ERROR ESTIMATES FOR TWO-PHASE STEFAN PROBLEMS IN SEVERAL SPACE VARIABLES, I: LINEAR BOUNDARY CONDITIONS

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ABSTRACT - The enthalpy formulation of two-phase Stefan problems, with linear boundary conditions, is approximated by C^0-piecewise linear finite elements in space and backward-differences in time combined with a regularization procedure. Error estimates of L^2-type are obtained. For general regularized problems an order $\varepsilon^{1/2}$ is proved, while the order is shown to be ε for non-degenerate cases. For discrete problems an order $h^2\varepsilon^{-1}+h+\tau\varepsilon^{-1/2}+\tau^{2/3}$ is obtained. These orders impose the relations $\varepsilon \sim \tau \sim h^{4/3}$ for the general case and $\varepsilon \sim h \sim \tau^{2/3}$ for non-degenerate problems, in order to obtain rates of convergence $h^{2/3}$ or h respectively. Besides, an order $h|\log h|+\tau^{1/2}$ is shown for finite element meshes with certain approximation property. Also continuous dependence of discrete solutions upon the data is proved.

1. Introduction.

Many physical processes involving phase change phenomena give rise to parabolic free boundary problems of the Stefan type. In this paper we analyze the numerical approximation of multidimensional two-phase Stefan problems via the enthalpy formulation. We use C^0-piecewise linear finite elements in space and backward-differences in time combined with a regularization procedure.

The enthalpy formulation has been extensively studied in recent years both from theoretical and numerical viewpoint. We refer to the survey [19] by E. Magenes where many references and comments are given. We only point out that

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one advantage of this weak formulation is that explicit tracking of the free boundary is unnecessary. Therefore Stefan problems for which the behavior of the free boundary is unknown can be successfully treated.

In this paper we obtain error estimates for temperature and enthalpy, using a new variational technique based on an integral test function. We present the results for parabolic problems involving non-homogeneous and anisotropic media, linear convective effects, non-linear internal heating term and linear mixed boundary conditions.

Error estimates of this type were also proved in [15]. However our approach permits us to simplify the analysis and to improve some results on one hand, and to extend the study to more general parabolic operators and non-linear flux conditions on the other hand. In particular we deal with non-quasiuniform finite element meshes (except in § 5).

The outline of our paper is the following. We state the problem and basic assumptions in § 1. Next we obtain error estimates of L^2-type for the regularized problems in § 2, while in § 3 we show some a priori estimates for discrete solutions, and some auxiliary error estimates. In § 4 we obtain L^2-type error estimates for discrete and semidiscrete solutions, and also a result of continuous dependence of discrete solutions upon the data. Finally, in § 5 we prove an essentially linear rate of convergence under additional assumptions upon the initial temperature and the finite element mesh.

1.1. *Statement of the Problem*

Roughly speeking our aim is to study the singular parabolic problem

$$(1.1)\qquad \frac{\partial}{\partial t}\gamma(u)-\nabla_x(K(x)\cdot\nabla_x u)+b(x)\cdot\nabla_x u = f(x,t,u), \qquad \text{in } \Omega\times(0,T)$$

$$(1.2)\qquad \frac{\partial}{\partial\nu}u+p(x)u = g_1(x,t), \qquad \text{on } \Gamma_1\times(0,T)$$

$$(1.3)\qquad u = g_2(x,t), \qquad \text{on } \Gamma_2\times(0,T)$$

$$(1.4)\qquad \gamma(u) = \gamma_0(x), \qquad \text{on } \Omega\times\{0\},$$

where $\Omega\subset\mathbb{R}^M$ ($M\geq 1$) is a bounded smooth domain ($\partial\Omega=\Gamma_1\cup\Gamma_2$), $u=u(x,t)$ is the unknown (physically, the temperature), $\gamma(u)=\gamma(x,t,u)$ is the enthalpy function defined by

$$\gamma(u) = c(x,t,u)+\chi(u). \tag{1.5}$$

The function χ is the characteristic of $\mathbb{R}^+$, and c, f, K, b, p, g_1, g_2 and γ_0 are given functions with regularity properties we shall state in § 1.2.

Problem (1.1)-(1.4) has to be understood in a weak sense (see § 1.3) and naturally arises from standard heat tranfer problems with phase change, after applying the Kirchhoff transformation (see [4], [15], [17], [21], [22]). Then, thermal properties of the medium are contained in the function c, while χ represents the latent heat content which we suppose normalized with a unitary jump. The matrix K takes the non-homogeneity and anisotropy of the medium into account, the vector b measures the convection, while the right hand side of (1.1) represents the internal heating term.

The side boundary conditions are of two types: on Γ_1 there is a linear flux condition and p ($p\geq 0$) measures the permeability to heat of the boundary, whereas on Γ_2 the temperature is imposed. We remark that either Γ_1 or Γ_2 could be empty.

1.2. *Basic Assumptions and Notation.*

(1.6) $\Omega\subset\mathbb{R}^M$ ($M\geq 1$) is a bounded domain with $\partial\Omega\in C^2$ (or optionally Ω is convex). We denote $Q=\Omega\times(0,T)$, where $0<T<\infty$ is fixed.

(1.7) $c(u)=c(x,t,u)\in C^{0,1}(\bar{Q}\times\mathbb{R})$, $0<\lambda_1\leq c_u(x,t,u)\leq\lambda_2$, for $(x,t,u)\in\bar{Q}\times\mathbb{R}$.

(1.8) $f(u) = f(x,t,u)$ is Hölder continuous with respect to (t,u) uniformly in $x\in\Omega$, more precisely there exists a constant $F>0$ such that

$$|f(x,t_1,u_1)-f(x,t_2,t_2)| \leq F(|u_1-u_2|+|t_1-t_2|^{2/3}).$$

(1.9) $K = K(x)\in C^{0,1}(\bar{\Omega},\mathbb{R}^{M\times M})$ is a symmetric and uniformly positive definite matrix.

(1.10) $b = b(x)\in L^\infty(\Omega,\mathbb{R}^M)$.

(1.11) $p = p(x)\in L^\infty(\Gamma_1)$, $p\geq 0$.

(1.12) $g_1 = g_1(x,t) \in H^1(0,T;L^2(\Gamma_1))$.

(1.13) $g_2 = g_2(x,t) \in H^2(Q)$ (we consider an extension to Q of the Dirichlet datum g_2).

(1.14) $u_0 = u_0(x) \in H^1(\Omega) \cap L^\infty(\Omega)$.

(1.15) $\gamma_0 = \gamma_0(x) \in \gamma(x,0,u_0(x))$, i.e. γ_0 is a section of the maximal monotone operator γ.
Finally we denote

(1.16) $V(g) = \{\phi \in H^1(\Omega): \phi = g \text{ on } \Gamma_2\}$, $\quad V^*$ dual of $V(0)$,

(1.17) $(\cdot,\cdot)_{L^2(\Omega)}, <\cdot,\cdot>_{L^2(\Gamma_1)}$ scalar products in $L^2(\Omega)$ and $L^2(\Gamma_1)$,

(1.18) $a(u,v) = (K\cdot\nabla u, \nabla v)_{L^2(\Omega)} + (b\cdot\nabla u, v)_{L^2(\Omega)} + <pu,v>_{L^2(\Gamma_1)}$, for $u,v \in H^1(\Omega)$.

1.3. *The Enthalpy Formulation*

This weak variational formulation was introduced by S. Kamenomostskaya [16], and then further studied by A. Friedman [13]. Both works consider Dirichlet boundary conditions and internal heating term independent of u (or linearly dependent on u). Afterwards A. Damlamian proved in [6] existence and uniqueness results for problems with mixed boundary conditions of the type (1.2)-(1.3) and non-linear heating terms, by a method of Lipschitz perturbations. More recently another type of nonlinearities have been considered ([3], [7], [8], [21], [30]).

We now state rigorously the Stefan problem (1.1)-(1.4). We say that *a pair of functions* $\{u,\omega\}$ *defined in* Q *is a weak solution of problem* (P) *iff*

(1.19) $$u \in L^2(0,T;V(g_2(\cdot,t))), \quad \omega \in L^2(Q),$$

(1.20) $$\omega(x,t) \in \gamma(x,t,u(x,t)), \text{ for a.e. } (x,t) \in Q,$$

(1.21) $$\int_0^T a(u,\phi) - (\omega,\phi_t)_{L^2(\Omega)} = \int_0^T <g_1,\phi>_{L^2(\Gamma_1)} + (f(u),\phi)_{L^2(\Omega)} + (\gamma_0,\phi(\cdot,0))_{L^2(\Omega)},$$

holds for all $\phi \in V(0)$ *with* $\phi(\cdot,T) = 0$.

For existence, uniqueness and global regularity we refer to [3], [6], [13], [14], [16], [17], [20], [21] and [30], and also to the survey [19] where a wide bibliography is given. It is known that

$$u \in L^\infty(Q) \cap L^\infty(0,T;H^1(\Omega)) \cap H^1(0,T;L^2(\Omega)). \tag{1.22}$$

Local regularity was recently proved by L. Caffarelli and L. Evans [2], and E. Di Benedetto ([7], [8]), who have obtained a modulus of continuity for weak solutions of (P). Although these regularity results seem to be optimal for a general two-phase Stefan problem, they do not imply the existence of a smooth interface, and in fact it could vary in a discontinuous manner (see [13]) even if the data are very smooth. In [23] the author has given a characterization of a class of non-degenerate problems of Stefan type for which the free boundary is a Lipschitz manifold; in this paper we will show how this further regularity improves the rate of convergence of regularized problems, obtaining an order ε instead of the order $\varepsilon^{1/2}$ valid for general problems (see § 2).

In order to achieve the error estimates the basic idea is to use an adequate integral test function in the weak formulation of the continuous problems (P) and (P_ε) (see (2.14)), and also in the discrete ($P_{\varepsilon,h,\tau}$) and semidiscrete problems ($P_{\varepsilon,h}$), ($P_{\varepsilon,\tau}$) (see (4.1), (4.24), (4.27)). In [15] a mapping defined by the inverse of $-\Delta_x$ subject to homogeneous Neumann boundary conditions is used, and the notion of weak solution is adapted in terms of this operator. This technique seems not to be applicable for non-linear flux conditions.

We finally point out that using the so-called freezing index, introduced by G. Duvaut [10], and M. Frémond [12], it is possible to obtain another weak formulation involving variational inequalities (see also [19] for details and references). Recently some results in the analysis of error estimates have been obtained (see [25]). Besides the non-linear semigroup of contractions in $L^1(\Omega)$ provides another theoretical approach and suggests some converging algorithms (see [1], [29]).

2. **Error Estimates for the Regularized Problems.**

2.1. *The Regularized Problems*

We begin this section by defining a family of nonlinear parabolic boundary value problems, regularized approximations to the Stefan problem (P): they are obtained by smoothing the enthalpy function. This procedure was proposed in some theoretical papers (see [2], [6], [13], [17], [21], [23], [30]) in order to prove existence, uniqueness and regularity of (P), but also in numerical works (see [4], [11], [14], [15], [19], [20], [22], [25], [29], [32], [33], [34]).

We consider for $\varepsilon>0$ (the regularization parameter) the function

$$\chi_\varepsilon(s) = \max(0,\min(s/\varepsilon,1)), \tag{2.1}$$

and then we approximate γ by the smooth function γ_ε defined by

$$\gamma_\varepsilon(s) = \gamma_\varepsilon(x,t,s) = c(x,t,s)+\chi_\varepsilon(s), \qquad \text{for } (x,t,s)\in Q\times\mathbb{R}. \tag{2.2}$$

We say that u_ε *is a weak solution of the regularized problem* (P_ε) *iff*

$$u_\varepsilon\in L^2(0,T;V(g_{2,\varepsilon}(\cdot,t))), \tag{2.3}$$

$$\int_0^T a(u_\varepsilon,\phi)-(\gamma_\varepsilon(u_\varepsilon),\phi_t)_{L^2(\Omega)} = \int_0^T <g_1,\phi>_{L^2(\Gamma_1)}+(f(u_\varepsilon),\phi)_{L^2(\Omega)} \tag{2.4}$$

$$+(\gamma_\varepsilon(u_{0,\varepsilon}),\phi(\cdot,0))_{L^2(\Omega)},$$

holds for all $\phi\in V(0)$ *with* $\phi(\cdot,T)=0$. Here $u_{0,\varepsilon}$ and $g_{2,\varepsilon}$ indicate approximations of u_0 and g_2 respectively (see (2.12) and (2.31)).

(2.5) *Remark*: problems (P_ε) have been defined using the same functions f, g_1, p, K and b as in (P) in order to simplify the argument but without loss of generality, being possible to consider some perturbations of these functions. At the same time $g_{2,\varepsilon}$ is a regularization of g_2 in the sense of Strang [28, p. 93]. Therefore $g_{2,\varepsilon}$ has some local regularity properties which shall be used in §§3-5 to define and estimate its interpolant. Clearly if g_2 is regular enough we can take $g_{2,\varepsilon}=g_2$//.

