 3, 4, and 5 some discretization results are proved. In section 6 the branch and bound based heuristic algorithm used in obtaining an ε-optimal solution to the problem of finding the attractiveness levels given the location is analysed. The computational results for the resolution of the different problems are presented in section 7. Finally, section 8 includes some remarks and possible extensions.

2 The model

Let $N(V, E)$ be a weighted network with node set $V = \{v_i\}_{i=1}^n$ and edge set E, where each node v has associated a weight $w(v)$ (≥ 0), and each edge $e \in E$ has associated a length $l(e)$ (≥ 0). The network $N(V, E)$ represents a market where the demand (or buying power) at node v is $w(v)$, and $l(e)$ is the unitary transportation cost along the edge e. A segment $[x_1, x_2]$ of the edge $[v_i, v_j]$ is the subset of points of $[v_i, v_j]$ between x_1 and x_2 including x_1 and x_2. The open segment $]x_1, x_2[$ is $[x_1, x_2] - \{x_1, x_2\}$, and $]x_1, x_2] = [x_1, x_2] - \{x_1\}$ and $[x_1, x_2[= [x_1, x_2] - \{x_2\}$. Each facility j is characterized by its location x_j and its attractiveness a_j, and the attraction felt by customers at node v towards facility j at x_j is given by

$$a_{vj} = \frac{a_j}{f_v(d(v, x_j))}$$

where f_v is a non-decreasing and positive function, $d(v, x_j)$ is the distance between node v and point x_j, and $a_j \in A \subseteq \Re^+$, with A being an interval.

In the rest of the paper, the following conditions are assumed.

Assumption 2.1. Functions $f_v : \Re_0^+ \to \Re^+$ are concave and non-decreasing, $\forall v \in V$.

Assumption 2.2. For each facility (new or existing) its attractiveness is on $[I, S]$, where $0 < I < S$.

Assumption 2.3. There exists a cost function of the attractiveness, F, which is a positive, continuous and non-decreasing function of a non-negative real variable.

Given $Y_r = (y_1, y_2, \ldots, y_r)$ and $X_p = (x_1, x_2, \ldots, x_p)$, the locations of the facilities belonging to the entry firm, F_Y, and the existing one, F_X, respectively, with attractiveness levels $A_r = (a_1, a_2, \ldots, a_r)$ and $A_p = (b_1, b_2, \ldots, b_p)$, where a_i is the attractiveness of the facility at y_i and b_j is the attractiveness of the facility at x_j, the profit function for the entry firm is formulated as

$$W(Y_r, A_r|X_p, A_p) = \sum_{v \in V} w_Y(v) - \sum_{j=1}^{r} F(a_j),$$

with $w_Y(v)$ the demand at node v which is captured by F_Y.

The pair $(Y_r^*, A_r^*) \in N^r \times [I, S]^r$ is an $(r|X_p, A_p)$-medianoid if it verifies that

$$W(Y_r^*, A_r^*|X_p, A_p) = \max_{Y_r \in N^r,\ A_r \in [I,S]^r} W(Y_r, A_r|X_p, A_p).$$

In this paper, essential demand and three different customer choice rules are considered, binary, partially binary, and proportional preferences. Binary preferences are assumed when each consumer selects the most attractive facility with respect to their own individual preferences. In partially binary preferences, customers' demand is distributed among the most attractive facilities of each competing firm. Finally, in proportional preferences, customers' demand is shared among all the facilities in the market.

3 Binary preferences

For each $X = (x_1, \ldots, x_k) \in N^k$ with attractiveness levels $A_X = (a_1, \ldots, a_k) \in [I, S]^k$, let $G(v, X, A_X) = \max\{\frac{a_i}{f_v(d(v,x_i))},\ i = 1, 2, \ldots, k\}$, and let $G_v = G(v, X_p, A_p)$. In binary preferences, the demand of a node v is captured by the most attractive facility. Then, given X_p and A_p, the set of nodes captured by F_Y when its new facilities are located at Y_r with attractiveness levels A_r, is

$$V(Y_r, A_r|X_p, A_p) = \{v \in V \ : \ G(v, Y_r, A_r) > G_v\}.$$

Therefore, the profit function for F_Y is

$$W(Y_r, A_r|X_p, A_p) = \sum_{v \in V(Y_r, A_r|X_p, A_p)} w(v) - \sum_{j=1}^{r} F(a_j).$$

For simplicity, the following notation is used:

$$V(Y_r, A_r) = V(Y_r, A_r|X_p, A_p) \text{ and } W(Y_r, A_r) = W(Y_r, A_r|X_p, A_p).$$

The binary model has been defined in such a way that if attraction perceived from firm F_X and firm F_Y are equal, demand at v is captured by an existing facility, which means that firm F_X has the advantage in case of equal attraction.

The revenue function is a discontinuous piecewise linear function with respect to the attractiveness level. This condition can be used to prove the existence of an ε-optimal solution to the problem of obtaining the attractiveness levels given Y_r.

