# ERROR ESTIMATES FOR FICTITIOUS DOMAIN/PENALTY/FINITE ELEMENT METHODS

R. GLOWINSKI (1) – TSORG-WHAY PAN (2)

ABSTRACT - We obtain error estimates for the finite element solution of elliptic problems with Neumann boundary conditions for domains with curved boundaries using fictitious domain/penalty methods.

## 1. Introduction

Fictitious domain methods for partial differential equations have shown recently a most interesting potential for solving complicated problems from Science and Engineering [e.g., 6]. One of the main reasons of this popularity of fictitious domain methods (they are sometimes called domain embedding methods; cf. [1]) is that they allow the use of fast solvers on fairly structured meshes in a simple shape auxiliary domain containing the actual one.

For solving elliptic problems on a domain with a curved boundary (i.e., the domain is no longer assumed to be polygonal) by finite element methods, there are usually two ways to handle the curved boundary: The first approach consists in using a polygonal domain to approximate the domain with a curved boundary. The second one consists in using isoparametric finite elemement which have «curved» face and are used to approximate «as well as possible» the curved boundary of the domain [e.g., 2].

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(1) Department of Mathematics, University of Houston, Houston, Texas 77204 USA and Université P. et M. Curie, Paris, France.

(2) Department of Mathematics, University of Houston, Houston, Texas 77204 USA.

In [4], a fictitious domain/penalty solution method for elliptic problems with Neumann boundary conditions for domains with curved boundaries was proposed. The approximation of the curved boundary is not necessary any more. In this article we study error estimates for the finite element solution of these problems, using a fictitious domain/penalty method. In Section 2, we describe the elliptic problems with Neumann boundary conditions. In Section 3, we introduce the fictitious domain/penalty treatment of Neumann problems for domains with curved boundaries and its finite element approximation. In Section 4, we obtain an $H^1$ error estimate for domains with curved boundaries by using finite element of type ($k$) on $n$-simplices, for integers $k > 0$; by a duality argument, we derive a $L^2$ error estimate. In Section 5, the results of numerical experiments are presented.

## 2. **Elliptic problems with Neumann boundary conditions**

We consider the following elliptic problem with a Neumann boundary condition

$$\alpha u - \triangle u = f \text{ in } \omega, \tag{2.1}$$

$$\frac{\partial u}{\partial n} = g \text{ on } \gamma, \tag{2.2}$$

where in (2.1), (2.2), $\alpha > 0$, $\omega$ is a bounded domain in $\mathbf{R}^n$ with a $C^{1,1}$ boundary $\gamma$, $f \in L^2(\omega)$ and $g \in H^{1/2}(\gamma)$.

Problem (2.1), (2.2) has a unique solution $u$ in $H^2(\omega)$ [e.g., 5] and $u$ is also the solution of the following variational problem

*Find $u$ in $H^1(\omega)$ such that*

$$a_\omega(u, v) = \int_\omega fv\, dx + \int_\gamma gv\, d\gamma, \forall v \in H^1(\omega), \tag{2.3}$$

where

$$a_\omega(u, v) = \int_\omega (\alpha uv + \nabla u \cdot \nabla v)\, dx.$$

## 3. A fictitious domain formulation

### 3.1. *A Fictitious domain/penalty method*

A fictitious domain/penalty method was proposed for problem (2.1), (2.2) in [4]. Let us consider a «box» $\Omega$ which is an open set in $\mathbf{R}^n$ such that $\omega \subset\subset \Omega$ (see Figure 3.1) and denote by $\Gamma$ the boundary of $\Omega$.

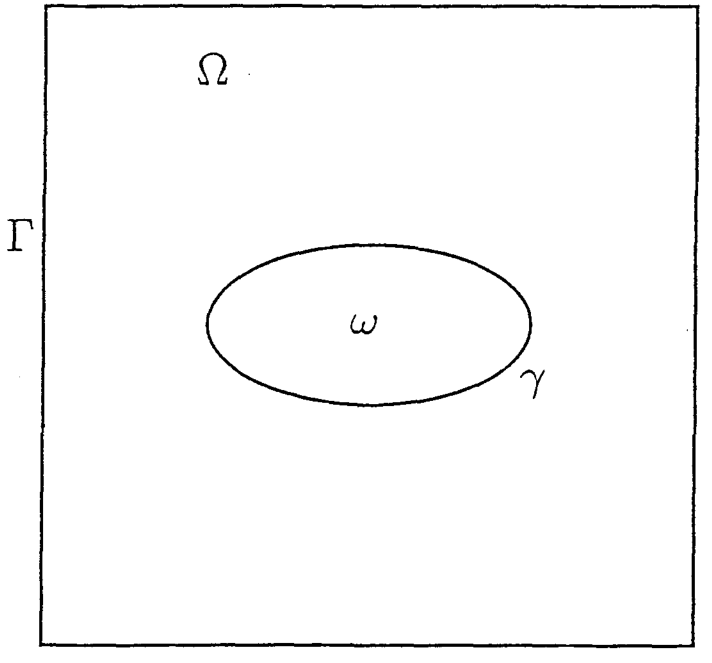


*Figure 3.1*

Let $\varepsilon > 0$ be a parameter which will tend to zero. We consider the following problem