We now briefly comment on the global regularity results at our knowledge available in the literature. Problems (P_ε) have one and only one solution (see [3], [6], [13], [14], [16], [17], [19]. [21], [30] for details). In these works some a priori estimates for u_ε are derived and, by compactness methods, the existence of a solution u of problem (P) is proved. We now recall some of these regularity results for u_ε:

There exists a constant $C>0$ independent of ε such that the following estimates are fulfilled,

$$(2.6)\qquad \|u_\varepsilon\|_{L^\infty(Q)} \leqslant C,$$

$$(2.7)\qquad \|\nabla_x u_\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}+\|\frac{\partial}{\partial t} u_\varepsilon\|_{L^2(Q)} \leqslant C,$$

$$(2.8)\qquad \|\gamma_\varepsilon(u_\varepsilon)_t\|_{L^2(Q)} \leqslant C\varepsilon^{-1/2},$$

$$(2.9)\qquad \|\gamma_\varepsilon(u_\varepsilon)_t\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C, \qquad \text{if} \qquad \|\Delta u_{0,\varepsilon}\|_{L^1(\Omega)} \leqslant C^*,$$

where $C^*>0$ is also a constant independent of ε.

Estimates (2.6)-(2.7) were proved by A. Friedman [13], for Dirichlet boundary conditions and without internal heating term. The same proof with minor modifications can be extended to other boundary conditions and nonlinear internal sources as it was shown in [6] and [21] (see also [14], [15], [17], [19], [22], [30]). The relation (2.8) is an immediate consequence of the proof of (2.7). The estimate (2.9) was obtained by J. Jerome and M. Rose in [15] (see also [19], where an easier proof by A. Visintin is given). We finally point out that under the hypotheses of §1.2 the solution u_ε is Hölder continuous in $\bar{Q}$ (see [17, chapter V]), but its norm depends on ε.

2.2. *The Error Estimate*

In this paragraph we obtain an error estimate stronger than those proved in [11], [15] and [25] (see also [14, chapter 2]). We shall use the notation

$$(2.10)\qquad A_\varepsilon(u) = \{(x,t)\in Q : 0\leqslant u(x,t)\leqslant\varepsilon\},$$

$$(2.11)\qquad A_\varepsilon(u_0) = \{x\in\Omega : 0\leqslant u_0(x)\leqslant\varepsilon\}.$$

(2.12) *Remark*: suppose the initial non-degeneracy condition $|A_\varepsilon(u_0)|\leqslant C\varepsilon$ to hold, where C is a constant independent of ε. So we can choose $u_{0,\varepsilon}=u_0$ and obtain

$$\|\gamma(u_0)-\gamma_\varepsilon(u_{0,\varepsilon})\|_{L^2(\Omega)} \leqslant C\varepsilon^{1/2}.$$

Notice that this condition holds if $F_0=\{x\in\Omega: u_0(x)=0\}$ is locally a Lipschitz manifold and there exists $C>0$ such that $\frac{\partial}{\partial\nu} u_0^+\geq C$ on F_0//.

In what follows we shall denote $\chi(u)=\omega-c(x,t,u)$ (see (1.20)), thus $\chi=0$ ($=1$) if $u<0$ ($u>0$). Now we derive an elementary inequality which will be useful to obtain the desired error estimate.

(2.13) LEMMA: $(\chi(u)-\chi_\varepsilon(u_\varepsilon))(u-u_\varepsilon)\geq-\varepsilon\zeta(A_\varepsilon(u))$, *where* $\zeta(A)$ *denotes the characteristic function of A.*

Proof: Consider the decomposition of $\mathbb{R}$ into the intervals $(-\infty,0)$, $[0,\varepsilon]$ and $(\varepsilon,+\infty)$. If u and u_ε do not belong to the same interval, then either u or u_ε does not belong to $[0,\varepsilon]$. If $u\notin[0,\varepsilon]$ (or $u_\varepsilon\notin[0,\varepsilon]$) then $\chi(u)=\chi_\varepsilon(u)$ (or $\chi_\varepsilon(u_\varepsilon)=\chi(u_\varepsilon)$) and the inequality is satisfied, with zero on the right hand side, because χ_ε (or χ) is a monotone function. If u and u_ε belong to the same interval, the only non trivial case is: u, $u_\varepsilon\in[0,\varepsilon]$. Then we get,

$$(\chi(u)-\chi_\varepsilon(u_\varepsilon))(u-u_\varepsilon)\geq-\varepsilon\zeta(A_\varepsilon(u)\cap A_\varepsilon(u_\varepsilon))\geq-\varepsilon\zeta(A_\varepsilon(u)),$$

that completes the proof.//

Now we establish convergence rates for the regularized problems in a general setting, i.e. without having an a priori control on $|A_\varepsilon(u)|$. Notice that this is the case when the problem (P) has a mushy region. In what follows we shall use the notation:

$$\sigma_1(\varepsilon)=\|\gamma(u_0)-\gamma_\varepsilon(u_{0,\varepsilon})\|^2_{L^2(\Omega)},\qquad \sigma_2(\varepsilon)=\|g_2-g_{2,\varepsilon}\|^2_{L^2(0,T;H^1(\Omega))}.$$

(2.14) THEOREM. *Let* u *and* u_ε *be the solutions of the Stefan problem* (P) *and of the regularized problems* (P_ε) *respectively. Then there exists a constant* $C>0$ *independent of* ε, *such that*

$$\|u-u_\varepsilon\|_{L^2(Q)}\leq C(\varepsilon|A_\varepsilon(u)|+\sigma_1(\varepsilon)+\sigma_2(\varepsilon))^{1/2},\tag{2.15}$$

$$\|\gamma(u)-\gamma_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;V^*)}\leq C(\varepsilon|A_\varepsilon(u)|+\sigma_1(\varepsilon)+\sigma_2(\varepsilon))^{1/2}.\tag{2.16}$$

Proof: We shall give the proof in three steps. In the first one we assume $g_{2,\varepsilon}=g_2$ and obtain the error estimate (2.15) for temperatures. In the second step we deal with the case $g_{2,\varepsilon} \neq g_2$. Finally in the third step we achieve the error estimate (2.16) for enthalpies.

1st Step-Estimate (2.15): $g_{2,\varepsilon}=g_2$.

Let us consider the weak formulations of (P) and (P_ε), i.e. expressions (1.21) and (2.4). Then, after subtraction and reordering, we have the relation

$$(2.17) \quad \int_0^T \Big[(K\cdot\nabla_x(u-u_\varepsilon),\nabla_x\phi)_{L^2(\Omega)} - (c(u)-c(u_\varepsilon),\phi_t)_{L^2(\Omega)} + \langle p(u-u_\varepsilon),\phi\rangle_{L^2(\Gamma_1)} \Big]$$
$$= I + II + III$$
$$= \int_0^T \Big[(f(u)-f(u_\varepsilon),\phi)_{L^2(\Omega)} - (b\nabla_x(u-u_\varepsilon),\phi)_{L^2(\Omega)} + (\chi(u)-\chi_\varepsilon(u_\varepsilon),\phi_t)_{L^2(\Omega)} \Big]$$
$$+ (\gamma(u_0)-\gamma_\varepsilon(u_{0,\varepsilon}),\phi(\cdot,0))_{L^2(\Omega)} = IV + V + VI + VII,$$

for all $\phi \in V(0)$ with $\phi(\cdot,T)=0$. The idea of the proof is to use a suitable test function in (2.17). For t_0 fixed $(0 \leqslant t_0 \leqslant T)$, we propose the integral test function

$$(2.18) \qquad \phi(x,t) = \int_t^{t_0} (u-u_\varepsilon)(x,s)ds, \qquad \text{if } 0\leqslant t\leqslant t_0$$
$$= 0, \qquad \text{if } t_0 \leqslant t \leqslant T.$$

Clearly this function is admissible because u, $u_\varepsilon \in H^1(Q)$, and then $\phi \in H^1(Q)$, and besides $\phi=0$ on $\Gamma_2\times(0,T) \cup \{t=T\}$.

Now, inserting (2.18) in (2.17), the rest of this step consists in analysing each term. In this sense, recalling that K is a symmetric positive definite matrix independent of $t \in (0,T)$, we have

$$I = \int_0^{t_0} (K^{1/2}\cdot\nabla_x(u-u_\varepsilon), \int_t^{t_0} K^{1/2}\cdot\nabla_x(u-u_\varepsilon))_{L^2(\Omega)} = \frac{1}{2}\Big\|\int_0^{t_0} K^{1/2}\cdot\nabla_x(u-u_\varepsilon)\Big\|^2_{L^2(\Omega)}.$$

For II, using the assumption (1.7), we obtain

$$II = \int_0^{t_0} (c(u)-c(u_\varepsilon),u-u_\varepsilon)_{L^2(\Omega)} \geqslant \lambda_1 \int_0^{t_0} \|u-u_\varepsilon\|^2_{L^2(\Omega)}.$$

Recalling that p is $\geqslant 0$ and independent of t, we get

$$III = \int_0^{t_0} \langle p^{1/2}(u-u_\varepsilon), \int_t^{t_0} p^{1/2}(u-u_\varepsilon)\rangle_{L^2(\Gamma_1)} = \frac{1}{2}\Big\|\int_0^{t_0} p^{1/2}(u-u_\varepsilon)\Big\|^2_{L^2(\Gamma_1)}.$$

The right hand side of (2.17) will be analyzed in the same manner. From the Lipschitz property of f (see (1.8)), we have

$$IV = \int_0^{t_0} (\int_0^t f(u)-f(u_\varepsilon),u-u_\varepsilon)_{L^2(\Omega)} \leqslant \frac{\lambda_1}{4}\int_0^{t_0} \|u-u_\varepsilon\|^2_{L^2(\Omega)}+C\int_0^{t_0}(\int_0^t \|u-u_\varepsilon\|^2_{L^2(\Omega)}),$$

where we have used the Cauchy-Schwarz inequality and the elementary relation

$$(2.19) \qquad 2ab \leqslant \alpha a^2 + \frac{1}{\alpha}b^2 \qquad (a,b\in \mathbb{R},\alpha>0),$$

for $\alpha=\frac{\lambda_1}{2}$. This inequality will be used extensively in the sequel.

Taking into account the assumption (1.10) upon b and applying again the Cauchy-Schwarz inequality and (2.19), we obtain

$$V = -\int_0^{t_0} (b\int_0^t \nabla_x(u-u_\varepsilon),u-u_\varepsilon)_{L^2(\Omega)} \leqslant \frac{\lambda_1}{4}\int_0^{t_0} \|u-u_\varepsilon\|^2_{L^2(\Omega)}+C\int_0^{t_0}\|\int_0^t \nabla_x(u-u_\varepsilon)\|^2_{L^2(\Omega)}$$

$$\leqslant \frac{\lambda_1}{4}\int_0^{t_0} \|u-u_\varepsilon\|^2_{L^2(\Omega)}+C\int_0^{t_0}\|\int_0^t K^{1/2}\cdot\nabla_x(u-u_\varepsilon)\|^2_{L^2(\Omega)},$$

where we have used again that K is positive definite. Due to (2.13) we get

$$VI = \int_0^{t_0} (\chi(u)-\chi_\varepsilon(u_\varepsilon),u_\varepsilon-u)_{L^2(\Omega)} \leqslant \varepsilon|A_\varepsilon(u)|.$$

Finally for the last term in (2.17), using again the Cauchy-Schwarz inequality and (2.19), we have

$$VII = (\gamma(u_0)-\gamma_\varepsilon(u_{0,\varepsilon}),\int_0^{t_0} u-u_\varepsilon)_{L^2(\Omega)} \leqslant \frac{\lambda_1}{4}\int_0^{t_0} \|u-u_\varepsilon\|^2_{L^2(\Omega)}+C\sigma_1(\varepsilon).$$

Hence, by substitution of the previous estimates in (2.17), we easily see that

$$\|\int_0^{t_0} K^{1/2}\cdot\nabla_x(u-u_\varepsilon)\|^2_{L^2(\Omega)}+\int_0^{t_0}\|u-u_\varepsilon\|^2_{L^2(\Omega)}+\|\int_0^{t_0} p^{1/2}(u-u_\varepsilon)\|^2_{L^2(\Gamma_1)}$$

$$\leqslant C(\varepsilon|A_\varepsilon(u)|+\sigma_1(\varepsilon)) + C\int_0^{t_0}(\|\int_0^t K^{1/2}\cdot\nabla_x(u-u_\varepsilon)\|^2_{L^2(\Omega)}+\int_0^t \|u-u_\varepsilon\|^2_{L^2(\Omega)}).$$

Now, an application of the Gronwall inequality allows us to conclude that the following relation holds

$$(2.20)\qquad \Big\|\int_0^{t_0} K^{1/2}\cdot\nabla_x(u-u_\varepsilon)\Big\|^2_{L^2(\Omega)}+\int_0^{t_0}\|u-u_\varepsilon\|^2_{L^2(\Omega)}+\Big\|\int_0^{t_0} p^{1/2}(u-u_\varepsilon)\Big\|^2_{L^2(\Gamma_1)}$$

$$\leq C(\varepsilon|A_\varepsilon(u)|+\sigma_1(\varepsilon)),$$

for all t_0, $0<t_0\leq T$, where $C>0$ is a constant depending only on the data and T. Clearly (2.20) implies the desired estimate (2.15), for $g_{2,\varepsilon}=g_2$, which completes the proof of this step.