Proposition 3.1. *Given X_p, A_p, and Y_r, there exists an ε-optimal solution to the problem of obtaining the optimal attractiveness levels of Y_r for the binary $(r|X_p, A_p)$-medianoid problem.*

Proof. For each location y_j, $j = 1, 2, \ldots, r$, let $P_j = \{a_v^j = G_v f_v(d(v, y_j)) : I \leq a_v^j < S, v \in V\}$, and reindex the points in P_j by increasing values, $I \leq a_{v_1}^j \leq a_{v_2}^j \leq \cdots \leq a_{v_l}^j < S$. The set of nodes captured by Y_r on $(a_{v_k}^j, a_{v_{k+1}}^j)$ is constant, and the cost function F is non-decreasing. Therefore, the profits are higher for values closer to $a_{v_k}^j$. Something similar happens on

$(I, a^j_{v_1})$ and $(a^j_{v_l}, S)$. Note that if $I < a^j_{v_1}$, the maximum on $[I, a^j_{v_1})$ occurs at I. Consequently, an ε-optimal attractiveness levels a^*_j can be obtained investigating the sets $\{I\} \cup \{a + \delta : a \in P_j\}$, $j = 1, 2, .., r$, with $\delta > 0$ sufficiently small. □

From proposition 3.1 it follows that an ε-optimal solution A^*_r given Y_r, can be obtained evaluating at most $\Pi^r_{j=1}(1 + |P_j|) \leq (1 + |V|)^r$ r-tuples. Henceforth a binary $(r|X_p, A_p)$-medianoid will be understood in terms of an ε-optimal solution for the attractiveness levels.

Now consider the problem of determining the locations for the new facilities, given their attractiveness levels. In this case, firm F_Y captures the demand of v if a facility at y_j with attractiveness a_j exists, such that

$$f_v\left(d\left(v, y_j\right)\right) < \frac{a_j}{G_v}.$$

If f_v is increasing, the previous inequality can be expressed as

$$r^j_v = d(v, y_j) < f^{-1}_v\left(\frac{a_j}{G_v}\right).$$

If a new facility with attractiveness a_j exists within a distance r^j_v from v, then firm F_Y captures the demand of v.

Definition 3.1. Given X_p, A_p, and $a \in [I, S]$, a point $x \in N(V, E)$ is a (v, a)-isoattractive point if

$$f_v(d(v, x)) = \frac{a}{G_v}.$$

Let the following sets,

$$\begin{aligned}
&ISOA(v, a) = \{x \in N(V, E) : x \text{ is a } (v, a)\text{-isoattractive point}\},\\
&ISOA(a) = \textstyle\bigcup_{v \in V} ISOA(v, a),\\
&ISOA_{ij}(a) = ISOA(a) \cap]v_i, v_j[, \text{ with } [v_i, v_j] \in E,\\
&ISOA(A_r) = \bigcup_{j=1}^{r} ISOA\left(a_j\right), \text{ where } A_r = (a_1, \ldots, a_r).
\end{aligned}$$

Each element in $ISOA(A_r)$ is called an isoattractive point. The concept of isoattractive point in the location-attractiveness problem has a sense similar to the notion of isodistant point defined by Peeters and Plastria (1998).

Proposition 3.2. *Given X_p, A_p, and A_r, the set $V((y_1, \ldots, y_j, \ldots, y_r), A_r)$ is constant when y_j varies on the open segment $]s,t[$, with*

$$]s,t[\cap ISOA(a_j) = \emptyset \text{ and } \{s,t\} \subset V \cup ISOA(a_j).$$

Proof. Let $Y_r = (y_1, y_2, \ldots, y_r)$. Suppose, without loss of generality, that y_1 is the location that varies on $]s,t[$. If $V(Y_r, A_r)$ is not constant when $y_1 \in]s,t[$, then there exist y_1', $y_2'' \in]s,t[$, such that $V((y_1', y_2, \ldots, y_r), A_r) \neq V((y_1'', y_2, \ldots, y_r), A_r)$. Therefore, there exists $v_0 \in V$ such that $v_0 \in V((y_1', y_2, \ldots, y_r), A_r)$ and $v_0 \notin V((y_1'', y_2, \ldots, y_r), A_r)$ (or vice versa). Then

$$G(v_0, (y_1', y_2, \ldots, y_r), A_r) > G_{v_0} \geq G(v_0, (y_1'', y_2, \ldots, y_r), A_r),$$

from which it follows that $G_{v_0} \geq G(v_0, (y_2, \ldots, y_r), (a_2, \ldots, a_r))$ and therefore, as $y_1 \notin ISOA(a_1)$,

$$\frac{a_1}{f_{v_0}(d(v_0, y_1'))} > G_{v_0} > \frac{a_1}{f_{v_0}(d(v_0, y_1''))}.$$

As the function $f_{v_0}(d(v_0, y))$ is continuous when $y \in]s,t[$, it follows that there exists $z \in (y_1', y_2'')$ such that

$$\frac{a_1}{f_{v_0}(d(v_0, z))} = G_{v_0}.$$

But this is not possible because isoattractive points do not exist on $]s,t[$. Therefore, $V(Y_r, A_r)$ is constant on $]s,t[$. □