*Find $u^\varepsilon$ in $H_0^1(\Omega)$ such that*

$$a_\omega(u^\varepsilon, v) + \varepsilon a_\Omega(u^\varepsilon, v)$$

$$= \int_\omega fv\, dx + \varepsilon \int_\Omega \tilde{f}v\, dx + \int_\gamma gv\, d\gamma,\ \forall v \in H_0^1(\Omega), \tag{3.1}$$

where

$$a_\Omega(u^\varepsilon, v) = \int_\Omega (au^\varepsilon v + \nabla u^\varepsilon \cdot \nabla v)\, dx,$$

and $\tilde{f} \in L^2(\Omega)$. Problem (3.1) has a unique solution in $H_0^1(\Omega)$. The finite element approximation of problem (3.1) is described in the following Section 3.2.

REMARK 3.1. *The fact that $H^1(\omega)$ is «embedded» in $H_0^1(\Omega)$ is not critical, we could have chosen $H^1(\Omega)$ or $H_p^1(\Omega) = \{v | v \in H^1(\Omega),\ v$ periodic at $\Gamma\}$.*

3.2. *A fictitious domain/penalty/finite element method*

Let $V_h$ be a finite dimensional subspace of $H_0^1(\Omega)$. We approximate the variational problem (3.1) by

*Find $u_h^\varepsilon$ in $V_h$ such that*

$$a_\omega(u_h^\varepsilon, v_h) + \varepsilon a_\Omega(u_h^\varepsilon, v_h)$$

$$= \int_\omega f v_h\, dx + \varepsilon \int_\Omega \tilde{f} v_h\, dx + \int_\gamma g v_h\, d\gamma,\ \forall v_h \in V_h. \tag{3.2}$$

Problem (3.2) has a unique solution in $V_h$.

## 4. Error estimates

### 4.1. $H^1$-error

In order to estimate the errors, $\|u - u_h^\varepsilon\|_{1,\omega}$ and $\|u - u_h^\varepsilon\|_{0,\omega}$, we need to extend the solution $u$ of problem (2.3) from $H^2(\omega)$ into $H_0^2(\Omega)$. In [3], there is a basic extension result. We state it in the following.

THEOREM 4.1. *Let $\omega$ be a bounded connected domain in $\mathbf{R}^n$ with a $C^{k,1}$ boundary for some integer $k \geqslant 0$ and $\omega \subset\subset \Omega$ where $\Omega$ is an open set. Then there is a bounded linear extension operator $E$ from $H^{k+1}(\omega)$ into $H_0^{k+1}(\Omega)$ such that $Ev|_\omega = v$ and*

$$\|Ev\|_{k+1,\Omega} \leqslant C(k, \omega, \Omega)\|u\|_{k+1,\omega} \tag{4.1}$$

*for all $v \in H^{k+1}(\omega)$.*

Thus for the polygonal fictitious domain $\Omega$, $\omega \subset\subset \Omega$, we can extend the solution $u$ of the problem (2.3) from $H^2(\omega)$ to $H_0^2(\Omega)$ by Theorem 4.1.

Now in (2.3) let $v = v_h \in V_h$ and use the extension $Eu$ instead of $u$. Then we have

$$a_\omega(Eu, v_h) = \int_\omega f v_h \, dx + \int_\gamma g v_h \, d\gamma, \; \forall v_h \in V_h. \tag{4.2}$$

Substracting (4.2) from (3.2), we have

$$a_\omega(u_h^\varepsilon - Eu, v_h) + \varepsilon a_\Omega(u_h^\varepsilon, v_h) = \varepsilon \int_\Omega \tilde{f} v_h \, dx, \; \forall v_h \in V_h. \tag{4.3}$$

Since $Eu$ is in $H_0^2(\Omega)$, we can consider

$$\begin{aligned}
&a_\omega(u_h^\varepsilon - Eu, u_h^\varepsilon - Eu) + \varepsilon a_\Omega(u_h^\varepsilon - Eu, u_h^\varepsilon - Eu) \\
&= a_\omega(u_h^\varepsilon - Eu, v_h - Eu) + \varepsilon a_\Omega(u_h^\varepsilon - Eu, v_h - Eu) \\
&+ a_\omega(u_h^\varepsilon - Eu, u_h^\varepsilon - v_h) + \varepsilon a_\Omega(u_h^\varepsilon - Eu, u_h^\varepsilon - v_h) \\
&= a_\omega(u_h^\varepsilon - Eu, u_h^\varepsilon - Eu) + \varepsilon a_\Omega(u_h^\varepsilon - Eu, v_h - Eu) \\
&+ \{a_\omega(u_h^\varepsilon - Eu, u_h^\varepsilon - v_h) + \varepsilon a_\Omega(u_h^\varepsilon, u_h^\varepsilon - v_h)\} \\
&- \varepsilon a_\Omega(Eu, u_h^\varepsilon - v_h).
\end{aligned} \tag{4.4}$$