2nd Step-Estimate (2.15): $g_{2,\varepsilon}\neq g_2$

Instead of (2.18) we take the following integral test function

$$\phi(x,t) = \phi_1-\phi_2 = \int_t^{t_0}[(u-u_\varepsilon)-(g_2-g_{2,\varepsilon})](x,s)ds, \qquad \text{if } 0\leq t\leq t_0$$

$$\phi(x,t) = 0, \qquad \text{if } t_0\leq t\leq T.$$

The analysis of each term in (2.17) for $\phi=\phi_1$ is exactly the same as in the previous step. For $\phi=\phi_2$ we proceed in almost the same manner as before, obtaining the inequality (2.20) with $\sigma_2(\varepsilon)$ on the right hand side.

3rd Step-Estimate (2.16)

Let η be a function belonging to V(0), and t_0 be an arbitrary point in (0,T). For $\delta>0$ small enough, consider the function $\phi(x,t)=\eta(x)\xi_\delta(t)$, where ξ_δ is the piecewise-linear function equal to one in $[0,t_0-\delta]$ and zero in $[t_0+\delta,T]$. Obviously ϕ is an admissible test function in (2.4), whence we obtain

$$\frac{1}{2\delta}\int_{t_0-\delta}^{t_0+\delta}(\gamma(u)-\gamma_\varepsilon(u_\varepsilon),\eta)_{L^2(\Omega)}$$

$$=\int_0^T \xi_\delta[-(K^{1/2}\cdot\nabla_x(u-u_\varepsilon),K^{1/2}\cdot\nabla_x\eta)_{L^2(\Omega)}-(b\nabla_x(u-u_\varepsilon),\eta)_{L^2(\Omega)}$$

$$-\langle p(u-u_\varepsilon),\eta\rangle_{L^2(\Gamma_1)}+(f(u)-f(u_\varepsilon),\eta)_{L^2(\Omega)}]+(\gamma(u_0)-\gamma_\varepsilon(u_{0,\varepsilon}),\eta)_{L^2(\Omega)}.$$

Now taking into account that ξ_δ converges (in L^2) to the characteristic function of

$(0,t_0)$, and employing the Lebesgue differentiation lemma on the left hand side, we have

$$((\gamma(u)-\gamma_\varepsilon(u_\varepsilon))(\cdot,t_0),\eta)_{L^2(\Omega)} \leqslant (\|\int_0^{t_0} K^{1/2}\nabla_x(u-u_\varepsilon)\|_{L^2(\Omega)}+\|\int_0^{t_0} p^{1/2}(u-u_\varepsilon)\|_{L^2(\Gamma_1)}$$

$$+\int_0^{t_0} \|u-u_\varepsilon\|_{L^2(\Omega)}+\sigma_1(\varepsilon)^{1/2})\cdot\|\eta\|_{H^1(\Omega)},$$

for a.e. t_0, $0<t_0<T$. Next, from the last inequality and (2.20), we obtain

$$((\gamma(u)-\gamma_\varepsilon(u_\varepsilon))(\cdot,t_0),\eta)_{L^2(\Omega)} \leqslant C(\varepsilon|A_\varepsilon(u)|+\sigma_1(\varepsilon)+\sigma_2(\varepsilon)))^{1/2}\|\eta\|_{H^1(\Omega)},$$

which implies the desired estimate (2.16). Then the theorem has been proved//.

This theorem says that the error introduced by the regularization procedure itself is of order $(\varepsilon|A_\varepsilon(u)|)^{1/2}$. In particular if the behavior of u near the free boundary is unknown or mushy regions can appear there is not an estimate on $|A_\varepsilon(u)|$, and the error is $\varepsilon^{1/2}$ (see [15], [25] where the same order was obtained). At the same time it shows that $u_{0,\varepsilon}$ and $g_{2,\varepsilon}$ have to be chosen in such a way that $\sigma_1(\varepsilon)$, $\sigma_2(\varepsilon)\leqslant C\varepsilon$, in order to get an error of order $\varepsilon^{1/2}$. By virtue of (1.13), $g_{2,\varepsilon}$ defined in (2.5) verifies this condition. Besides, if u_0 satisfies the non-degeneracy property $|A_\varepsilon(u_0)|\leqslant C\varepsilon$, the choice $u_{0,\varepsilon}=u_0$ is possible (see Remark (2.12)).

Now we summarize this latter comment.

(2.21) COROLLARY. *Assume* $\sigma_1(\varepsilon)$, $\sigma_2(\varepsilon)\leqslant C\varepsilon$. *Then there exists a constant* $C>0$ *independent of* ε, *such that*

$$\|u-u_\varepsilon\|_{L^2(Q)}+\|\gamma(u)-\gamma_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;V^*)} \leqslant C\varepsilon^{1/2}.$$

2.3. *The Non-Degenerate Case*: $|A_\varepsilon(u)|\leqslant C\varepsilon$.

We say that (P) *is a non-degenerate problem if there exists a constant* $C>0$ *such that*

$$|A_\varepsilon(u)| \leqslant C\varepsilon. \tag{2.22}$$

Clearly this condition excludes explicitly mushy region problems, and also other cases with almost regular free boundary (for instance, consider the harmonic function $u=xy$ in the unit ball of $\mathbb{R}^2$, for which $|A_\varepsilon(u)|\leqslant C\varepsilon|\log\varepsilon|$).

A characterization of a class of such problems is given in [23]. Now we are going to state the essential assumptions upon the data and recall the main result of that paper.

Assume, for the sake of simplicity, that Ω is an annulus with boundaries Γ_1 and Γ_2 and that a Dirichlet boundary condition g_1 is imposed on Γ_1. Suppose that the following hypotheses hold: *there exist constants* α, θ_1, θ_2, μ_1, μ_2 *with* α, $\theta_1>0$, $\theta_2<0$, $0<\mu_2<\mu_1$, *such that*

(2.23) $\dfrac{\partial}{\partial t} g_i \geqslant \alpha$, *on* Γ_i $(i=1,2)$,

(2.24) $(-1)^i(g_i-\theta_i) \geqslant 0$, *on* Γ_i $(i=1,2)$,

(2.25) $F_0 = \{x \in \Omega: u_0(x)=0\}$ *is locally a Lipschitz manifold,*

(2.26) $u_0 \in C^{0,1}(\bar{\Omega})$,

(2.27) $\Delta_x u_0 + f(u_0) \geqslant \alpha$, in $D'(\Omega)$,

(2.28) u_0^+ (*resp.* u_0^-) $\in C^1(\bar{G}^+)$ (resp. $\bar{G}^-$), *where* G *is a neighborhood of* F_0, *and*

$$\frac{\partial}{\partial \nu} u_0^+ \geqslant \mu_1, \quad \frac{\partial}{\partial \nu} u_0^- \leqslant \mu_2 \qquad \text{on } F_0,$$

(2.29) $f(0) \leqslant 0$.

Then we have,

(2.30) THEOREM ([23]). *There exists a constant* $C>0$ *such that* $|A_\varepsilon(u)| \leqslant C\varepsilon$. *Moreover the free boundary is a Lipschitz manifold*: $t=\sigma(x)$.

From (2.14) and (2.30) we conclude that the rate of convergence could be of order ε if $u_{0,\varepsilon}$ and $g_{2,\varepsilon}$ are chosen so that: $\sigma_1(\varepsilon)$, $\sigma_2(\varepsilon) \leqslant C\varepsilon^2$.

(2.31) *Remark*: from (2.25) and (2.28), u_0 satisfies: $|A_\varepsilon(u_0)| \leqslant C\varepsilon$. Therefore, $u_{0,\varepsilon}$ can be taken as

$$u_{0,\varepsilon} = \begin{cases} u_0, & \text{if } u_0 \geq \varepsilon \text{ or } u_0 < 0 \\ \dfrac{\varepsilon}{1+\varepsilon}(u_0+1), & \text{if } \varepsilon > u_0 \geq \varepsilon^2 \\ \dfrac{1+\varepsilon^2}{\varepsilon(1+\varepsilon)} u_0, & \text{if } \varepsilon^2 > u_0 \geq 0, \end{cases}$$

and verifies $\sigma_1(\varepsilon) \leq C\varepsilon^2$, $\|u_{0,\varepsilon}\|_{H^1(\Omega)} \leq C$//.

Now we state the order of convergence for non-degenerate cases.

(2.32) COROLLARY. *Assume that* $\sigma_1(\varepsilon)$, $\sigma_2(\varepsilon) \leq C\varepsilon^2$ *and* (P) *is non-degenerate. Then there exists a constant* $C>0$ *independent of* ε, *such that*

$$\|u-u_\varepsilon\|_{L^2(Q)} + \|\gamma(u)-\gamma_\varepsilon(u_\varepsilon)\|_{L^\infty(0,T;V^*)} \leq C\varepsilon.$$

2.4. *Stability of Solutions*

We end this section with another application of the variational technique developed in (2.14). In fact it is easy to prove the continuous dependence of u (and u_ε) upon the data. Results of this type are well known (see [13, p. 65]; [6, p. 1032]; [21, p. 205]), but our approach seems to be easier.

We shall omit the proof because it follows the lines of (2.14).

(2.33) THEOREM. *Let* u *and* û *be the solution of* (P) *with data* u_0, f, g_2 *and* $\hat{u}_0$, $\hat{f}$, $\hat{g}_2$ *respectively. Then there exists a constant* $C>0$ *independent of the data, such that*

$$\int_0^T (\gamma(u)-\gamma(\hat{u}), u-\hat{u})_{L^2(\Omega)} + \sup_{0<t<T} \|\gamma(u)-\gamma(\hat{u})\|^2_{V^*}$$

$$\leq C\Big[\|\gamma(u_0)-\gamma(\hat{u}_0)\|^2_{L^2(\Omega)} + \int_0^T \|f(u)-\hat{f}(u)\|^2_{L^2(\Omega)} + \Big(\int_0^T \|g_2-\hat{g}_2\|^2_{H^1(\Omega)}\Big)^{1/2}\Big].$$

We only point out that the adequate integral test function is:

$$\phi(x,t) = \int_t^{t_0} (u-\hat{u})-(g_2-\hat{g}_2), \qquad \text{if } 0<t<t_0$$

$$= 0, \qquad \text{if } t_0 \leq t \leq T.$$

3. **The Discrete Problems.**

This section is devoted to define and analyse the discrete problems. We shall prove some a priori estimates for discrete solutions and also some auxiliary error estimates we need in § 4 to obtain the desired L^2-rate of convergence.

3.1. *Notation and Basic Assumptions*

Let $\mathscr{D}_h$ be a family of finite elements and let h_s (resp. ϱ_s) be the diameter of the smallest ball containing $S \in \mathscr{D}_h$ (resp. greatest ball contained in S). Then the size of the mesh $\mathscr{D}_h$ is defined as $h = \max_{S\in\mathscr{D}_h} h_s$.

We suppose that $\mathscr{D}_h$ is *regular* (see [5, p. 132 or 247]). If $\mathscr{D}_h$ is a family of n-simplexes the regularity means that there exists $\sigma>0$ independent of h such that $h_s \leqslant \sigma\varrho_s$ for all $S \in \mathscr{D}_h$.

For the sake of simplicity we shall assume that

$$\Omega = \Omega_h = \bigcup_{S\in\mathscr{D}_h} S$$

The case $\Omega \neq \Omega_h$ involves some technical calculations near the fixed boundary $\partial\Omega$ which are omitted in this work (see [22], where a Stefan problem with Dirichlet boundary conditions and $\partial\Omega \in C^2$ was considered).

Let us now define the following finite element spaces,

$$(3.1) \qquad V_h = \{\phi \in C^0(\bar{\Omega}) \colon \phi|_s \text{ is linear for all } S \in \mathscr{D}_h\},$$

$$(3.2) \qquad V_h(g) = V_h \cap \{\phi \in C^0(\bar{\Omega}) \colon \phi = g^I \text{ on } \Gamma_2\},$$

where g^I is the C^0-piecewise linear interpolant of g.

The regularity of $\mathscr{D}_h$ yields the following *approximation property* of V_h (see [28], and also [5, chapter 3]),

$$(3.3) \qquad \inf_{\eta\in V_h} \|v-\eta\|_{H^k(\Omega)} \leqslant Ch^{2-k}\|v\|_{H^2(\Omega)}, \qquad \text{for all } v \in H^2(\Omega),\ k=0,1.$$

(3.4) *Remark*: we do not need to *assume* that $\mathscr{D}_h$ is quasiuniform (i.e. there exists $\gamma>0$ fixed such that $h \leqslant \gamma\varrho_s$ for all $S \in \mathscr{D}_h$ (see [5, p. 140])) as done in [15]//.

(3.5) *Remark*: taking into account the global regularity of u_ε (and u) stated in (2.6)-(2.9), we see that it is not strong enough to justify the use of interpolant polinomials of degree greater than one//.