For each $a_k \in A_r$ and each edge $[v_i, v_j]$, let $ISOA_{ij}(a_k) = \{x_{ijk}^1, x_{ijk}^2, \ldots, x_{ijk}^{q_{ijk}}\}$ where the points are ordered by increasing value of the distance to v_i. Let the segments $]v_i, x_{ijk}^1[,]x_{ijk}^l, x_{ijk}^{l+1}[$, $l = 1, 2, \ldots, q_{ijk} - 1$, and $]x_{ijk}^{q_{ijk}}, v_j[$, and the set $C_{ij}(a_k) = \{y_{ijk}^l\}_{l=0}^{q_{ijk}}$ with $y_{ijk}^l \in]x_{ijk}^l, x_{ijk}^{l+1}[$, where $v_i = x_{ijk}^0$ and $v_j = x_{ijk}^{q_{ijk}+1}$. If v_i is not an isoattractive point then set $y_{ijk}^0 = v_i$. Analogously, if v_j is not an isoattractive point then set $y_{ijk}^{q_{ijk}} = v_j$. If isoattractive points do not exist on $[v_i, v_j]$ then $C_{ij}(a_k) = \{v_i, v_j\}$. Clearly, the set C_{ij} is not necessarily unique. Let $C(a_k) = \bigcup_{i,j} C_{ij}(a_k)$.

Proposition 3.3. *Given X_p, A_p, and A_r, then there exists $Y_r = (y_1, y_2, \ldots, y_r)$ with $y_j \in C(a_j)$, $j = 1, 2, \ldots r$, such that Y_r is an r-tuple of optimal locations for the binary $(r|X_p, A_p)$-medianoid problem.*

Proof. Let $Y_r^* = (y_1^*, y_2^*, \ldots, y_r^*)$ be an r-tuple of optimal locations such that $y_k^* \notin C(a_k)$. Suppose, without loss of generality, that $y_1^* \notin C(a_1)$ and that $y_1^* \in [s,t]$, with $[s,t] \subset [v_i, v_j]$, such that $s,t \in V \cup ISOA_{ij}(a_1)$ and $]s,t[\cap ISOA_{ij}(a_1) = \emptyset$.

If $y_1^* \notin ISOA_{ij}(a_1)$, applying proposition 3.2, y_1^* can move to a point belonging to $C(a_1)$ on $]s,t[$ without varying $V(Y_r^*, A_r)$, that is, y_1^* can be replaced by $y_{ij1}^l \in C(a_1) \cap [s,t]$ and $V((y_{ij1}^l, y_2^*, \ldots, y_r^*), A_r) = V(Y_r^*, A_r)$.

If $y_1^* \in ISOA_{ij}(a_1)$, then $y_1^* = s$ or $y_1^* = t$. Furthermore, there exists $v_0 \in V$ such that

$$G_{v_0} = \frac{a_1}{f_{v_0}(d(v_0, y_1^*))},$$

that is, y_1^* is a (v_0, a_1)-isoattractive point. Suppose that $y_1^* = s$ (the case $y_1^* = t$ is similar) and let $\{v_0^1, v_0^2, \ldots, v_0^H\}$ be the set of nodes such that y_1^* is (v, a_1)-isoattractive. As $]s,t[\cap A_{ij}(a_1) = \emptyset$, for each node v_0^h, $h = 1, 2, \ldots, H$, only two situations can ocurr:

1. $\frac{a_1}{f_{v_0^h}(d(v_0^h, y))} < G_{v_0^h}$, $\forall y \in]s,t[$.
2. $\frac{a_1}{f_{v_0^h}(d(v_0^h, y))} > G_{v_0^h}$, $\forall y \in]s,t[$.

Then only the following two cases can occur:

(i) $\frac{a_1}{f_{v_0^h}(d(v_0^h, y))} < G_{v_0^h}$, $\forall y \in]s,t[$, $\forall h$.

In this case, no points on $[s,t]$ capture v_0^h, therefore, y_1^* can be replaced by $y_{ij1}^l \in C(a_1) \cap [s,t]$ and $V(\{y_{ij1}^l, y_2^*, \ldots, y_r^*\}, A_r) = V(Y_r^*, A_r)$.

(ii) $\exists$ h such that $\frac{a_1}{f_{v_0^h}(d(v_0^h, y))} > G_{v_0^h}$, $\forall y \in]s,t[$.

In this case, the node v_0^h is captured by every point on $]s,t[$. Therefore, a movement from s to t implies the capture of v_0^h without the loss of any nodes already captured. Therefore, choosing $y_{ij1}^l \in C(a_1) \cap [s,t]$, it holds that $V(Y_r^*, A_r) \subset V(\{y_{ij1}^l, y_2^*, \ldots, y_r^*\}, A_r)$, which means that Y_r^* is not a set of optimal locations.