Then by (4.3) and (4.4), we have

$$\begin{aligned}
&a_\omega(u_h^\varepsilon - Eu, u_h^\varepsilon - Eu) + \varepsilon a_\Omega(u_h^\varepsilon - Eu, u_h^\varepsilon - Eu) \\
&= a_\omega(u_h^\varepsilon - Eu, v_h - Eu) + \varepsilon a_\Omega(u_h^\varepsilon - Eu, v_h - Eu) \\
&+ \varepsilon \int_\Omega (u_h^\varepsilon - v_h) \tilde{f} \, dx - \varepsilon a_\Omega(Eu, u_h^\varepsilon - v_h) \\
&= a_\omega(u_h^\varepsilon - Eu, v_h - Eu) + \varepsilon a_\Omega(u_h^\varepsilon - Eu, v_h - Eu) \\
&+ \varepsilon \int_\Omega (u_h^\varepsilon - Eu) \tilde{f} \, dx + \varepsilon \int_\Omega (Eu - v_h) \tilde{f} \, dx \\
&- \varepsilon a_\Omega(Eu, u_h^\varepsilon - Eu) - \varepsilon a_\Omega(Eu, Eu - v_h).
\end{aligned} \tag{4.5}$$

Then from (4.5), we have the following inequality

$$\|u_h^\varepsilon - Eu\|_{1,\omega}^2 + \varepsilon\|u_h^\varepsilon - Eu\|_{1,\Omega}^2$$

$$\leqslant c_1 \{ \|u_h^\varepsilon - Eu\|_{1,\omega}\| v_h - Eu\|_{1,\omega}$$

$$+ \varepsilon\|u_h^\varepsilon - Eu\|_{1,\Omega}\| v_h - Eu\|_{1,\Omega} \tag{4.6}$$

$$+ \varepsilon\|\tilde{f}\|_{0,\Omega}\|u_h^\varepsilon - Eu\|_{1,\Omega} + \varepsilon\|\tilde{f}\|_{0,\Omega}\|v_h - Eu\|_{1,\Omega}$$

$$+ \varepsilon\|Eu\|_{1,\Omega}\|u_h^\varepsilon - Eu\|_{1,\Omega} + \varepsilon\|Eu\|_{1,\Omega}\|v_h - Eu\|_{1,\Omega}\}.$$

Using (4.6) and the inequality $ab \leqslant \eta a^2 + \frac{1}{\eta}b^2$ valid for all $\eta > 0$, we have

$$\|u_h^\varepsilon - Eu\|_{1,\Omega}^2 + \varepsilon\|u_h^\varepsilon - Eu\|_{1,\omega}^2$$

$$\leqslant c_1\{\eta_1\|u_h^\varepsilon - Eu\|_{1,\omega}^2 + \frac{1}{\eta_1}\|v_h - Eu\|_{1,\omega}^2$$

$$+ \varepsilon\eta_2\|u_h^\varepsilon - Eu\|_{1,\Omega}^2 + \frac{\varepsilon}{\eta_2}\|v_h - Eu\|_{1,\Omega}^2$$

$$+ \varepsilon\eta_3\|u_h^\varepsilon - Eu\|_{1,\Omega}^2 + \frac{\varepsilon}{\eta_3}\|\tilde{f}\|_{0,\Omega}^2 \tag{4.7}$$

$$+ \varepsilon\eta_4\|v_h - Eu\|_{1,\Omega}^2 + \frac{\varepsilon}{\eta_4}\|\tilde{f}\|_{0,\Omega}^2$$

$$+\varepsilon\eta_5\|u_h^\varepsilon - Eu\|_{1,\Omega}^2 + \frac{\varepsilon}{\eta_5}\|Eu\|_{1,\Omega}^2$$

$$+ \varepsilon\eta_6\|v_h - Eu\|_{1,\Omega}^2 + \frac{\varepsilon}{\eta_6}\|Eu\|_{1,\Omega}^2\}.$$

Let $\eta_2 = \eta_3 = \eta_5 = \frac{1}{3c_1}$, $\eta_1 = \frac{1}{2c_1}$, $\eta_4 = \eta_6 = 1$ in (4.7), then we have

$$\begin{aligned}(4.8)\quad \|u_h^\varepsilon - Eu\|_{1,\Omega}^2 \leqslant 2c_1\{&2c_1\|v_h - Eu\|_{1,\omega}^2 + \varepsilon(3c_1 + 2)\|v_h - Eu\|_{1,\Omega}^2 \\ &+\varepsilon(3c_1 + 1)\|\tilde{f}\|_{0,\Omega}^2 + \varepsilon(3c_1 + 1)\|Eu\|_{1,\Omega}^2\}.\end{aligned}$$

Thus we have

$$(4.9)\qquad \|u_h^\varepsilon - Eu\|_{1,\omega} \leqslant c_2\{\|v_h - Eu\|_{1,\Omega}^2 + \varepsilon\|\tilde{f}\|_{0,\Omega}^2 + \varepsilon\|Eu\|_{1,\Omega}^2\}^{\frac{1}{2}},\ \forall v_h \in V_h.$$