We consider now an auxiliary elliptic boundary value problem, that we shall use to obtain the desired error estimate. Let $b(\cdot,\cdot)$ be the following bilinear, continuous, coercive ($p \geqslant 0$) and symmetric form in $H^1(\Omega)$

$$b(\eta,\phi) = (K(x)\nabla\eta,\nabla\phi)_{L^2(\Omega)}+(\eta,\phi)_{L^2(\Omega)}+\langle p\eta,\phi\rangle_{L^2(\Gamma_1)}.$$

We say that a function *v is the solution of the auxiliary problem* (A) *iff*

$$(3.6) \qquad v \in V(g_2)$$

$$(3.7) \qquad b(v,\phi) = (\Psi,\phi)_{L^2(\Omega)}+\langle g_1,\phi\rangle_{L^2(\Gamma_1)}, \qquad \textit{for all } \phi \in V(0).$$

Let us suppose that (A) is *regular* (see [5, p. 138]), i.e. if $\Psi \in L^2(\Omega)$, $g_1 \in H^{1/2}(\Gamma_1)$, $g_2 \in H^{3/2}(\Gamma_2)$ then $v \in H^2(\Omega)$ and there exists $C>0$ such that

$$(3.8) \qquad \|v\|_{H^2(\Omega)} \leqslant C(\|\Psi\|_{L^2(\Omega)}+\|g_1\|_{H^{1/2}(\Gamma_1)}+\|g_2\|_{H^{3/2}(\Gamma_2)}).$$

(3.9) *Remark*: the latter condition imposes certain regularity restrictions on $\partial\Omega$ and the sets Γ_1, Γ_2. In fact, being either Ω convex or $\partial\Omega \in C^2$, if $\Gamma_1=\emptyset$ (or $\Gamma_2=\emptyset$) then (A) is regular (see [18, p. 148 – Vol. I]). In this case we would actually have a Dirichlet or a Neumann condition respectively, in the whole fixed boundary $\partial\Omega$. If Ω is an annulus and Γ_1 (resp. Γ_2) is the interior (resp. exterior) fixed boundary, (A) is also regular. For mixed boundary conditions (A) is not regular. However such conditions may be taken into account (see remark (4.14))//.

(3.10) *Remark*: clearly a.e. in (0,T) the function $u_\varepsilon(\cdot,t)$ satisfies (A) with

$$g_1 = g_1(\cdot,t), \qquad g_2 = g_2(\cdot,t),$$

$$\Psi = f(u_\varepsilon(\cdot,t))+u_\varepsilon(\cdot,t)-b\cdot\nabla_x u_\varepsilon(\cdot,t)-\gamma_\varepsilon(u_\varepsilon(\cdot,t))_t.$$

Due to (2.6)-(2.8), after an integration in (0,T) it is easily seen that

(3.11) $\|u_\varepsilon\|_{L^2(0,T;H^2(\Omega))} \leq C\cdot\varepsilon^{-1/2}$,

where $C>0$ is a constant independent of ε//.

Finally we shall introduce some notation for the time-discretization. Let $\tau=\dfrac{T}{N}$ be the size of a uniform partition of $(0,T)$, with $N\in\mathbb{N}$ arbitrary. Then let us denote $t_n=n\tau$, $n=1,\dots,N$, and

$$(3.12)\qquad \begin{aligned} \Psi^n &= \Psi(\cdot,t_n), & \bar{\Psi}^n &= \frac{1}{\tau}\int_{t_{n-1}}^{t_n} \Psi(\cdot,t)dt,\\ f^n(\Psi) &= f(\cdot,t_n,\Psi^n), & \bar{f}^n(\Psi) &= \frac{1}{\tau}\int_{t_{n-1}}^{t_n} f(\cdot,t,\Psi(\cdot,t))dt, \end{aligned}$$

where Ψ is any continuous function defined in Q.

3.2. *The Discrete-Time Galerkin Scheme*

Let us consider the discrete scheme obtained from (P_ε) using C^0-piecewise linear finite elements in space and backward-differences in time.

We say that *a family* $\{U_n\}_{n=1}^N$ *is a solution of the discrete problem* $(P_{\varepsilon,h,\tau})$ *iff*

(3.13) $U_n\in V_h(g^n_{2,\varepsilon})$

(3.14) $a(U_n,\phi)+\dfrac{1}{\tau}(\gamma_\varepsilon(U_n)-\gamma_\varepsilon(U_{n-1}),\phi)_{L^2(\Omega)} = \langle g^n_1,\phi\rangle_{L^2(\Gamma_1)}+(f^n(U_n),\phi)_{L^2(\Omega)}$,

holds for all $\phi\in V_h(0)$, $1\leq n\leq N$.

Here U_0 is chosen «close» to $u_{0,\varepsilon}$ in $L^2(\Omega)$, for instance so that

(3.15) $P_h\gamma_\varepsilon(U_0) = P_h\gamma_\varepsilon(u_{0,\varepsilon})$,

where P_h is the L^2-orthogonal projection operator onto $V_h(u_{0,\varepsilon})$. However if u_0 satisfies further regularity properties, U_0 may be chosen equal to u^I_0 (see (3.36), (3.38) and (4.19)).

(3.16) *Remark*: the discrete problem $(P_{\varepsilon,h,\tau})$ is a recursive nonlinear system of algebraic equations, associated with a continuous monotone operator in

$\mathbb{R}^N$. Then applying the theory of monotone operators (see [24, p. 167]), we conclude that $(P_{\varepsilon,h,\tau})$ has a unique solution. For the same reason there exists a unique function U_0 satisfying (3.15)//.

Let us denote by $u_{\varepsilon,h,\tau}$, and call it *the discrete solution*, the function defined in Q by

$$u_{\varepsilon,h,\tau}(\cdot,t) = U_n, \qquad \text{if } t_{n-1}<t\leq t_n \ (1\leq n\leq N).$$

3.3. *A Priori Estimate in $H^1(Q)$*

Now we shall derive an energy estimate for discrete problems which is the discrete analogue of an a priori estimate proved by A. Friedman for the continuous problem (see [13, p. 56]; see also [30, p. 69] and [21, p. 221], where more general problems are considered).

First we shall prove an elementary inequality we need in the sequel.

(3.17) LEMMA. *Let* $a_n \in \mathbb{R}^M$, $0\leq n\leq N$. *Then* $\sum_{n=1}^{N} a_n(a_n-a_{n-1}) \geq \frac{1}{2}(a_N^2-a_0^2)$.

Proof: $\sum_{n=1}^{N} a_n(a_n-a_{n-1}) = \frac{1}{2}\sum_{n=1}^{N} (a_n+a_{n-1})(a_n-a_{n-1}) + \frac{1}{2}\sum_{n=1}^{N} (a_n-a_{n-1})^2$

$$\geq \frac{1}{2}\sum_{n=1}^{N} a_n^2-a_{n-1}^2 = \frac{1}{2}(a_N^2-a_0^2)//.$$

(3.18) *Remark*: let $\alpha_n \in \mathbb{R}^M$, $1\leq n\leq N$. Then the following inequality holds

$$\sum_{n=1}^{N} \alpha_n \cdot \sum_{k=n}^{N} \alpha_k \geq \frac{1}{2}\Big(\sum_{n=1}^{N} \alpha_n\Big)^2.$$

In fact, denoting $a_n = \sum_{k=n}^{N} \alpha_k$, $1\leq n\leq N$ and $a_{N+1}=0$, and proceeding as in (3.17), we easily obtain the desired inequality//.

Now we are ready to prove the a priori estimate in $H^1(Q)$.

(3.19) THEOREM. *There exists a constant* $C>0$ *independent of* ε, h, *and* τ, *such that*

$$\max_{1\leq n\leq N} \|\nabla U_n\|^2_{L^2(\Omega)} + \sum_{n=1}^{N} \tau^{-1}(\gamma_\varepsilon(U_n)-\gamma_\varepsilon(U_{n-1}), U_n-U_{n-1})_{L^2(\Omega)} \leq C, \tag{3.20}$$

where $\{U_n\}_{n=1}^{N}$ *is the solution of* $(P_{\varepsilon,h,\tau})$.

Proof: Let us consider the family of test functions

$$\phi_n = \frac{1}{\tau}\,[(U_n-U_{n-1})-(G_n-G_{n-1})], \qquad 1\leq n\leq N,$$

where we have denoted $G_n=G^{n,I}_{2,\varepsilon}$. Clearly $\phi_n\in V_h(0)$ for $1\leq n\leq N$, and therefore ϕ_n is an admissible test function in (3.14).

Then we replace ϕ_n in (3.14), add on n for $1\leq n\leq k\leq N$ (k fixed) and after some calculations we will explain in the sequel, we obtain the inequality

$$(3.21)\qquad \|\nabla U_k\|^2_{L^2(\Omega)}+\sum_{n=1}^{k}\tau^{-1}(\gamma_\varepsilon(U_n)-\gamma_\varepsilon(U_{n-1}),U_n-U_{n-1})_{L^2(\Omega)}$$

$$\leq C_1+C_2\sum_{n=1}^{k}\tau(\|\nabla U_n\|^2_{L^2(\Omega)}+\sum_{i=1}^{n}\tau^{-1}(\gamma_\varepsilon(U_i)-\gamma_\varepsilon(U_{i-1}),U_i-U_{i-1})_{L^2(\Omega)}).$$

The latter expression has the appropriate form for applying the discrete Gronwall inequality, from which we easily get the desired bound (3.20).

Now we briefly describe how to estimate the various terms of (3.14) in order to get (3.21). We begin by observing that, by (3.17) and properties of K, we have

$$\sum_{n=1}^{k}(K.\nabla U_n,\nabla(U_n-U_{n-1}))_{L^2(\Omega)}\geq C_1\|\nabla U_k\|^2_{L^2(\Omega)}-C_2\|\nabla U_0\|^2_{L^2(\Omega)},$$

$$\sum_{n=1}^{k}(K.\nabla U_n,\nabla(G_n-G_{n-1}))_{L^2(\Omega)}\leq C\tau\sum_{n=1}^{k}\|\nabla U_n\|^2_{L^2(\Omega)}$$

$$+C\tau\sum_{n=1}^{k}\|\frac{1}{\tau}\nabla(G_n-G_{n-1})\|^2_{L^2(\Omega)}.$$

At the same time, recalling (2.2), we get

$$\sum_{n=1}^{k}\tau^{-1}(\gamma_\varepsilon(U_n)-\gamma_\varepsilon(U_{n-1}),G_n-G_{n-1})_{L^2(\Omega)}=$$

$$=\sum_{n=1}^{k}\tau^{-1}(c(U_n)-c(U_{n-1}),G_n-G_{n-1})_{L^2(\Omega)}$$

$$+\sum_{n=1}^{k}\tau^{-1}(\chi_\varepsilon(U_n)-\chi_\varepsilon(U_{n-1}),G_n-G_{n-1})_{L^2(\Omega)}$$

$$\leqslant \frac{\lambda_1}{5} \sum_{n=1}^{k} \tau^{-1} \|U_n - U_{n-1}\|^2_{L^2(\Omega)} + C \sum_{n=1}^{k} \tau^{-1} \|G_n - G_{n-1}\|^2_{L^2(\Omega)}$$

$$+ \sum_{n=1}^{k-1} \tau^{-1} \|G_{n-1} + G_{n+1} - 2G_n\|_{L^1(\Omega)} + \tau^{-1} \|G_k - G_{k-1}\|_{L^1(\Omega)} + \tau^{-1} \|G_1 - G_0\|_{L^1(\Omega)}$$

$$\leqslant C + \frac{\lambda_1}{5} \sum_{n=1}^{k} \tau^{-1} \|U_n - U_{n-1}\|^2_{L^2(\Omega)},$$

where we have used the Cauchy-Schwarz inequality, a formula of addition by parts and the regularity assumption $g_2 \in H^2(Q)$ combined with the choice of $g_{2,\varepsilon}$ (see (2.5)). For the convective term we have

$$\sum_{n=1}^{k} (b \nabla U_n, U_n - U_{n-1})_{L^2(\Omega)} \leqslant \frac{\lambda_1}{5} \sum_{n=1}^{k} \tau^{-1} \|U_n - U_{n-1}\|^2_{L^2(\Omega)} + C\tau \sum_{n=1}^{k} \|\nabla U_n\|^2_{L^2(\Omega)}.$$

Now taking into account that $U_n = U_0 + \sum_{i=1}^{n} U_i - U_{i-1}$, and the Lipschitz continuity of f, the nonlinear internal source term can be bounded as follows,

$$\sum_{n=1}^{k} (f(U_n), U_n - U_{n-1})_{L^2(\Omega)} \leqslant C + \frac{\lambda_1}{5} \sum_{n=1}^{k} \tau^{-1} \|U_n - U_{n-1}\|^2_{L^2(\Omega)}$$

$$+ C\tau \sum_{n=1}^{k} \left(\sum_{i=1}^{n} \tau^{-1} \|U_i - U_{i-1}\|^2_{L^2(\Omega)} \right).$$

Finally, in order to analyse the boundary terms, we note that there exists $C>0$ depending only on Ω, such that

$$\|U_n\|^2_{L^2(\Gamma_1)} \leqslant C(\|\nabla U_n\|^2_{L^2(\Omega)} + \|U_n\|^2_{L^2(\Omega)})$$

$$\leqslant C_1 + C_2(\|\nabla U_n\|^2_{L^2(\Omega)} + \sum_{i=1}^{n} \tau^{-1} \|U_i - U_{i-1}\|^2_{L^2(\Omega)}).$$

To complete the proof observe that terms $\sum_{n=1}^{k} \tau^{-1} \|U_n - U_{n-1}\|^2_{L^2(\Omega)}$ may be absorved into the left hand side of (3.21) (recall that $\gamma'_\varepsilon \geqslant \lambda_1 > 0$), while the remaining terms yield the right hand side of (3.21)//.