□

On each edge $[v_i, v_j]$ at most two isoattractive points for each node exist and thus $|C_{ij}(a_k)| \leq 2 + |V|$ and $|C(a_k)| \leq |V|(1 + |E|)$. Therefore, given A_r, optimal locations, y_k^*, $k = 1, 2, \ldots, r$, can be found in the set of r-tuples $\{(y_1, y_2, \ldots, y_r) : y_k \in C(a_k)\}$ which has, at most, $|V|^r(1 + |E|)^r$ points.

4 Partially binary preferences

For partially binary preferences the demand of a node v is shared between the most attractive facilities of F_X and F_Y, and the amount captured by each firm is directly proportional to G_v and $G(v, Y_r, A_r)$, respectively. Then

$$W(Y_r, A_r|X_p, A_p) = \sum_{v \in V} \frac{w(v)G(v, Y_r, A_r)}{G(v, Y_r, A_r) + G_v} - \sum_{j=1}^{r} F(a_j).$$

Proposition 4.1. *Under assumption 2.1, the profit function for the partially binary- essential $(r|X_p, A_p)$-medianoid, $W(Y_r, A_r|X_p, A_p)$, where $Y_r = (y_1, \ldots, y_i, \ldots, y_r)$, is convex with respect to y_i, when y_i varies along an edge of the network and A_r, X_p, and A_p are fixed.*

Proof. The profit function is

$$W(Y_r, A_r) = \sum_{v \in V} \frac{w(v)G(v, Y_r, A_r)}{G(v, Y_r, A_r) + G_v} - \sum_{j=1}^{r} F(a_j).$$

Suppose, without loss of generality, that the location which varies is y_1. Then, the function W can be expressed as

$$W(y_1) = \sum_{v \in V} \frac{w(v)K_v(y_1)}{K_v(y_1) + G_v} - F = \sum_{v \in V} w(v) \left(1 - \frac{G_v}{K_v(y_1) + G_v}\right) - F,$$

where $F = \sum_{j=1}^{r} F(a_j)$, and $K_v(y_1) = \max\{\frac{a_1}{f_v(d(v,y_1))}, G_v^*\}$, with $G_v^* = \max\{\frac{a_j}{f_v(d(v,y_j))} : j = 2, 3, \ldots r\}$.

Suppose that y_1 is on edge $[s, t]$ and its location is given by the variable z which represents the distance between the facility and the node s. The distance between any node $v \in V$ and the location y_1 is a concave function

of z denoted by $\delta_v(z)$. Therefore it follows that $f_v(\delta_v(z))$ is concave and thus $\frac{G_v}{\frac{a_1}{f_v(\delta_v(z))}+G_v}$ is also a concave function of z. Then,

$$\frac{G_v}{K_v(y_1)+G_v} = \min\{\frac{G_v}{G_v^*+G_v}, \frac{G_v}{\frac{a_1}{f_v(\delta_v(z))}+G_v}\}$$

is concave. Therefore, $1-\frac{G_v}{K_v(y_1)+G_v}$ is a convex function on $[s,t]$, $\forall v \in V$, and it follows that $W(y_1)$ is convex on $[s,t]$. □

Proposition 4.2. *Under assumptions 2.1 to 2.3, there exists (V_r, A_r), with $V_r \in V^r$, which is a partially binary $(r|X_p, A_p)$-medianoid on $N(V,E)$.*

Proof. From proposition 4.1, since the profit function is convex when a location varies on an edge and the rest of locations and all the attractiveness levels are fixed, any non-node location of the solution can be moved to one of the nodes of the edge where it is located without decreasing the profits. □

The following results will be taken into account in the algorithm used to find the optimal attractiveness levels.

Definition 4.1. The function $f : C \subset \Re^r \longrightarrow \Re$ is non-decreasing if

$$a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_r) \in C, \text{ with } a_i \leq b_i \Rightarrow f(a) \leq f(b),$$

and it is increasing if

$$a = (a_1, a_2, \ldots, a_n), b = (b_1, b_2, \ldots, b_r) \in C, \\ \text{with } a_i \leq b_i, a \neq b \Rightarrow f(a) < f(b).$$

Proposition 4.3. *Under assumptions 2.1 and 2.2, given the locations Y_r, the revenue function in the partially binary $(r|X_p, A_p)$-medianoid problem is non-decreasing on $[I,S]^r$ and quasiconcave with respect to each attractiveness component a_i.*

Proof. Given the locations, the revenue function is

$$REV(A_r) = REV(a_1, a_2, \ldots, a_r) = \sum_{v \in V} w_Y(v)$$

where $w_Y(v) = h_v(G(v, Y_r, A_r))$, with $h_v(x) = \frac{w(v)x}{x+G_v}$. The function $G(v, Y_r, A_r)$ is non-decreasing on $[I, S]^r$ and h_v is increasing for $x > 0$, therefore w_Y is non-decreasing on $[I, S]^r$ and, consequently, $REV(A_r)$ is non-decreasing.