And since $\|u_h^\varepsilon - Eu\|_{1,\omega} = \|u_h^\varepsilon - u\|_{1,\omega}$, we have the following

LEMMA 4.2. *Assume that $\omega$ is a bounded connected domain in $\mathbf{R}^n$ with a $C^{1,1}$ boundary and $\omega \subset\subset \Omega$ where $\Omega$ is a polygonal domain. Then there exists a constant $C_1$ such that*

$$(4.10)\qquad \|u_h^\varepsilon - u\|_{1,\omega} \leqslant C_1 \inf_{v_h \in V_h} \{\|v_h - Eu\|_{1,\Omega}^2 + \varepsilon\|\tilde{f}\|_{0,\Omega}^2 + \varepsilon\|Eu\|_{1,\Omega}^2\}^{\frac{1}{2}},$$

*where $u$ is the solution of problem (2.3) and $u_h^\varepsilon$ is the solution of problem (3.2).*

Let us give a brief description of the finite element spaces used to obtaine the error estimates in the following. Let $T_h$ be a regular triangulation of the polygonal domain $\Omega$. We would like to use finite elements, $n$-simplices of type $(k)$, for integer $k > 0$ (other finite elements can be used). The finite element spaces $V_h$ associated with the finite elements $n$ simplices of type $(k)$ are given by

$$V_h = \{v_h | v_h \in H_0^1(\Omega) \cap \mathbf{C}^0(\bar{\Omega}),\ v_h|_T \in P_k,\ \forall T \in T_h\}$$

where $P_k$ is the space of the polynomials in $n$ variables of degree $\leqslant k$.

Let $\{\psi_i\}_{i=1}^N$ be the bases of the finite element spaces $V_h$ where $N$ is the dimension of $V_h$ and $\{d_i\}_{i=1}^N$ be the mesh nodes; they satisfy the following relation

$$\psi_i(d_j) = \delta_{ij} = \begin{cases}1, & \text{if } i = j,\\ 0, & \text{if } i \neq j,\end{cases}$$

for $1 \leqslant i, j \leqslant N$. Also for any $v_h$ in $V_h$, we have

$$v_h = \sum_{i=1}^{N} v_h(d_i)\psi_i,$$

The $X_h$-interpolant is defined by

$$\Pi_\eta \omega = = \sum_{i=1}^{N} v(d_i)\psi_i,$$

for any $v \in C^0(\Omega)$. There are estimates of the interpolation error in [2]. We will use the following.

THEOREM 4.3. *If* $k > \frac{n}{2} - 1$, *then there exists a constant* $C_2$ *independent of* h *such that, for any function* $v \in H^{k+1}(\Omega) \cap H_0^1(\Omega)$,

$$\|v - \Pi_h v\|_{1,\Omega} \leqslant C_2 h^k |v|_{k+,\Omega}. \tag{4.11}$$

Let us assume that the solution $u$ of problem (2.3) verifies $u \in H^{k1+1}(\omega)$ for an integer $k > \max\{0, \frac{n}{2} - 1\}$. Suppose also that $\omega$ is a bounded connected domain in $\mathbf{R}^n$ with a $C^{k,1}$ boundary, strictly contained in the polygonal open set $\Omega$. By (4.10) we have

$$\|u_h^\varepsilon - u\|_{1,\omega} \leqslant C_1 \{\|\Pi_h Eu - Eu\|_{1,\Omega}^2 + \varepsilon\|\tilde{f}\|_{0,\Omega}^2 + \varepsilon\|Eu\|_{1,\Omega}^2\}^{\frac{1}{2}}. \tag{4.12}$$

Using (4.12) and inequality $(a+b)^{\frac{1}{2}} \leqslant a^{\frac{1}{2}} + b^{\frac{1}{2}}$ valid for $a, b \geqslant 0$, we have

$$\begin{aligned}
\|u_h^\varepsilon - u\|_{1,\omega} &\leqslant C_1\{\|\Pi_h Eu - Eu\|_{1,\Omega} + \sqrt{\varepsilon}\,\|\tilde{f}\|_{0,\Omega} + \sqrt{\varepsilon}\,\|Eu\|_{1,\Omega}\} \\
&\leqslant c_3\{h^k\|Eu\|_{k+1,\Omega} + \sqrt{\varepsilon}\,\|\tilde{f}\|_{0,\Omega} + \sqrt{\varepsilon}\,\|Eu\|_{1,\Omega}\} \\
&\leqslant c_4\{h^k\|u\|_{k+1,\omega} + \sqrt{\varepsilon}\,\|\tilde{f}\|_{0,\Omega} + \sqrt{\varepsilon}\,\|u\|_{k+1,\Omega}\}.
\end{aligned} \tag{4.13}$$

by (4.1) and Theorems 4.1 and 4.3. Thus we proved the following

THEOREM 4.4. *Let $\omega$ be a bounded connected domain in $\mathbf{R}^n$ with a $C^{k,1}$ boundary for integer $k > \max\{0, \frac{n}{2} - 1\}$, strictly contained in the polygonal open set $\Omega$.*

*If the solution* u *of problem* (2.3) *verifies* $u \in H^{k+1}(\omega)$, *then there exists a constant $C_3$ independent of h such that*

$$(4.14) \qquad \|u_h^\varepsilon - u\|_{1,\omega} \leqslant C_3\{h^k\|u\|_{k+1,\omega} + \sqrt{\varepsilon}\ \|\tilde{f}\|_{0,\Omega} + \sqrt{\varepsilon}\ \|u\|_{k+1,\omega}\},$$

*where $u_h^\varepsilon$ is the solution of problem (3.2) and* $\varepsilon > 0$.