Let us now remark that (3.20) implies trivially an a priori estimate in $L^\infty(0,T;L^2(\Omega))$ for the discrete enthalpy. In fact, using again the relation

$U_n = U_0 + \sum_{i=1}^{n} (U_i - U_{i-1})$, and recalling that $\gamma_\varepsilon(U_n) = c(U_n) + \chi_\varepsilon(U_n)$ with $0 \leq \chi_\varepsilon \leq 1$, we easily obtain,

(3.22) COROLLARY. $\max_{1\leq n\leq N} \|\gamma_\varepsilon(U_n)\|_{L^2(\Omega)} \leq C.$

(3.23) Remark: (3.22) was considered as an *assumption* in [15].

3.4. *Some Auxiliary Results*

In this paragraph we compare u_ε with its H^1-projection onto $V_h(g_{2,\varepsilon}(\cdot,t))$ using $b(\cdot,\cdot)$ as scalar product, and we also obtain some error estimates for the initial enthalpies.

Let $Y(\cdot,t)$ be the function that satisfies a.e. in $(0,T)$

$$Y(\cdot,t) \in V_h(g_{2,\varepsilon}(\cdot,t)) \tag{3.24}$$

$$b((u_\varepsilon - Y)(\cdot,t), \Phi) = 0, \qquad \text{for all } \Phi \in V_h(0). \tag{3.25}$$

Our aim is to estimate in $L^2(Q)$ the error between u_ε and Y.

(3.26) LEMMA. *There exists a constant* $C>0$ *independent of* ε *and* h, *such that*

$$\|Y - u_\varepsilon\|_{L^2(Q)} \leq C \frac{h^2}{\varepsilon^{1/2}} \tag{3.27}$$

$$\left\|\int_0^t \nabla_x (Y - u_\varepsilon)\right\|_{L^\infty(0,T;L^2(\Omega))} \leq Ch. \tag{3.28}$$

Proof: It is well known from the elliptic theory (see [5, p. 139], for homogeneous boundary conditions) that

$$\|(Y - u_\varepsilon)(\cdot,t)\|_{L^2(\Omega)} \leq Ch^2 \|u_\varepsilon(\cdot,t)\|_{H^2(\Omega)}.$$

Then, by integration in $(0,T)$ and using (3.11), we immediately obtain the estimate (3.27). In order to derive (3.28) we have to notice that the function $v = \int_0^t u_\varepsilon$ satisfies a problem $\Delta_x v = \Psi \in L^\infty(0,T;L^2(\Omega))$, with regular mixed boundary conditions. Applying again the elliptic theory we easily get the desired estimate (3.28), hence the proof is complete//.

Now we define a time discretization of Y. For $1 \leqslant n \leqslant N$ we consider the function

$$(3.29) \qquad Y_n(x) = \frac{1}{\tau} \int_{t_{n-1}}^{t_n} Y(x,t)dt.$$

It is easy to verify from (3.24)-(3.25) that

$$(3.30) \qquad Y_n \in V_h\Big(\frac{1}{\tau} \int_{t_{n-1}}^{t_n} g_{2,\varepsilon}(\cdot,t)dt\Big) = V_h(\tilde{g}^n_{2,\varepsilon}),$$

$$(3.31) \qquad a(Y_n,\Phi) + \frac{1}{\tau}(\gamma_\varepsilon(u_\varepsilon^n)-\gamma_\varepsilon(u_\varepsilon^{n-1}),\Phi)_{L^2(\Omega)} = <\tilde{g}^n_1,\Phi>_{L^2(\Gamma_1)}$$

$$+ (\tilde{f}^n(u_\varepsilon),\Phi)_{L^2(\Omega)} + (\tilde{u}_\varepsilon^n - Y_n,\Phi)_{L^2(\Omega)} + (b\cdot\nabla_x(Y_n-\tilde{u}_\varepsilon^n),\Phi)_{L^2(\Omega)},$$

for all $\Phi \in V_h(0)$, $1 \leqslant n \leqslant N$.

In the following lemma we compare the families $\{Y_n\}$ and $\{u_\varepsilon^n\}$, obtaining a discrete version of (3.26).

(3.32) LEMMA. *There exists a constant* $C>0$ *independent of* ε *and* h, *such that*

$$(3.33) \qquad \Big(\sum_{n=1}^{N} \tau\|Y_n-u_\varepsilon^n\|^2_{L^2(\Omega)}\Big)^{1/2} \leqslant C\Big(\frac{h^2}{\varepsilon^{1/2}}+\tau\Big),$$

$$(3.34) \qquad \Big\|\tau \sum_{k=1}^{n} \nabla_x(Y_n-\tilde{u}_\varepsilon^n)\Big\|_{L^2(\Omega)} \leqslant Ch, \qquad 1 \leqslant n \leqslant N.$$

Proof: Notice that

$$Y_n-u_\varepsilon^n = \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (Y(\cdot,t)-u_\varepsilon(\cdot,t_n))dt$$

$$= \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (Y(\cdot,t)-u_\varepsilon(\cdot,t))dt + \frac{1}{\tau}\int_{t_{n-1}}^{t_n} (u_\varepsilon(\cdot,t)-u_\varepsilon(\cdot,t_n))dt = I_n + II_n.$$

Now we analyse each term of the latter expression. Using (3.26) we get

$$\sum_{n=1}^{N} \tau\|I_n\|^2_{L^2(\Omega)} \leqslant \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|Y(\cdot,t)-u_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}dt = \|Y-u_\varepsilon\|^2_{L^2(Q)} \leqslant C(\frac{h^2}{\varepsilon^{1/2}})^2.$$

For the second term II_n we have

$$\sum_{n=1}^{N} \tau\|II_n\|^2_{L^2(\Omega)} \leqslant C\tau^2 \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} \|\frac{\partial}{\partial t}u_\varepsilon(\cdot,t)\|^2_{L^2(\Omega)}dt \leqslant C\tau^2,$$

where we have used (2.7). Proceeding in the same manner, (3.34) is an easy consequence of (3.28), which completes the proof//.

We shall finish the paragraph by giving two error estimates for the initial enthalpies. We assume that u_0 satisfies the non-degeneracy condition

(3.35) $\quad |A_\varepsilon(u_0)| \leqslant C\varepsilon,$

and denote by Ω^+ and Ω^- the following sets:

$$\Omega^+ = \{x\in\Omega\colon u_0(x)>0\}, \qquad \Omega^- = \{x\in\Omega\colon u_0(x)<0\}.$$

(3.36) LEMMA. *Suppose that* F_0 *and* u_0 *satisfy* (2.25), (2.26) *and* (3.35). *Then there exists a constant* $C>0$ *independent of* ε *and* h, *such that*

$$\|\gamma_\varepsilon(u_0)-\gamma_\varepsilon(u^I_0)\|_{L^2(\Omega)} \leqslant C(\varepsilon+h)^{1/2}.$$

Proof: Consider the set $F_h=\{S\in\mathscr{D}_h\colon S\cap F_0\neq\emptyset\}$. Then we have $|F_h|\leqslant Ch$, because $F_0\in C^{0,1}$. Clearly $\chi_\varepsilon(u_0)=\chi_\varepsilon(u^I_0)=0$ in Ω^-/F_h. At the same time we get

$$\{x\in\Omega^+\setminus F_h\colon \chi_\varepsilon(u_0(x))-\chi_\varepsilon(u^I_0(x))\neq0\} \subset \{x\in\Omega^+\colon u_0(x)\leqslant\varepsilon+Ch\} = A_{\varepsilon+Ch}(u_0),$$

because if $x\notin A_{\varepsilon+Ch}(u_0)$, then

$$u^I_0(x) \geqslant u_0(x)-Ch \geqslant \varepsilon+Ch-Ch = \varepsilon,$$

which implies $\chi_\varepsilon(u_0(x))=\chi_\varepsilon(u^I_0(x))=1$. Now, applying the non-degeneracy condition on u_0, we obtain

$$\|\chi_\varepsilon(u_0)-\chi_\varepsilon(u^I_0)\|_{L^2(\Omega)} \leqslant |F_h|^{1/2}+|A_{\varepsilon+Ch}(u_0)|^{1/2} \leqslant C(\varepsilon+h)^{1/2}.$$

Finally recalling that $\gamma_\varepsilon(s)=c(s)+\chi_\varepsilon(s)$ with c regular, it is easy to conclude the desired result//.

We now suppose that the finite element mesh follows the initial free boundary F_0, in the sense that:

(3.37) there exist two subsets Ω^+_h and Ω^-_h of finite elements such that $\Omega=\Omega^+_h\cup\Omega^-_h$ and the nodes of $\partial\Omega^+_h$ (resp. $\partial\Omega^-_h$) belong to $\partial\Omega^+$ (resp. $\partial\Omega^-$).

(3.38) LEMMA. *Assume* (3.37), $F_0\in C^{1,1}$ *and* $u^+_0\in W^{2,\infty}(G^+)$ (*resp.* $u^-_0\in W^{2,\infty}(G^-)$) *where* G *is a neighborhood of* F_0. *Then there exists a constant* $C>0$ *independent of* h *and* ε *such that*

$$\|\gamma_\varepsilon(u_0)-\gamma_\varepsilon(u^I_0)\|_{L^2(\Omega)} \leq C(\varepsilon+h^2)^{1/2}\cdot\min(1,\frac{h^2}{\varepsilon}).$$

Proof: Notice that $\chi_\varepsilon(u^I_0)=0$ in Ω^-_h, and then

$$\|\chi_\varepsilon(u_0)-\chi_\varepsilon(u^I_0)\|_{L^2(\Omega^-_h)} \leq (\int_{\Omega^-_h\setminus\Omega^-} |\chi_\varepsilon(u_0)|^2)^{1/2} \leq Ch\cdot\min(1,\frac{h^2}{\varepsilon}).$$

In order to obtain the estimate in Ω^+_h, let us call $\bar u^+_0$ the $W^{2,\infty}$-extension of u^+_0 to Ω. Then

$$\|\chi_\varepsilon(u_0)-\chi_\varepsilon(u^I_0)\|_{L^2(\Omega^+_h)} \leq \|\chi_\varepsilon(u_0)-\chi_\varepsilon(\bar u^+_0)\|_{L^2(\Omega^+_h)} + \|\chi_\varepsilon(\bar u^+_0)-\chi_\varepsilon(u^I_0)\|_{L^2(\Omega^+_h)} = I+II.$$

For I, by virtue of $|\Omega^+_h\setminus\Omega^+|\leq Ch^2$, we have

$$I = (\int_{\Omega^+_h\setminus\Omega^+} |\chi_\varepsilon(u_0)-\chi_\varepsilon(\bar u^+_0)|^2)^{1/2} \leq Ch\cdot\min(1,\frac{h^2}{\varepsilon}).$$

For the other term, taking into account that u^I_0 is the interpolant of $\bar u^+_0$ in Ω^+_h, we get

$$\{x\in\Omega^+_h: \chi_\varepsilon(u^I_0)-\chi_\varepsilon(\bar u^+_0)\neq 0\}\subset\{x\in\Omega^+_h: \bar u^+_0(x)<\varepsilon+Ch^2\}\subset A_{\varepsilon+Ch^2}(u_0)\cup(\Omega^+_h\setminus\Omega^+),$$

which can be proved as in (3.36). Hence

$$II\leq|A_{\varepsilon+Ch^2}(u_0)\cup(\Omega^+_h\setminus\Omega^+)|^{1/2}\min(1,\frac{1}{\varepsilon}\|\bar u^+_0-u^I_0\|_{L^\infty(G^+_h)})\leq C(\varepsilon+h^2)^{1/2}\min(1,\frac{h^2}{\varepsilon}).$$

that completes the proof//.

4. **Error Estimates for Discrete Problems.**

In this section we derive some L^2-error estimates for solutions of $(P_{\varepsilon,h,\tau})$. The main idea in our proof is to use a discrete version of the integral test function (2.18) (see (4.7)).

At the end we will make some remarks about some usual semidiscrete schemes in space and time, and we also state L^2-rates of convergence, leaving the details to the reader.

4.1. *Error Estimates for a Discrete-Time Galerkin Scheme*

Observe that $(P_{\varepsilon,h,\tau})$ is a mildly non-linear parabolic problem, for which an order of convergence $C_1h^2+C_2\tau$ is known (see [31, p. 753]; see also [9] for more general results). Actually C_1 and C_2 depend on the regularity of the continuous solution u_ε, i.e. they depend on ε. Then the difficulty lies in finding that dependence, for which the theory developed in [31] and [9] seems not to be applicable.

Now we are ready to prove the main result of this section.

(4.1) THEOREM. *Let* u_ε *and* $u_{\varepsilon,h,\tau}$ *be the solutions of* (P_ε) *and* $(P_{\varepsilon,h,\tau})$ *respectively. Then there exists a constant* $C>0$ *independent of* ε, h *and* τ, *such that*

$$\|u_\varepsilon-u_{\varepsilon,h,\tau}\|_{L^2(Q)} \leq C\left(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}}+\tau^{2/3}\right), \tag{4.2}$$

$$\|\gamma_\varepsilon(u_\varepsilon)-\gamma_\varepsilon(u_{\varepsilon,h,\tau})\|_{L^\infty(0,T;V^*)} \leq C\left(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}}+\tau^{2/3}\right), \tag{4.3}$$

where the initial datum U_0 *satisfies* (3.15).