Suppose, without loss of generality, that $a_i = a_1$ varies on $[I, S]$ and the rest of the attractiveness levels are fixed. Then $G(a_1) = G(v, Y_r, A_r) = \max\{\frac{a_1}{f_v(d(v,y_1))}, G_v^*\}$ where $G_v^* = \max\{\frac{a_j}{f_v(d(v,y_j))}, j = 2, \ldots, r\}$, and

$$w_Y(v) = \frac{w(v)G(a_1)}{G(a_1) + G_v} = \begin{cases} \frac{w(v)G_v^*}{G_v^* + G_v} & if \quad \frac{a_1}{f_v(d(v,y_1))} \leq G_v^* \\ \frac{w(v)a_1}{a_1 + f_v(d(v,y_1))G_v} & if \quad \frac{a_1}{f_v(d(v,y_1))} > G_v^*. \end{cases}$$

The demand $w_Y(v)$ is a continuous function of a_1, it is constant if $\frac{a_1}{f_v(d(v,y_1))} \leq G_v^*$, and increasing and concave if $\frac{a_1}{f_v(d(v,y_1))} > G_v^*$. Therefore $w_Y(v)$ is a quasiconcave and non-decreasing function of a_1. Therefore the revenue function $REV(A_r)$ is non-decreasing on $[I, S]^r$ and quasiconcave with respect to a_i, $i = 1, 2, \ldots, r$.

□

5 Proportional preferences

For proportional preferences the demand of a node is shared among all the facilities operating in the market and the amount of demand captured by each of them is directly proportional to the attraction felt by the demand node towards the facility. Let $X_{r+p} = (x_1, x_2, \ldots, x_r, x_{r+1}, \ldots, x_{r+p})$ be the r locations of F_Y, $Y_r = (x_1, x_2, \ldots, x_r)$, and the p locations of F_X, $X_p = (x_{r+1}, \ldots, x_{r+p})$. Let $A_{r+p} = (a_1, a_2, \ldots, a_r, a_{r+1}, \ldots, a_{r+p})$ be the attractiveness levels associated to Y_r (A_r), and X_p (A_p). Let $f_v(d(v, x_j)) = f_{vj}$ and $a_{vj} = \frac{a_j}{f_{vj}}$, and let $w_j(v)$ be the part of the demand of v captured by the facility located at x_j, $\forall v \in V$, $j = 1, \ldots, r + p$. Then

$$w_j(v) = w(v)\frac{a_{vj}}{\sum_{k=1}^{r+p} a_{vk}}, \quad \forall v \in V, \ j = 1, \ldots, r + p.$$

Proposition 5.1. *Let $N(V, E)$ be a network and assume that $p + r \leq |V|$. Then, under assumptions 2.1 to 2.3, there exists (V_r, A_r), with $V_r \in V^r$, which is a proportional $(r|X_p, A_p)$-medianoid on $N(V, E)$.*

Proof. It is an extension to multiple facilities of the node optimality result given by Peeters and Plastria (1998). □

Proposition 5.2. *Under assumptions 2.1 and 2.2, given the locations Y_r, the revenue function in the proportional $(r|X_p, A_p)$-medianoid problem is increasing and strictly concave on $[I, S]^r$.*

Proof. Given Y_r, the revenue function is $REV(A_r) = \sum_{v \in V} REV_v(A_r)$ where

$$REV_v(A_r) = w(v) \frac{\sum_{j=1}^{r} \frac{a_j}{f_{vj}}}{\sum_{j=1}^{r} \frac{a_j}{f_{vj}} + K_v}$$

with $K_v = \sum_{j=r+1}^{r+p} \frac{a_j}{f_{vj}}$.

The function REV_v can be expressed as $REV_v = F_v \circ g_v$, where

$$g_v(A_r) = \sum_{j=1}^{r} \frac{a_j}{f_{vj}} \text{ and } F_v(x) = w(v) \frac{x}{x + K_v}.$$

Since g_v is linear and increasing, and F_v is increasing and strictly concave for $x > 0$, it follows that $REV_v(A_r)$ is increasing and strictly concave. Therefore $REV(A_r) = \sum_{v \in V} REV_v(A_r)$ is strictly concave. □

6 Resolution of the attractiveness problem

Depending on the properties of the attractiveness cost function, the problem of obtaining the optimal attractiveness levels when the new facilities locations are given can be multiextremal. To solve this problem a branch and bound based algorithm is proposed.

Let the general non-linear program

$$\begin{array}{ll} \max & f(x) \\ s.t. & x \in D \subset \Re^r \end{array}$$

such that an optimal solution exists. The branch and bound method consists of generating a partition of D to determine the subset where the optimal solution is. Although this algorithm can converge to the optimal solution, for computational efficiency, an ε-optimal solution, with prescribed $\varepsilon > 0$, is accepted.

The branch and bound algorithm applied in this paper is an adaptation of the one proposed by Horst and Tuy (1993).