Therefore the optimal choice of $\varepsilon$ is $\varepsilon = Mh^{2s}$ where $M$ is constant and $s \geqslant k$ so that $\|u - u_h^\varepsilon\|_{1,\omega}$ is of order $h^k$.

4.2. *$L^2$-error*

Let us consider the following auxiliary problem

*Find $\phi$ in $H^1(\omega)$ such that*

$$(4.15) \qquad a_\omega(\phi, v) = \int_\omega (u - u_h^\varepsilon)v\, dx,\ \forall v \in H^1(\omega).$$

Since $u - u_h^\varepsilon \in H^1(\omega)$, we have

$$(4.16) \qquad \|\phi\|_{2,\omega} \leqslant c_5\|u - u_h^\varepsilon\|_{0,\omega}.$$

Also we consider the auxiliary problem with fictitious domain/penalty approach

*Find $\phi^\varrho$ in $H_0^1(\Omega)$ such that*

$$(4.17) \qquad a_\omega(\phi^\varrho, v) + \varrho a_\Omega(\phi^\varrho, v) = \int_\omega (u - u_h^\varepsilon)v\, dx,\ \forall v \in H_0^1(\Omega).$$

and the finite element approximation of problem (4.17)

*Find $\phi_h^\varrho$ in $V_h$ such that*

$$a_\omega(\phi_h^\varrho, v_h) + \varrho a_\Omega(\phi_h^\varrho, v_h) = \int_\omega (u - u_h^\varepsilon) v_h \, dx, \ \forall v_h \in V_h. \tag{4.18}$$

Problems (4.17) and (4.18) have a unique solution in $H_0^1(\Omega)$ and $V_h$ respectively.
Let $v = u - u_h^\varepsilon$ in (4.15); we have then

$$\begin{aligned} \|u - u_h^\varepsilon\|_{0,\omega}^2 &= a_\omega(\phi, u - u_h^\varepsilon) \\ &= a_\omega(\phi - \phi_h^\varrho, u - u_h^\varepsilon) + a_\omega(\phi_h^\varrho, u - u_h^\varepsilon). \end{aligned} \tag{4.19}$$

Let $v_h = \phi_h^\varrho$ in (4.3), then (4.19) becomes

$$\|u - u_h^\varepsilon\|_{0,\omega}^2 = a_\omega(\phi - \phi_h^\varrho, u - u_h^\varepsilon) + \varepsilon a_\Omega(\phi_h^\varrho, u_h^\varepsilon) - \varepsilon \int_\omega \tilde{f} \phi_h^\varrho \, dx. \tag{4.20}$$

Thus we have

$$\begin{aligned} \|u - u_h^\varepsilon\|_{0,\omega}^2 \leq c_6 \{ &\|\phi - \phi_h^\varrho\|_{1,\omega} \|u - u_h^\varepsilon\|_{1,\omega} \\ &+ \varepsilon \|\phi_h^\varrho\|_{1,\Omega} \|u_h^\varepsilon\|_{1,\Omega} + \varepsilon \|\tilde{f}\|_{0,\Omega} \|\phi_h^\varrho\|_{1,\Omega} \}. \end{aligned} \tag{4.21}$$

Let us now assume that $n \leq 3$. Then for $\phi \in H^2(\omega)$ by Theorem 4.4, we have

$$\|\phi - \phi_h^\varrho\|_{1,\omega} \leq c_7 \{ h \|\phi\|_{2,\omega} + \sqrt{\varrho} \ \|\phi\|_{2,\omega} \} \tag{4.22}$$

since $\phi$, $\phi_h^\varrho$ are the solutions of problems (4.15) and (4.18) respectively. Then by (4.17), we have

$$\|\phi - \phi_h^\varrho\|_{1,\omega} \leq c_8 \, (h + \sqrt{\varrho}) \| u - u_h^\varepsilon\|_{0,\omega}. \tag{4.23}$$

For the estimates of $\|\phi_h^\varrho\|_{1,\Omega}$ and $\|u_h^\varepsilon\|_{1,\Omega}$ in (4.21), we have

$$\|\phi_h^\varrho\|_{1,\Omega} \leqslant \|E\phi - \phi_h^\varrho\|_{1,\Omega} + \|E\phi\|_{1,\Omega}, \tag{4.24}$$

$$\|u_h^\varepsilon\|_{1,\Omega} \leqslant \|Eu - u_h^\varepsilon\|_{1,\Omega} + \|Eu\|_{1,\Omega}, \tag{4.25}$$

since $\phi$ is in $H^2(\omega)$, it has an extension $E\phi$ in $H_0^2(\Omega)$.