Proof: We give the proof in three steps. In the first we suppose $\{U_n\}_{n=1}^N$ verifying (3.13)-(3.14) with $\tilde g^n_{2,\varepsilon}$, $\tilde g^n_1$ and $\tilde f^n$, and we obtain an L^2-rate of convergence for the temperature. In the second step we give the essential modifications in order to consider the exact conditions $g^n_{2,\varepsilon}$, g^n_1 and f^n, while in the last step we obtain the enthalpy error estimates.

1st Step: Suppose $\{U_n\}_{n=1}^N$ satisfying

$$U_n \in V_h(\tilde g^n_{2,\varepsilon}) \tag{4.4}$$

$$(4.5)\qquad a(U_n,\Phi)+\frac{1}{\tau}(\gamma_\varepsilon(U_n)-\gamma_\varepsilon(U_{n-1}),\Phi)_{L^2(\Omega)} = <\tilde{g}^n_1,\Phi>_{L^2(\Gamma_1)}+(\tilde{f}^n(U_n),\Phi)_{L^2(\Omega)}$$

for all $\Phi\in V_h(0)$, $1\leq n\leq N$.

The idea of the proof is to reproduce in the discrete the technique of (2.14). Then we first obtain an integral version of (4.5) by addition on n for $1\leq n\leq n_0$, with n_0 fixed ($1\leq n_0\leq N$). Denoting $\Phi_{n_0+1}=0$, and reordering, we have

$$\tau\sum_{n=1}^{n_0}\ [a(U_n,\Phi_n)+(\gamma_\varepsilon(U_n),\frac{1}{\tau}(\Phi_n-\Phi_{n+1}))_{L^2(\Omega)}$$

$$= (\gamma_\varepsilon(U_0),\Phi_1)_{L^2(\Omega)}+\tau\sum_{n=1}^{n_0}\ [<\tilde{g}^n_1,\Phi_n>_{L^2(\Gamma_1)}+(\tilde{f}^n(U_n),\Phi_n)_{L^2(\Omega)}]$$

where $\Phi_n\in V_h(0)$, $1\leq n\leq n_0$. Proceeding in the same way with (3.31), we can obtain the following relation for $\{Y_n\}_{n=1}^N$,

$$\tau\sum_{n=1}^{n_0}\ [a(Y_n,\Phi_n)+(\gamma_\varepsilon(u_\varepsilon^n),\frac{1}{\tau}(\Phi_n-\Phi_{n+1}))_{L^2(\Omega)}$$

$$= (\gamma_\varepsilon(u_{0,\varepsilon}),\Phi_1)_{L^2(\Omega)} + \tau\sum_{n=1}^{n_0}\ [<\tilde{g}^n_1,\Phi_n>_{L^2(\Gamma_1)}+(\tilde{f}^n(u_\varepsilon),\Phi_n)_{L^2(\Omega)}$$

$$+ (\tilde{u}_\varepsilon^n-Y_n,\Phi_n)_{L^2(\Omega)}+(b\cdot\nabla(Y_n-\tilde{u}_\varepsilon^n),\Phi_n)_{L^2(\Omega)}].$$

Now, subtracting the last two expressions and reordering, we get the equality

$$(4.6)\qquad \tau\sum_{n=1}^{n_0}\ [(K\cdot\nabla(U_n-Y_n),\nabla\Phi_n)_{L^2(\Omega)}+<p(U_n-Y_n),\Phi_n>_{L^2(\Gamma_1)}$$

$$+ (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u_\varepsilon^n),\frac{1}{\tau}(\Phi_n-\Phi_{n+1}))_{L^2(\Omega)}] = I+II+III$$

$$= (\gamma_\varepsilon(U_0)-\gamma_\varepsilon(u_{0,\varepsilon}),\Phi_1)_{L^2(\Omega)}$$

$$+ \tau\sum_{n=1}^{n_0}\ (b\cdot\nabla(Y_n-U_n),\Phi_n)_{L^2(\Omega)}$$

$$+ \tau \sum_{n=1}^{n_0} (\tilde{f}^n(U_n)-\tilde{f}^n(u_\varepsilon),\Phi_n)_{L^2(\Omega)}$$

$$- \tau \sum_{n=1}^{n_0} [(\tilde{u}_\varepsilon^n-Y_n,\Phi_n)_{L^2(\Omega)}+(b\cdot\nabla(Y_n-\tilde{u}_\varepsilon^n),\Phi_n)_{L^2(\Omega)}]$$

$$= IV + V + VI + VII + VIII.$$

In order to analyse each term of (4.6) we shall use a variational technique. In fact we propose the following family $\{\Phi_n\}_{n=1}^{N+1}$ of discrete test functions:

$$\Phi_n = \tau \sum_{k=n}^{n_0} (U_k-Y_k), \qquad 1\leq n\leq n_0$$
(4.7)
$$\Phi_n = 0, \qquad n_0<n\leq N+1.$$

Clearly $\Phi_n \in V_h(0)$, $1\leq n\leq N+1$, because both Y_n and U_n attain the same boundary value $\tilde{g}^n_{2,\varepsilon}$ in Γ_2 (see (3.30) and (4.4)). Thus Φ_n is an admissible test function for (4.6).

We now start estimating the left hand side of (4.6). For I, using Remark (3.18), we have

$$I = \tau \sum_{n=1}^{n_0} (K^{1/2}\cdot\nabla(U_n-Y_n),\tau \sum_{k=n}^{n_0} K^{1/2}\cdot\nabla(U_k-Y_k))_{L^2(\Omega)}$$

$$\geq \|\tau \sum_{n=1}^{n_0} K^{1/2}\cdot\nabla(U_n-Y_n)\|^2_{L^2(\Omega)} = A_{n_0}.$$

For II, using again (3.18), we get

$$II = \tau \sum_{n=1}^{n_0} <p^{1/2}(U_n-Y_n),\tau \sum_{k=n}^{n_0} p^{1/2}\cdot(U_k-Y_k)> \geq \|\tau \sum_{n=1}^{n_0} p^{1/2}(U_n-Y_n)\|_{L^2(\Gamma_1)} = B_{n_0}.$$

The third term requires a different analysis, namely

$$III = \tau \sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u_\varepsilon^n),U_n-Y_n)_{L^2(\Omega)} = \tau \sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u_\varepsilon^n),U_n-u_\varepsilon^n)_{L^2(\Omega)}$$

$$+ \tau \sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u_\varepsilon^n),u_\varepsilon^n-Y_n)_{L^2(\Omega)} = D_{n_0} + E_{n_0}.$$

We now consider the auxiliary function $a_n(x)$ defined by

$$a_n(x) = (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u_\varepsilon^n))(U_n-u_\varepsilon^n)^{-1}, \qquad \text{if } U_n \neq u_\varepsilon^n$$

$$= 0, \qquad \text{if } U_n = u_\varepsilon^n.$$

From (1.7) and (2.2), it is easy to verify that $0 \leq a_n(x) \leq C\cdot\varepsilon^{-1}$. Hence, applying the Cauchy-Schwarz inequality, we get

$$E_{n_0} \leq [\tau \sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u_\varepsilon^n), U_n-u_\varepsilon^n)_{L^2(\Omega)}]^{1/2} \cdot [\tau \sum_{n=1}^{n_0} (a_n(u_\varepsilon^n-Y_n), u_\varepsilon^n-Y_n)_{L^2(\Omega)}]^{1/2}$$

$$\leq C\, D_{n_0}^{1/2} \cdot \frac{1}{\varepsilon^{1/2}} (\frac{h^2}{\varepsilon^{1/2}}+\tau) \leq \frac{1}{6} D_{n_0} + C(\frac{h^2}{\varepsilon}+\frac{\tau}{\varepsilon^{1/2}})^2,$$

where we have used (3.33). Therefore $III \geq \frac{5}{6} D_{n_0} - C(\frac{h^2}{\varepsilon}+\frac{\tau}{\varepsilon^{1/2}})^2$.

In order to estimate the right hand side of (4.6) we notice that by virtue of (3.15), $\gamma_\varepsilon(U_0)-\gamma_\varepsilon(u_{0,\varepsilon})$ is orthogonal to $V_h(0)$. Thus, IV=0.

The term V can be bounded by making use of (3.33), the elementary inequality $(a+b)^2 \leq 2(a^2+b^2)$ $a,b \in \mathbb{R}$, and reordering the double sum. Namely,

$$V = \tau \sum_{n=1}^{n_0} (\tau \sum_{k=1}^{n} b\cdot\nabla(Y_k-U_k), U_n-Y_n)_{L^2(\Omega)}$$

$$\leq \frac{\lambda_1}{6} \tau \sum_{n=1}^{n_0} \|U_n-u_\varepsilon^n\|^2_{L^2(\Omega)} + C\tau \sum_{n=1}^{n_0} \|Y_n-u_\varepsilon^n\|^2_{L^2(\Omega)}$$

$$+ C\tau \sum_{n=1}^{n_0} \|\tau \sum_{k=1}^{n} K^{1/2}\cdot\nabla(Y_k-U_k)\|^2_{L^2(\Omega)} \leq \frac{1}{6} D_{n_0} + C\tau \sum_{n=1}^{n_0} A_n + C(\frac{h^2}{\varepsilon^{1/2}}+\tau)^2.$$

Now recalling the Lipschitz continuity of f, the a priori estimate (2.7), and proceeding in the same manner as before, we can estimate VI. Indeed,

$$VI = \tau \sum_{n=1}^{n_0} (\tau \sum_{k=1}^{n} \bar{f}^k(U_k)-\bar{f}^k(u_\varepsilon), U_n-Y_n)_{L^2(\Omega)}$$

$$\leq \frac{\lambda_1}{6} \tau \sum_{n=1}^{n_0} \|U_n-u_\varepsilon^n\|^2_{L^2(\Omega)} + C\tau \sum_{n=1}^{n_0} \|Y_n-u_\varepsilon^n\|^2_{L^2(\Omega)}$$

$$+ C\tau \sum_{n=1}^{n_0} [\tau \sum_{k=1}^{n} (\|U_k-u_\varepsilon^k\|^2_{L^2(\Omega)}+\|\int_{t_{k-1}}^{t_k} |\frac{\partial}{\partial t}u_\varepsilon|\|^2_{L^2(\Omega)})]$$

$$\leqslant \frac{1}{6}D_{n_0}+C\tau \sum_{n=1}^{n_0} D_n+C(\frac{h^2}{\varepsilon^{1/2}}+\tau)^2.$$

Repeating again the previous analysis we can bound VII and VIII (use (3.33) and (3.34) respectively), obtaining

$$VII = -\tau \sum_{n=1}^{n_0} (\tau \sum_{k=1}^{n} \tilde{u}_\varepsilon^k-Y_k, U_n-Y_n)_{L^2(\Omega)} \leqslant \frac{1}{6}D_{n_0}+C(\frac{h^2}{\varepsilon^{1/2}}+\tau)^2,$$

$$VIII = \tau \sum_{n=1}^{n_0} (b\cdot\tau \sum_{k=1}^{n} \nabla(\tilde{u}_\varepsilon^k-Y_k), U_n-Y_n)_{L^2(\Omega)} \leqslant \frac{1}{6}D_{n_0}+C(\frac{h^2}{\varepsilon^{1/2}}+h+\tau)^2.$$

Now we are ready to conclude our analysis. Combining all of the previous estimates we easily see from (4.6) that the following relation is fulfilled

$$B_{n_0}+(A_{n_0}+D_{n_0}) \leqslant C(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}})^2+C\tau \sum_{n=1}^{n_0} (A_n+D_n)$$

for all n_0, $1\leqslant n_0\leqslant N$ (notice that terms D_{n_0} on the right hand side of (4.6) are absorbed into the left hand side). The last expression has an appropriate form for applying the discrete Gronwall inequality. This yields

$$\tau \sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u_\varepsilon^n), U_n-u_\varepsilon^n)_{L^2(\Omega)} \tag{4.8}$$

$$+ \|\tau \sum_{n=1}^{n_0} K^{1/2}\cdot\nabla(U_n-Y_n)\|^2_{L^2(\Omega)}+\|\tau \sum_{n=1}^{n_0} p^{1/2}(U_n-Y_n)\|^2_{L^2(\Gamma_1)}$$

$$\leqslant C(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}})^2,$$

for all n_0, $1\leqslant n_0\leqslant N$. Taking $n_0=N$ in (4.8), and recalling that $\gamma_\varepsilon'(s)\geqslant\lambda_1>0$ a.e. $s\in\mathbb{R}$, (4.8) clearly implies the following L^2-rate of convergence for the temperature,

$$(\tau \sum_{n=1}^{N} \|u_\varepsilon^n-U_n\|^2_{L^2(\Omega)})^{1/2} \leqslant C(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}}).$$

Now due to (2.7), i.e. recalling that $\|\frac{\partial}{\partial t}u_\varepsilon\|_{L^2(Q)}\leq C$, it is easy to achieve the desired estimate (4.2). Then the first step is proved.