Branch and Bound

Step 0:

Let $D_0 = D$, $k = 0$. Obtain $P_0 = \{D_i : i \in I_0\}$ a partition of D_0.

For each $D_i \in P_0$ calculate the vertex set of D_i, $V(D_i)$, and the bounds

(i) $\beta(D_i, x_i) = \max\limits_{v_i \in V(D_i)} f(v_i)$ (lower bound of sup f in D_i reached at node x_i).

(ii) $\alpha(D_i)$ (upper bound of f in D_i).

Set the "overall" bounds

$$\alpha_0 = \max\{\alpha(D_i) : D_i \in P_0\} \text{ and } \beta_0 = \max\{\beta(D_i, x_i) : D_i \in P_0\} \text{ reached at } x_0.$$

If $\frac{\alpha_0 - \beta_0}{\alpha_0} = 0$ ($\leq \varepsilon$) stop. The (ε-)optimal solution is x_0. Otherwise, go to step 1.

Step 1:

(i) Delete from P_k all subsets satisfying $\alpha(D_i) < \beta_k$, $D_i \in P_k$.

(ii) Let $k = k + 1$ and select $D_k \in P_{k-1}$.
Obtain P'_k, a partition of D_k, and update the list of subsets changing D_k by P'_k, set $P_k = P'_k \cup (P_{k-1} - D_k)$.

(iii) Calculate the bounds for every subset $D'_i \in P'_k$ $(\alpha'(D'_i), \beta'(D'_i, x'_i))$. Let $\alpha(D'_i) = \min\{\alpha'(D'_i), \alpha(D_k)\}$.

(iv) Update the "overall" bounds

$$\alpha_k = \max\{\alpha(D_i) : D_i \in P_k\} \text{ and } \beta_k = \max\{\beta(D_i, x_i) : D_i \in P_k\} \text{ reached at } x_k.$$

If $\frac{\alpha_k - \beta_k}{\alpha_k} = 0$ ($\leq \varepsilon$), stop. The (ε-)optimal solution is x_k. Otherwise, go to step 1.

To adapt this algorithm to a particular problem it is necessary to decide three basic points: the choice of an appropriate partition, the calculation of the bounds, and the selection of the subset that must be partitioned in each iteration.

As in the attractiveness problem studied, the feasible set is an r-rectangle, $D = [I, S]^r$, a partition of D into 2^r r-rectangles has been chosen. These new subsets are obtained by perpendicular hyperplanes to each facet passing through their midpoints. This type of partition has been used by Plastria (1992), and Hansen, Peeters and Thisse (1995, 1997), among others. Henceforth, the notation $D(x, y)$ will be used to denote the r-rectangle determined by the vectors x (the lower left vertex) and y (the upper right vertex).

Sometimes, a prefixed tolerance in size is taken as a stopping rule. In these cases, a determined approximation of the optimal values for the decision variables is sufficient so the *minimum diameter* (*md*) parameter is defined to determine the smaller size of a set which can be partitioned. This condition can be incorporated into step 1*(i)* to delete all the sets $D(x, y)$, with $x, y \in \Re^r$, such that $\max\{|x_i - y_i| : i = 1, 2 \ldots r\} < md$.

Given the locations of the new facilities, for each customer choice rules analysed, the revenue function of the $(r|X_p, A_p)$-medianoid is non-decreasing with respect to the attractiveness level. Taking into account assumption 2.3, an upper bound of the profit function in $D(a, b)$ is

$$\alpha(D(a, b)) = REV(b) - C(a),$$

where $REV(\cdot)$ and $C(\cdot)$ are the revenue and cost functions, respectively. To obtain the lower bound, the higher objective value in $V(D(a, b))$ is selected.

The choice of the set to partition in each iteration is very important in relation to the speed of the algorithm convergence. In this case, following the rule employed by Plastria (1992), and Hansen, Peeters and Thisse (1995, 1997), the most "promising" rectangle is partitioned.

The convergence of the branch and bound algorithm proposed in this paper can be deduced from the results given by Horst and Tuy (1993). In the binary case, only an ε-optimal solution is guaranteed.

7 Computational results

In this section, some computational results are presented. First, in section 7.1, a branch and bound algorithm is used to solve the problem of obtaining the optimal attractiveness levels for the new facilities when their locations are given. In the binary case, this algorithm is compared to an enumerative algorithm based on proposition 3.1. In section 7.2, the location-attractivenes problem is treated combining the global search procedure with three combinatorial heuristics. First a greedy algorithm (GR) (Kuehn and Hamburger (1963)) is employed. Then, taking the greedy solution as the initial solution, an interchange algorithm (TB) (Teitz and Bart (1968)) and a tabu search algorithm (TS) (Glover (1993)) are applied.

To compare the behavior of the heuristics, the problems were solved on 13 different networks. These networks can be divided into two groups. The first group consisting of 10 randomly generated networks (5 containing 50 nodes and 5 containing 75 nodes) and the second formed by three networks already used in previous papers.

The first group of networks was randomly generated in a square of 500×500 units. The degree of the nodes varies between 3 and 8, and the distance is defined as the Euclidean norm between any pair of adjacent nodes.