For obtaining the estimates of $\|Eu - u_h^\varepsilon\|_{1,\Omega}$ and $\|E\phi - \phi_h^\varrho\|_{1,\Omega}$, we would like to use (4.7). Let $\eta_1 = \dfrac{1}{c_1}$, $\eta_2 = \eta_3 = \eta_5 = \dfrac{1}{6c_1}$, $\eta_4 = \eta_6 = 1$ in (4.7), then we have

$$\begin{aligned} \varepsilon\|Eu - u_h^\varepsilon\|_{1,\Omega}^2 \leqslant\ & 2c_1\{c_1\|v_h - Eu\|_{1,\omega}^2 + \varepsilon(6c_1 + 2)\|v_h - Eu\|_{1,\Omega}^2 \\ & + \varepsilon(6c_1 + 1)\|\tilde{f}\|_{0,\Omega}^2 + \varepsilon(6c_1 + 1)\|Eu\|_{1,\Omega}^2\} \\ \leqslant\ & c_9\{\|v_h - Eu\|_{1,\Omega}^2 + \varepsilon\|\tilde{f}\|_{0,\Omega}^2 + \varepsilon\|Eu\|_{1,\Omega}^2\}, \end{aligned} \tag{4.26}$$

for all $v_h \in V_h$. So

$$\|Eu - u_h^\varepsilon\|_{1,\Omega} \leqslant c_{10}\{\frac{1}{\sqrt{\varepsilon}}\|v_h - Eu\|_{1,\Omega} + \|\tilde{f}\|_{0,\Omega} + \|Eu\|_{1,\Omega}\},\ \forall v_h \in V_h. \tag{4.27}$$

Then let the conditions in Theorem 4.3 hold, we have

$$\begin{aligned} \|Eu - u_h^\varepsilon\|_{1,\Omega} & \leqslant c_{11}\{\frac{1}{\sqrt{\varepsilon}}\|Eu - \Pi_h Eu\|_{1,\Omega} + \|\tilde{f}\|_{0,\Omega} + \|Eu\|_{1,\Omega}\} \\ & \leqslant c_{12}\{\frac{h^k}{\sqrt{\varepsilon}}\|Eu\|_{k+1,\Omega} + \|\tilde{f}\|_{0,\Omega} + \|Eu\|_{1,\Omega}\} \\ & \leqslant c_{12}\{\left(\frac{h^k}{\sqrt{\varepsilon}} + 1\right)\|Eu\|_{k+1,\Omega} + \|\tilde{f}\|_{0,\Omega}\}. \end{aligned} \tag{4.28}$$

Similarly to (4.27), for $\|E\phi - \phi_h^\varrho\|_{1,\Omega}$ we have

$$\|E\phi - \phi_h^\varrho\|_{1,\Omega} \leqslant c_{13}\{\frac{1}{\sqrt{\varrho}}\|E\phi - v_h\|_{1,\Omega} + \|E\phi_{1,\Omega}\},\ \forall v_h \in V_h. \tag{4.29}$$

since there is no $\tilde{f}$ term in (4.18). Then for $E\phi \in H_0^2(\Omega)$ by Theorem 4.3, we have

$$\|E\phi - \phi_h^{\varrho}\|_{1,\Omega} \leqslant c_{13}\{\frac{1}{\sqrt{\varrho}} \| E\phi - \Pi_h E\phi\|_{1,\Omega} + \|E\phi\|_{1,\Omega}\}$$

(4.30)

$$\leqslant c_{14}\left(\frac{h}{\sqrt{\varrho}} + 1\right)\|E\phi\|_{2,\Omega}.$$

The by (4.24) and (4.30), we have

$$\|\phi^{\varrho}{}_h\|_{1,\Omega} \leqslant c^{14}\left(\frac{h}{\sqrt{\varrho}} + 1\right)\|E\phi\|_{2,\Omega} + \|E\phi\|_{1,\Omega}$$

$$\leqslant c_{15}\left(\frac{h}{\sqrt{\varrho}} + 1\right)\|E\phi\|_{2,\Omega}$$

$$\leqslant c_{16}\left(\frac{h}{\sqrt{\varrho}} + 1\right)\|\phi\|_{2,\omega}, \text{ by (4.1)}$$

(4.31)

$$\leqslant c_{17}\left(\frac{h}{\sqrt{\varrho}} + 1\right)\|u - u_h^{\varepsilon}\|_{0,\omega}, \text{ by (4.16)}.$$

Similarly by (4.25) and (4.28), we have

$$\|u_h^{\varepsilon}\|_{1,\Omega} \leqslant c_{12}\left\{\left(\frac{h^k}{\sqrt{\varepsilon}} + 1\right)\|Eu\|_{k+1,\Omega} + \|\tilde{f}\|_{0,\Omega}\right\} + \|Eu\|_{1,\Omega}$$