2nd Step: Suppose that $\{U_n\}_{n=1}^N$ satisfies the discrete problem $(P_{\varepsilon,h,\tau})$, i.e. (3.13)-(3.14). Now, the term $IX=\tau\sum_{n=1}^{n_0} <g^n_1-\bar{g}^n_1,\Phi_n>_{L^2(\Gamma_1)}$ has to be added to the right hand side of (4.5). Instead of (4.7) we propose the following family of test functions,

$$
(4.9)\qquad \begin{aligned}
\Phi_n &= \Phi_n^1-\Phi_n^2 = \tau\sum_{k=n}^{n_0} (U_k-Y_k)-(g^{k,I}_{2,\varepsilon}-\bar{g}^{k,I}_{2,\varepsilon}), && 1\leq n\leq n_0\\
\Phi_n &= 0, && n_0<n\leq N+1.
\end{aligned}
$$

Due to (3.13) and (3.30), it follows that $\Phi_n\in V_h(0)$.

The analysis of (4.6) for Φ_n^1 $(1\leq n\leq n_0)$ can be carried out as in the first step, except for IX. For this term we have,

$$
\begin{aligned}
IX &= \tau\sum_{n=1}^{n_0} <\tau\sum_{k=1}^{n} g^k_1-\bar{g}^k_1, U_n-Y_n>_{L^2(\Gamma_1)}\\
&\leq C\Big(\tau\sum_{n=1}^{n_0} \|g^n_1-\bar{g}^n_1\|^2_{L^2(\Gamma_1)}\Big)^{1/2}\cdot\Big(\tau\sum_{n=1}^{n_0} \|U_n-Y_n\|^2_{L^2(\Gamma_1)}\Big)^{1/2}\\
&\leq C\tau\Big[\Big(\tau\sum_{n=1}^{n_0} \|U_n-\bar{u}^n_\varepsilon\|^2_{L^2(\Gamma_1)}\Big)^{1/2}+\Big(\tau\sum_{n=1}^{n_0} \|\bar{u}^n_\varepsilon-Y_n\|^2_{L^2(\Gamma_1)}\Big)^{1/2}\Big]\\
&\leq C\tau\Big[\Big(\tau\sum_{n=1}^{n_0} \|U_n-\bar{u}^n_\varepsilon\|^2_{L^2(\Omega)}\Big)^{1/4}+\Big(\tau\sum_{n=1}^{n_0} \|\bar{u}^n_\varepsilon-Y_n\|^2_{L^2(\Omega)}\Big)^{1/4}\Big]\\
&\leq C\Big[\tau^{4/3}+\tau\Big(\frac{h^2}{\varepsilon^{1/2}}\Big)^{1/2}\Big]+\frac{1}{12}D_{n_0} \leq C\Big(\tau^{2/3}+\frac{h^2}{\varepsilon^{1/2}}\Big)^2+\frac{1}{12}D_{n_0}.
\end{aligned}
$$

Here we have used the assumption $\frac{\partial}{\partial t}g_1\in L^2(\Gamma_1)$, the uniform boundedness of $\{U_n\}_{n=1}^N$ in $H^1(\Omega)$ proved in (3.19), the analogous property of $\{Y_n\}_{n=1}^N$ (recall that Y_n is the H^1-projection of $\bar{u}^n_\varepsilon$, and the estimate (2.7)), the estimate $\|\frac{\partial}{\partial t}u_\varepsilon\|_{L^2(Q)}\leq C$ and, finally, the well known inequalities:

$$\|v\|^2_{L^2(\Gamma_1)} \leqslant C\|v\|_{L^2(\Omega)}(\|v\|_{L^2(\Omega)}+\|\nabla v\|_{L^2(\Omega)}), \qquad \text{for all } v\in H^1(\Omega),$$

$$ab \leqslant \frac{1}{p}(\alpha a)^p+\frac{1}{q}\left(\frac{b}{\alpha}\right)^q, \qquad a,b\in \mathbb{R}^+, \qquad \frac{1}{p}+\frac{1}{q} = 1.$$

For the remaining term Φ^2_n we first observe that by construction of $g_{2,\varepsilon}$ (see (2.5) and [28, p. 93]) the following estimate is valid:

$$\|g^I_{2,\varepsilon}(\cdot,t)\|_{H^1(\Omega)} \leqslant C\|g_{2,\varepsilon}(\cdot,t)\|_{H^1(\Omega)}, \qquad \text{for all } t\in(0,T)$$

Therefore, using that $g_2\in H^2(Q)$, we get

$$\tau\sum_{n=1}^{n_0} \|g^{n,I}_{2,\varepsilon}-\bar{g}^{n,I}_{2,\varepsilon}\|^2_{H^1(\Omega)} \leqslant \tau\sum_{n=1}^{n_0} \left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \|g^I_{2,\varepsilon}(\cdot,t)-g^I_{2,\varepsilon}(\cdot,t_n)\|_{H^1(\Omega)}\right)^2$$

$$\leqslant C\tau\sum_{n=1}^{n_0} \left(\frac{1}{\tau}\int_{t_{n-1}}^{t_n} \|g_{2,\varepsilon}(\cdot,t)-g_{2,\varepsilon}(\cdot,t_n)\|_{H^1(\Omega)}\right)^2$$

$$\leqslant C\tau^2\sum_{n=1}^{n_0} \int_{t_{n-1}}^{t_n} \|\frac{\partial}{\partial t}g_{2,\varepsilon}(\cdot,t)\|^2_{H^1(\Omega)} \leqslant C\tau^2\|g_{2,\varepsilon}\|_{H^2(Q)} \leqslant C\tau^2.$$

Finally, by virtue of this estimate and proceeding as in the first step, it is not difficult to find the relation

$$B_{n_0}+A_{n_0}+D_{n_0} \leqslant C\left(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}}+\tau^{2/3}\right)^2+C\tau\sum_{n=1}^{n_0} (A_n+D_n),$$

for all n_0, $1\leqslant n_0\leqslant N$. Then, applying the discrete Gronwall inequality, we obtain an expression like (4.8) with the right hand side $\left(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}}+\tau^{2/3}\right)^2$. This inequality implies the desired estimate (4.2) for the temperature, and provides an additional information we shall use in the next step. The second step is complete.

3rd Step: In this part we derive the error estimate (4.3) for the enthalpy. Let Ψ be an arbitrary function belonging to V, and Ψ_h be its H^1-projection on $V_h(0)$, using $b(\cdot,\cdot)$ as scalar product. Employing a duality argument based on the regularity of (A) (see [5, p. 138]), it is easy to see that

$$\|\Psi-\Psi_h\|_{L^2(\Omega)} \leqslant Ch\|\Psi\|_{H^1(\Omega)}.$$

We now propose the following family of test functions in (4.5):

$$\Phi_n = \Psi_h, \qquad 1\leq n\leq n_0$$

$$= 0, \qquad n_0<n\leq N+1,$$

with which we obtain the relation

$$(\gamma_\varepsilon(U_{n_0})-\gamma_\varepsilon(u_\varepsilon^{n_0}),\Psi_h)_{L^2(\Omega)}$$

$$= -(\tau\sum_{n=1}^{n_0} K^{1/2}\cdot\nabla(U_n-Y_n),K^{1/2}\cdot\nabla\Psi_h)_{L^2(\Omega)}-<\tau\sum_{n=1}^{n_0} p^{1/2}(U_n-Y_n),p^{1/2}\Psi_h>_{L^2(\Gamma_1)}$$

$$+ (\tau\sum_{n=1}^{n_0} b\cdot\nabla(Y_n-U_n),\Psi_h)_{L^2(\Omega)}+(\tau\sum_{n=1}^{n_0} f^n(U_n)-\bar{f}^n(u_\varepsilon),\Psi_h)_{L^2(\Omega)}$$

$$- (\tau\sum_{n=1}^{n_0} \bar{u}^n_\varepsilon-Y_n+b\cdot\nabla(Y_n-\bar{u}^n_\varepsilon),\Psi_h)_{L^2(\Omega)}+<\tau\sum_{n=1}^{n_0} g^n_1-\bar{g}^n_1,\Psi_h>_{L^2(\Gamma_1)}.$$

Then, applying the Cauchy-Schwarz inequality to each term on the right hand side, and taking into account (3.33), (3.34), the relation (4.8) obtained in the second step, and that $\frac{\partial}{\partial t}g_1\in L^2(\Gamma_1)$, we have the estimate

$$|(\gamma_\varepsilon(U_{n_0})-\gamma_\varepsilon(u_\varepsilon^{n_0}),\Psi_h)_{L^2(\Omega)}| \leq C\left(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}}+\tau^{2/3}\right)|\Psi_h|_{H^1(\Omega)}.$$

Finally, using (3.22) and (2.6), we get

$$|(\gamma_\varepsilon(U_{n_0})-\gamma_\varepsilon(u_\varepsilon^{n_0}),\Psi)_{L^2(\Omega)}| \leq |(\gamma_\varepsilon(U_{n_0})-\gamma_\varepsilon(u_\varepsilon^{n_0}),\Psi_h)_{L^2(\Omega)}|$$

$$+ |(\gamma_\varepsilon(U_{n_0})-\gamma_\varepsilon(u_\varepsilon^{n_0}),\Psi-\Psi_h)_{L^2(\Omega)}| \leq C\left(\frac{h^2}{\varepsilon}+h+\frac{\tau}{\varepsilon^{1/2}}+\tau^{2/3}\right)|\Psi|_{H^1(\Omega)},$$

which clearly implies (4.3) because n_0 ($1\leq n_0\leq N$) is arbitrary. With this estimate we have completed the proof of the theorem//.

The following remark concerning the structure of the previous proof is given as a reference for the next section.

(4.10) *Remark*: from the first step of (4.1) it clearly follows the relation,

(4.11)
$$\tau \sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u^n_\varepsilon),U_n-u^n_\varepsilon)_{L^2(\Omega)}$$

$$+ \|\tau \sum_{n=1}^{n_0} K^{1/2}\cdot\nabla(U_n-Y_n)\|^2_{L^2(\Omega)}+\|\tau \sum_{n=1}^{n_0} p^{1/2}(U_n-Y_n)\|^2_{L^2(\Gamma_1)}$$

$$\leqslant C\Big[\tau \sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u^n_\varepsilon),Y_n-u^n_\varepsilon)_{L^2(\Omega)}$$

$$+ \tau \sum_{n=1}^{n_0} \|u^n_\varepsilon-Y_n\|^2_{L^2(\Omega)}+\|\tau \sum_{n=1}^{n_0} \nabla(\bar{u}^n_\varepsilon-Y_n)\|^2_{L^2(\Omega)}\Big]//.$$

In the sequel we mention possible modifications and extensions of theorem (4.1).

(4.12) *Remark*: if $(P_{\varepsilon,h,\tau})$ is defined with $\tilde{g}^n_1$ and $\tilde{g}^n_{2,\varepsilon}$ instead of g^n_1 and $g^n_{2,\varepsilon}$, and without convective term, the rate of convergence is $O(h^2\varepsilon^{-1}+\tau\varepsilon^{-1/2})$. In this case the assumptions (1.12) and (1.13) upon g_1 and g_2 respectively can be relaxed according to the requirements for existence, uniqueness and regularity of the solution of (P) and (P_ε). Now we are not able to prove (3.20) and (3.22), and therefore to carry out the third step of (4.1), obtaining only an L^2-error estimate for the temperature//.

(4.13) *Remark*: observe that in (4.1) we supposed that U_0 satisfies (3.15), and then $\gamma_\varepsilon(U_0)-\gamma_\varepsilon(u_{0,\varepsilon})$ is orthogonal to $V_h(0)$. In general, the error between the initial enthalpies,

$$\sigma(\varepsilon,h) = \|\gamma_\varepsilon(U_0)-\gamma_\varepsilon(u_{0,\varepsilon})\|_{L^2(\Omega)},$$

has to be added to the right hand side of (4.8). Perhaps the natural choice is: $u_{0,\varepsilon}=u_0$, $U_0=u^I_0$. Hence, recalling (3.36) and (3.38), we get

$$\sigma(\varepsilon,h) \leqslant C(\varepsilon+h)^{1/2} \qquad \text{or} \qquad \sigma(\varepsilon,h) \leqslant C(\varepsilon+h^2)^{1/2}\cdot\min\left(1,\frac{h^2}{\varepsilon}\right),$$

respectively//.

(4.14) *Remark* (On Mixed Boundary Conditions). For mixed boundary conditions the auxiliary problem (A) is *not regular* (see (3.6)-(3.9)), and consequently we can not expect to have the error estimates stated in (3.26) and (3.32). In fact, it is well known (for instance, see [27]) that,

$$u_\varepsilon(\cdot,t) \in H^{3/2-\delta}(\Omega), \text{ for all } \delta>0.$$

Next, by virtue of (2.8) and (5.7), and a priori estimates for elliptic problems in $H^s(\Omega)$, $s<2$ (see [18, p. 189]), we get

$$\|u_\varepsilon\|_{L^2(0,T;H^{3/2-\delta}(\Omega))} \leqslant C_\delta \cdot \varepsilon^{-1/2(1/2-\delta)},$$

instead of (3.11). Now, using a duality argument (see [5, p. 139]), we are able to obtain the non-optimal error estimates,

$$\|Y-u_\varepsilon\|_{L^2(Q)} \leqslant C_\delta\left(\frac{h^2}{\varepsilon}\right)^{1/2-\delta}, \qquad \left\|\int_0^t \nabla_x(Y-u_\varepsilon)\right\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C_\delta h^{1/2-\delta},$$

and finally a *global L^2-rate of convergence* $O(h^{2/5-\delta})$, if the relationship $\varepsilon \sim \tau \sim h^{4/5}$ is satisfied//.