The second group of networks is formed by specific cases: a 32 node network (Suárez, Santos and Dorta (2001)), another of 55 nodes (Serra (1996)), and the last one made up of 79 nodes (Serra (1989)). These networks are inscribed in a square of 95×95, 60×60, and 7×7 units, respectively.

In this analysis, the parameters r and p vary with $r, p \in \{1, 2, 3\}$. Furthermore, three different demand distributions were used. In the first case, the demand is equal to 1 for all the nodes; in the second and third cases, the demand follows a uniform distribution with values between 9 and 10, and between 5 and 15, respectively.

The computational experience was carried out using the concave cost function $F(a) = a + 2\sqrt[3]{a}$. All the heuristics have been programmed in C++ and executed on a computer working at 450 Mhz.

7.1 The attractiveness problem

To obtain the attractiveness levels when the locations of the new facilities are given, a non-linear multiextreme problem must be solved. To analyse the computational times required for the branch and bound algorithm 102 problems have been solved on a 50 node network where three competing facilities exist. The attractiveness level of the entry facilities can be chosen on $[0.9, 9]$ while the attractiveness level for each existing facility is three. The problems were solved both for $r = 2$ and $r = 3$. The simulations were conducted varying the parameter ε (maximum percentage of error allowed) and taking values 0.0001(0.01%) and 0.001(0.1%). The maximum error for the attractiveness level (md) also varies between 0.0005 and 0.05.

Times, in seconds, and iterations (number of partitions made) needed to solve these problems are presented in table 1. Note that there exists a significant difference between times required by the proportional case and the other two models. To obtain the highest precision when $r = 3$, in the proportional case, 83.843 seconds are required. For the binary and partially binary times are 0.251 and 2.701 seconds, respectively. Times for $r = 2$ are significantly lower, 4.403, 0.593, and 0.035 seconds in the most difficult case for the proportional, partially binary and binary cases, respectively. The computational times show a high dependence on the number of new facilities while the influence on the number of nodes of the network is less significant.

The problem of obtaining the optimal attractiveness levels in the binary case, when the locations of the new facilities are given, can also be solved using an enumerative algorithm based on proposition 3.1. To compare both the enumerative and the branch and bound algorithms, 102 problems were solved in five networks with a different number of nodes. The problems were solved taking $\varepsilon = md = 0.000001$ for the branch and bound algorithm and $\delta = 0.000001$ for the enumerative approach. For problems with $r = 2$, times are very low and no difference exists between the two algorithms. For $r = 3$, the enumerative algorithm is competitive only when the number of nodes is small (upto $n = 32$). When n increases, the times required for this algorithm are much higher than the branch and bound procedure. This difference can be observed in table 2.

SECONDS (ITERATIONS)	md=0.0005	md=0.001	md=0.005	md=0.01	md=0.05
BINARY					
r=2, ε =0.0001	0.035 (25)	0.035 (25)	0.026 (22)	0.026 (20)	0.023 (18)
r=2, ε =0.001	0.028 (20)	0.022 (20)	0.026 (20)	0.029 (20)	0.022 (18)
r=3, ε =0.0001	0.251 (29)	0.239 (28)	0.230 (25)	0.219 (24)	0.201 (21)
r=3, ε =0.001	0.219 (24)	0.222 (24)	0.221 (24)	0.214 (24)	0.203 (21)
PARTIALLY BINARY					
r=2, ε =0.0001	0.593 (435)	0.492 (435)	0.263 (246)	0.190 (172)	0.093 (79)
r=2, ε =0.001	0.175 (126)	0.166 (126)	0.166 (126)	0.163 (126)	0.088 (75)
r=3, ε =0.0001	2.701 (486)	2.211 (485)	1.191 (270)	0.854 (190)	0.426 (92)
r=3, ε =0.001	0.835 (155)	0.829 (155)	0.830 (155)	0.757 (155)	0.417 (155)
PROPORTIONAL					
r=2, ε =0.0001	4.403 (3666)	3.680 (3530)	1.576 (1708)	1.166 (1267)	0.615 (685)
r=2, ε =0.001	1.534 (1233)	1.536 (1233)	1.332 (1229)	1.064 (1074)	0.595 (652)
r=3, ε =0.0001	83.843 (15499)	70.475 (14607)	31.480 (7316)	23.565 (5503)	12.753 (3075)
r=3, ε =0.001	30.479 (5351)	30.480 (5351)	26.237 (5244)	21.115 (4063)	12.293 (2901)

Table 1: *Computational times (secs.) and iterations (in brackets) for the branch and bound algorithm*

BINARY	n=32	n=50	n=55	n=75	n=79
ENUMERATIVE	0.3287	1.5916	1.9664	7.7827	7.7619
BRANCH AND BOUND	0.3943	0.1713	0.1993	0.2101	0.2214

Table 2: *Computational times (secs.) for the enumerative and the branch and bound algorithms for* $r = 3$.