(4.32)

$$\leqslant c_{18}\left\{\left(\frac{h^k}{\sqrt{\varrho}} + 1\right)\|Eu\|_{k+1,\Omega} + \|\tilde{f}\|_{0,\Omega}\right\}$$

$$\leqslant c_{19}\left\{\left(\frac{h^k}{\sqrt{\varepsilon}} + 1\right)\|u\|_{k+1,\omega} + \|\tilde{f}\|_{0,\Omega}\right\}, \text{ by (4.1)}.$$

Thus by (4.23), (4.31) and (4.32), (4.21) becomes

$$\|u - u_h^\varepsilon\|_{0,\omega}^2 \leqslant c_{20}\{(h + \sqrt{\varrho}\,)\|u - u_h^\varepsilon\|_{0,\omega}\|u - u_h^\varepsilon\|_{1,\omega}$$

$$+ \varepsilon\left(\frac{h}{\sqrt{\varrho}} + 1\right)\|u - u_h^\varepsilon\|_{0,\omega}\{\left(\frac{h^k}{\sqrt{\varepsilon}} + 1\right)\|u\|_{k+1,\omega} + \|\tilde{f}\|_{0,\Omega}\} \tag{4.33}$$

$$+ \varepsilon\left(\frac{h}{\sqrt{\varrho}} + 1\right)\|u - u_h^\varepsilon\|_{0,\omega}\|\tilde{f}\|_{0,\Omega}\}.$$

Hence we have

$$\|u - u_h^\varepsilon\|_{0,\omega} \leqslant c_{20}\ \{(h + \sqrt{\varrho}\,)\ \|u - u_h^\varepsilon\|_{0,\omega}$$

$$+ \varepsilon\left(\frac{h}{\sqrt{\varrho}} + 1\right)\{\left(\frac{h^k}{\sqrt{\varepsilon}} + 1\right)\|u\|_{k+1,\omega} + 2\|\tilde{f}\|_{0,\Omega}\}$$

$$\leqslant c_{20}\{(h + \sqrt{\varrho}\,)\left((h^k + \sqrt{\varepsilon}\,)\|u\|_{k+1,\omega} + \sqrt{\varepsilon}\,\|\tilde{f}\|_{0,\Omega}\right) \tag{4.34}$$

$$+ \varepsilon\left(\frac{h}{\sqrt{\varrho}} + 1\right)\{\left(\frac{h^k}{\sqrt{\varepsilon}} + 1\right)\|u\|_{k+1,\omega} + 2\|\tilde{f}\|_{0,\Omega}\}\}.$$

Therefore we have the following result.

THEOREM 4.5. *Assume that the conditions in Theorem 4.4 hold and* $n \leqslant 3$. *Then there exists a constant* $C_4$ *independent of* $h$ *such that*

$$\|u - u_h^\varepsilon\|_{0,\omega} \leqslant C_4\{(h + \sqrt{\varrho}\,)\left((h^k + \sqrt{\varepsilon}\,)\|u\|_{k+1,\omega} + \sqrt{\varepsilon}\,\|\tilde{f}\|_{0,\Omega}\right)$$

$$+ \varepsilon\left(\frac{h}{\sqrt{\varrho}} + 1\right)\{\left(\frac{h^k}{\sqrt{\varepsilon}} + 1\right)\|u\|_{k+1,\omega} + 2\|\tilde{f}\|_{0,\Omega}\}\} \tag{4.35}$$

*where* $u$ *is the solution of problem (2.3),* $u_h^\varepsilon$ *is the solution of problem (3.2) and* $\varepsilon, \varrho > 0$.

Let $\varepsilon = M_1 h^{2s}$ and $\varrho = M_2 h^{2l}$ where $M_1$, $M_2$ are constants and $s$, $l$ are integers. Then the optimal choices of $\varepsilon$ and $\varrho$ are $s \geqslant k$ and $s \geqslant l > 0$ so that $\|u - u_h^\varepsilon\|_{0,\omega}$ is of order $h^{k+1}$.

## 5. Numerical experiments

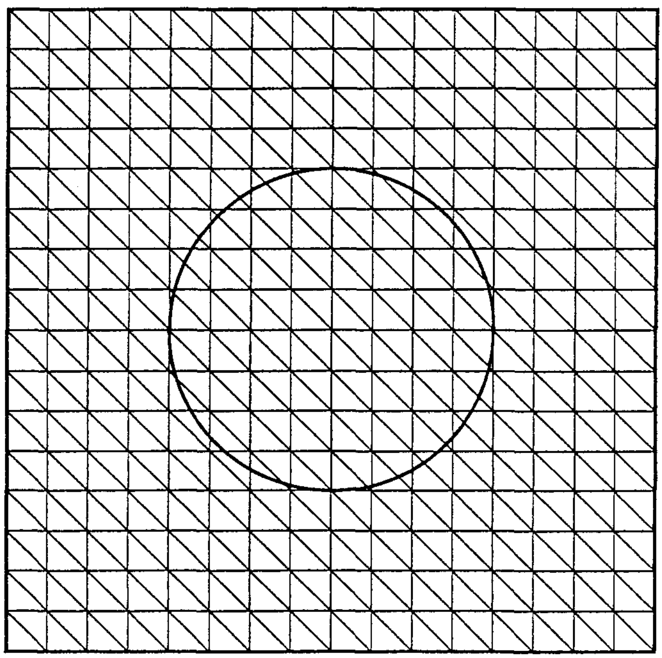

*Figure 5.1*

We consider the following test problem. Let $\omega = \{(x,y) | x^2 + y^2 \leqslant \frac{1}{4}\}$ and $\Omega = (-1, 1) \times (-1, 1)$, and let $u(x,y) = x^3 - y^3$ be the solution of the following Neumann problem