7.2 The location-attractiveness problem

To solve the problem of obtaining the optimal locations and the attractiveness levels for the new facilities, some combinatorial heuristic algorithms combined with the branch and bound algorithm are used. To compare the results of the three combinatorial heuristics two aspects were analysed: the percentage of failure and the average percentage error with respect to the best solution found for each scenario. Table 3 shows these results for the binary, partially binary and proportional cases. Note that results corresponding to different networks have been aggregated.

In all the cases, the solutions obtained by the GR are significantly improved by the TB and the TS. For these cases, the worst results are reached in the binary case where the percentage of failure, when $r = 3$, is 10.256%. Nevertheless, these percentages decreases when the error is analysed. The results for the partially binary and proportional cases are significantly better, with percentage of failure and error less than 4.273% and 0.0005%, respectively.

The TB usually needs, at most, one iteration to stop the search when the starting solution is derived from the greedy algorithm. This means that the number of pairs, or trios, of potential solutions evaluated, i.e., the number of non-linear programs solved using the branch and bound algorithm, in addition to the programs solved by the GR algorithm, is about $r \times n$. This implies that the problems solved by the TS, in addition to those by the GR, are approximately $(r \times n) \times 1.5$. Average times employed by the TB algorithm to solve the problems when the maximum precision is required ($\varepsilon = 0.0001$ and $dm = 0.0005$) for the 50 node networks are shown in the last column of table 3. The highest times are required to solve the proportional problem.

It seems that a general tendency with respect to the variations of the parameters p, r, and the demand distribution, does not exist. The number of failures is usually similar when $r = 2$ and $r = 3$, independently of the algorithm used and the customer choice rule considered. This difference is more significant in the proportional preferences case and the GR algorithm. On the other hand, the only cases where a tendency with respect to the distribution of the demand seems to exist, is for the GR applied to the binary and partially binary preferences. In contrast to the partially binary case, in the binary case the number of failures increases with demand variation.

	Fail. GR	Fail. TB	Fail. TS	Error GR	Error TB	Error TS	TB Times
BINARY							
r=2	35.897	7.692	4.273	0.0200	0.0057	0.0005	5.112
r=3	35.897	10.256	5.128	0.0109	0.0046	0.0004	49.512
PARTIALLY BINARY							
r=2	16.239	0	0.008	0.0005	0	$1.5\ 10^{-6}$	86.590
r=3	16.239	4.273	0	0.0005	$3.36\ 10^{-6}$	0	542.339
PROPORTIONAL							
r=2	17.094	0.342	0.008	0.0012	0.0002	$2.41\ 10^{-5}$	642.938
r=3	33.333	0.007	0.342	0.0042	0.0005	0.0001	16408.613

Table 3: *Results for the GR, TB(GR), and TS(GR)*

8 Conclusions

In this paper, the $(r|X_p)$-medianoid problem for essential demands (Hakimi (1983)) is generalized to the case where customers' choice is based not only on distance between demand points and facilities but on certain attributes of the facilities such as size. The attraction felt by a customer towards a facility is formulated as an increasing function of the *quality* of the facility and non-increasing with respect to distance. The objective of the entry firm is to determine the locations and the attractiveness levels in order to maximize its profits.

This problem has been analysed under different customer choice rules, binary, partially binary and proportional preferences. The existence of a nodal solution has been proved in the partially binary and proportional network problems. For binary preferences, a discretization of the location problem can be applied if the attractiveness levels are given.

To solve the discrete problem three combinatorial heuristics (greedy, interchange and tabu search) combined with a global search procedure based on branch and bound techniques are used. The location problem is solved by means of the combinatorial heuristics and, given the locations, the maximum profit is obtained using the branch and bound algorithm. In the binary case, the optimum attractiveness levels can be found evaluating, at most, $(|V|+1)^r$ candidates.

The three combinatorial heuristics were compared by evaluating the average percentage of error with respect to the best solution found. The error for the GR varied between 0.015%, for the binary preferences, and

0.0005%, for the partially binary preferences. The TB and TS algorithms were run taking the greedy solution as the initial solution. These algorithms reduce the error, especially the TS (with almost insignificant errors).

A possible extension to this work consists of considering other more realistic customer choice rules. In this sense, the Pareto-Huff model (Peeters and Plastria (1998)) is a modified Huff model where a customer patronizes a more distant facility only if that facility is more attractive. Another possibility is to consider that the attraction felt by a consumer towards a facility decreases when other facility exerting a higher attraction on the consumer exists. This assumption is based on the idea that if the difference between the best and the "second best" facilities is big, the consumer has not any incentive to patronize the second best facility. On the other hand, the formulation of the attractive as a function of a unique variable such as the facility size may be too simple. Usually, the behaviour of the consumer depends on several factors (size, parking space, price and others) which would be incorporated in the attraction functions. These functions would be associated to different groups of customers charaterized by certain properties such as the purchasing power. Finally, a dynamic formulation of the problem would be useful to solve situations where demand and other elements vary with the time.

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