$$u - \Delta u = f \text{ in } \omega,$$

$$\frac{\partial u}{\partial n} = g \text{ on } \gamma;$$

we then have $f(x,y) = x^3 - y^3 - 6(x-y)$ and $g(x,y) = 6(x^3 - y^3)$.

In numerical experiment, we use 2-simplices of type (1) finite elements, so

$$V_h = \{v_h | v_h \in H_0^1(\Omega) \cap \mathbf{C}^0(\bar{\Omega}),\ v_h|_T \in P_1,\ \forall T \in T_h\}$$

where $T_h$ is a regular uniform triangulations of $\Omega$ (e.g., see Figure 5.1) and $P_1$ is the space of the polynomials in 2 variables of degree 1. The linear systems have been solved via a *Cholesky factorization.* In Tables 5.1 and 5.2, we list the relative $H^1$ and $L^2$ errors with $\varepsilon$ going from $10^{-1}$ to $10^{-10}$. The $H^1$ errors are of order $h$, the $L^2$ errors being of order $h^2$ if $\varepsilon$ is small enough. These are what we expect from Theorems 4.4 and 4.5.

*Table 5.1*

| | $\|u - u_h^\varepsilon\|_{1,\omega}/\|u\|_{1,\omega}$ | | |
|---|---|---|---|
| $\varepsilon$ | $h = 1/8$ | $h = 1/16$ | $h = 1/32$ |
| $10^{-1}$ | 1.27759362699 | 1.62154953409 | 2.16374221811 |
| $10^{-2}$ | 0.32358749061 | 0.26197911257 | 0.29164753716 |
| $10^{-3}$ | 0.19417803703 | $8.3662335694\times10^{-2}$ | $5.0478305305\times10^{-2}$ |
| $10^{-4}$ | 0.18043329784 | $6.5094253508\times10^{-2}$ | $2.5674105973\times10^{-2}$ |
| $10^{-5}$ | 0.17899468270 | $6.3208218876\times10^{-2}$ | $2.3200437070\times10^{-2}$ |
| $10^{-6}$ | 0.17884885920 | $6.3018799334\times10^{-2}$ | $2.2953287236\times10^{-2}$ |
| $10^{-7}$ | 0.17883425279 | $6.2999847513\times10^{-2}$ | $2.2928574657\times10^{-2}$ |
| $10^{-8}$ | 0.17883279190 | $6.2997952230\times10^{-2}$ | $2.2926103423\times10^{-2}$ |
| $10^{-9}$ | 0.17883264581 | $6.2997762701\times10^{-2}$ | $2.2925853300\times10^{-2}$ |
| $10^{-10}$ | 0.17883263120 | $6.2997743748\times10^{-2}$ | $2.2925831587\times10^{-2}$ |

*Table 5.2*

| | $\|u - u_h^\varepsilon\|_{0,\omega}/\|u\|_{0,\omega}$ | | |
|---|---|---|---|
| $\varepsilon$ | $h = 1/8$ | $h = 1/16$ | $h = 1/32$ |
| $10^{-1}$ | 1.48056340340 | 1.32677581622 | 1.23291488149 |
| $10^{-2}$ | 0.37017875130 | 0.21327731659 | 0.16603820621 |
| $10^{-3}$ | 0.21946578156 | $6.6884846802\times 10^{-2}$ | $2.8487414845\times 10^{-2}$ |
| $10^{-4}$ | 0.20345479970 | $5.1556945153\times 10^{-2}$ | $1.4246838113\times 10^{-2}$ |
| $10^{-5}$ | 0.20178058788 | $4.9997028789\times 10^{-2}$ | $1.2813815642\times 10^{-2}$ |
| $10^{-6}$ | 0.20161092886 | $4.9840331005\times 10^{-2}$ | $1.2670335191\times 10^{-2}$ |
| $10^{-7}$ | 0.20159393549 | $4.9824652763\times 10^{-2}$ | $1.2655985068\times 10^{-2}$ |
| $10^{-8}$ | 0.20159223587 | $4.9823084852\times 10^{-2}$ | $1.2654550035\times 10^{-2}$ |
| $10^{-9}$ | 0.20159206590 | $4.9822928061\Omega 0^{-2}$ | $1.2654406531\times 10^{-2}$ |
| $10^{-10}$ | 0.20159204898 | $4.9822912381\times 10^{-2}$ | $1.2654392181\times 10^{-2}$ |

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