We end this paragraph by showing the *continuous dependence* of discrete solutions upon the data. This result reproduces in the discrete a well known property of continuous problems (see § 2.4). The proof is based on the variational technique used in (4.1); thus the details are omitted.

For the sake of simplicity we only consider perturbations of the initial and Dirichlet boundary datum, and of the internal heating term.

(4.15) THEOREM. *Let $\{U_n\}_{n=1}^N$ and $\{\hat{U}_n\}_{n=1}^N$ be the solutions of the discrete problems $(P_{\varepsilon,h,\tau})$ with data u_0, f, g_2 and $\hat{u}_0$, $\hat{f}$, $\hat{g}_2$ respectively. Then there exists a constant $C>0$ independent of the data, ε, h and τ, such that*

$$\sum_{n=1}^N \tau(\gamma_\varepsilon(U_n)-\gamma_\varepsilon(\hat{U}_n), U_n-\hat{U}_n)_{L^2(\Omega)} + \max_{1\leqslant n\leqslant N} \|\gamma_\varepsilon(U_n)-\gamma_\varepsilon(\hat{U}_n)\|^2_{V^*}$$

$$\leqslant C\Big[\|\gamma_\varepsilon(U_0)-\gamma_\varepsilon(\hat{U}_0)\|^2_{L^2(\Omega)} + \tau\sum_{n=1}^N \|f^n(U_n)-\hat{f}^n(U_n)\|^2_{L^2(\Omega)}$$

$$+\Big(\tau\sum_{n=1}^N \|G_n-\hat{G}_n\|^2_{L^2(\Omega)}+\|\nabla(G_n-\hat{G}_n)\|^2_{L^2(\Omega)}\Big)^{1/2}\Big],$$

where we have set $G_n=g^{n,I}_{2,\varepsilon}$. If g_2, $\hat{g}_2$ are sufficiently regular we can take $G_n=g_2^{n,I}$.

Sketch of the Proof: We take the difference between the identities (3.14) corresponding to each discrete solution. Then we propose the following family of integral test functions

$$\Phi_n = \tau \sum_{k=n}^{n_0} (U_n - \hat{U}_n) - (G_n - \hat{G}_n), \qquad \text{if } 1 \leqslant n \leqslant n_0 \leqslant N$$

$$\Phi_n = 0, \qquad \text{if } n_0 < n \leqslant N+1,$$

and proceed as in (4.1)//.

(4.16) *Remark*: for $\varepsilon=0$, (4.15) can also be obtained, as a limit case, because C is independent of ε.

4.2. *Relationship between* ε, h, *and* τ

From (2.21) and (4.2)-(4.3) it is easily seen that for general two-phase Stefan problems the suitable relationship between ε, h and τ is

$$h^{4/3} \sim \varepsilon \sim \tau, \tag{4.17}$$

which yields a *global* L^2*-rate of convergence of order* $h^{2/3}$ for the whole approximation procedure. This condition was also obtained by J. Jerome and M. Rose in [15], for homogeneous Neumann conditions, but using a different variational technique, assuming the validity of (3.22) and dealing with quasiuniform finite element meshes.

If problem (P) is of non-degenerate type, i.e. $|A_\varepsilon(u)| \leqslant C\varepsilon$, by (2.30) the rate of convergence for the regularized problems is linear in ε. Therefore the adequate relationship between the parameters is

$$h \sim \varepsilon \sim \tau^{2/3}, \tag{4.18}$$

which yields a *global* L^2*-rate of convergence of order h.*

The previous conditions are true if U_0 is chosen according to (3.15). Let us now point out which relationship and rates are valid *for the choice* $U_0 = u^I_0$. By virtue of (4.1) and (4.13), under the assumptions of Lemma (3.36), we should impose the relation

$$h \sim \varepsilon \sim \tau, \tag{4.19}$$

to get a *global L^2-rate of convergence of order $h^{1/2}$*. Clearly (4.19) is more restrictive than (4.17) and the rate is worse than in that case. If we consider (3.38), instead of (3.36), then the relationship (4.17) and the rate $O(h^{2/3})$ are preserved.

In the sequel we give a final remark concerning the choice $U_0=u^I_0$. Let us consider the continuous problem for initial data u_0 and u^I_0, and call the corresponding solutions u and $\hat{u}$. In view of a well known stability result (see (2.33); and also [13, p. 65], [21, p. 205]), it follows that

$$\|u-\hat{u}\|_{L^2(Q)} \leqslant C\|\gamma(u_0)-\gamma(u^I_0)\|_{L^2(\Omega)} \leqslant Ch^{1/2}, \text{ (resp. h)},$$

under the assumptions stated in (3.36) (resp. (3.38)). Then we should not expect an L^2-rate better than $O(h^{1/2})$ to hold in general, but if the regularity properties of (3.38) are satisfied the order should be h.

4.3. *Error Estimates for some Semidiscrete Schemes*

In this section we deal with two semidiscrete schemes: the Continuous-Time Galerkin Scheme and the Discrete-Time Scheme. These schemes are commonly used in the theoretical analysis of the numerical approximation of time-dependent problems.

We point out that the main idea to obtain the error estimates is again the use of an integral test function similar to (2.18) and (4.7), and the same variational technique explained in §§ 2, 4. Therefore the proofs are omitted, leaving them to the reader.

4.3.1. *Continuous-Time Galerkin Scheme*

We say that *a function* $u_{\varepsilon,h}$: $[0,T]\rightarrow V_h$ *is a solution of the Continuous-Time Galerkin Scheme* $(P_{\varepsilon,h})$ *iff*

(4.20) $\quad u_{\varepsilon,h}(\cdot,0) = U_0 \quad$ (given by (3.28))

(4.21) $\quad u_{\varepsilon,h}(\cdot,t) \in V_h(g_{2,\varepsilon}(\cdot,t)), \qquad 0<t<T$

(4.22)
$$a(u_{\varepsilon,h}(\cdot,t),\Phi)+(\gamma_\varepsilon(u_{\varepsilon,h}(\cdot,t))_t,\Phi)_{L^2(\Omega)} = \langle g_1(\cdot,t),\Phi\rangle_{L^2(\Gamma_1)}+(f(u_{\varepsilon,h}(\cdot,t)),\Phi)_{L^2(\Omega)},$$

holds for all $\Phi \in V_h(0)$ *and a.e. in* (0,T).

(4.23) *Remark*: (4.22) is an ordinary differential equation, that cannot be understood in the classical sense, at least for γ_ε Lipschitz continuous. However it is not difficult to prove existence and uniqueness of a Lipschitz solution (for instance using a C^∞-regularization of γ_ε and compactness methods, or via monotone operator theory as in [22])//.

Prior to the statement of the result, we observe that the following a priori estimate is valid

$$\|\gamma_\varepsilon(u_{\varepsilon,h})\|_{L^\infty(0,T;L^2(\Omega))} \leqslant C.$$

Moreover, it can be proved in an easier way than (3.22) (see [15, p. 399]). Then using a variational procedure based on the integral test function

$$\Phi(x,t) = \int_t^{t_0} (u_{\varepsilon,h}-Y)(x,s)ds, \qquad 0<t<t_0\leqslant T$$

$$= 0, \qquad t_0\leqslant t\leqslant T,$$

(Y defined in (3.24)-(3.25)), we can prove

(4.24) THEOREM. *Let* u_ε *and* $u_{\varepsilon,h}$ *be the solutions of* (P_ε) *and* $(P_{\varepsilon,h})$ *respectively. Then there exists a constant* $C>0$ *independent of* ε *and* h, *such that*

$$\|u_\varepsilon-u_{\varepsilon,h}\|_{L^2(Q)}+\|\gamma_\varepsilon(u_\varepsilon)-\gamma_\varepsilon(u_{\varepsilon,h})\|_{L^\infty(0,T;V^*)} \leqslant C\left(\frac{h^2}{\varepsilon}+h\right).$$

The comments given in § 4.2 are also applicable here. In particular if $u_{\varepsilon,h}(\cdot,0)=u^I_0$, the error $\sigma(\varepsilon,h)$ (see (4.13)) has to be added to the right hand side of the latter estimate.

4.3.2. *Discrete-Time Scheme*

We say that *a family of functions* $\{U_n\}_{n=1}^N$ *is a solution of the Discrete-Time Scheme* $(P_{\varepsilon,\tau})$ (also called Horizontal Line Method or Rothe method) *iff*

(4.25) $\quad U_n \in V(g^n_{2,\varepsilon})$

(4.26) $\quad a(U_n,\Phi)+\frac{1}{\tau}(\gamma_\varepsilon(U_n)-\gamma_\varepsilon(U_{n-1}),\Phi)_{L^2(\Omega)} = \langle g^n_1,\Phi\rangle_{L^2(\Gamma_1)}+(f^n(U_n),\Phi)_{L^2(\Omega)},$

holds for all $\Phi \in V$, $1 \leq n \leq N$, *where* $U_0 = u_{0,\varepsilon}$.

The existence and uniqueness of the solution of $(P_{\varepsilon,\tau})$ is well known (see for example [14, p. 104]), as well as the a priori estimate (see [14, p. 57]): $\|\gamma_\varepsilon(U^n)\|_{L^\infty(\Omega)} \leq C$.

Then we take the family of test functions (n_0 is fixed, $1 \leq n_0 \leq N$)

$$\begin{aligned} \Phi_n &= \tau \sum_{k=n}^{n_0} (U_k - u^k_\varepsilon), && 1 \leq n \leq n_0 < N+1 \\ &= 0, && n_0 < n \leq N+1. \end{aligned}$$

Next, we can prove:

(4.27) THEOREM. *Let* u_ε *and* $\{U_n\}_{n=1}^N$ *be the solutions of* (P_ε) *and* $(P_{\varepsilon,\tau})$ *respectively. Then there exists a constant* $C>0$ *independent of* ε *and* τ *such that*

$$\left(\tau \sum_{n=1}^N \|u^n_\varepsilon - U_n\|^2_{L^2(\Omega)}\right)^{1/2} + \max_{1 \leq n \leq N} \|\gamma_\varepsilon(u^n_\varepsilon) - \gamma_\varepsilon(U_n)\|_{V^*} \leq C\left(\frac{\tau}{\varepsilon^{1/2}} + \tau^{2/3}\right).$$

5. A Quasioptimal L^2-Error Estimate

In this section we shall prove, under certain restrictions on the initial datum and on the finite element mesh, that the discrete scheme $(P_{\varepsilon,h,\tau})$ is essentially $O(h+\tau^{1/2})$-accurate (independent of ε). Our result improves the one obtained by J. Jerome and M. Rose in [15], because we do not need the *assumption*

$$\max_{1 \leq n \leq N} \|\gamma_\varepsilon(U_n)\|_{L^\infty(\Omega)} \leq C,$$

(see [15, p. 401 and 408]; or [14, p. 152]).

Let us consider the following assumption upon the initial temperature:

(5.1) Δu_0 *is a finite regular Baire measure.*

Consequently, if $u_{0,\varepsilon}$ is a regularization of u_0, there exists a constant $C>0$ independent of ε, such that

(5.2) $$\|\Delta u_{0,\varepsilon}\|_{L^1(\Omega)} \leq C.$$

We also suppose that the auxiliary problem (A) has the *approximation property*

(5.3) $$\|v-v_h\|_{L^\infty(\Omega)} \leq C \log \frac{1}{h} \cdot \min_{\phi \in V_h} \|v-\phi\|_{L^\infty(\Omega)},$$

where v_h is the H^1-projection of v onto $V_h(v)$.

If the finite element mesh $\mathcal{S}_h$ satisfies an *inverse property*, i.e.

$$h \leq C\varrho_s, \qquad \text{for all } S \in \mathcal{S}_h,\ C>0$$

and the boundary conditions are of Dirichlet type, the property (5.3) was proved by A. Schatz and L. Wahlbin in [26].

Prior to the statement of the main result we need some auxiliary error estimates. Recall that $Y(\cdot,t)$ is the H^1-projection of $u_\varepsilon(\cdot,t)$ onto $V_h(g_2(\cdot,t))$ and Y_n its mean value in (t_n,t_{n-1}) (see § 3.4).

(5.4) LEMMA. *There exists a constant* $C>0$ *independent of* ε, h *and* τ, *such that*

(5.5) $$\left(\tau \sum_{n=1}^{N} \|Y_n-u^n_\varepsilon\|^2_{L^2(\Omega)}\right)^{1/2} \leq C(h+\tau),$$

(5.6) $$\max_{1\leq n\leq N} \|Y_n-u^n_\varepsilon\|_{L^1(\Omega)} \leq C(h^2|\log h|^2+\tau).$$

Proof: Due to (2.6)–(2.7) it easily follows that

(5.7) $$\|\gamma_\varepsilon(u_\varepsilon)_t\|_{L^\infty(0,T;V^*)} \leq C \text{ (independent of } \varepsilon).$$

Now, taking into account that (A) is regular, a standard duality argument (see [5, p. 139]) yields the estimate

$$\|(Y-u_\varepsilon)(\cdot,t)\|_{L^2(\Omega)} \leq Ch\|(Y-u_\varepsilon)(\cdot,t)\|_{H^1(\Omega)} \leq Ch\|u_\varepsilon(\cdot,t)\|_{H^1(\Omega)} \leq Ch.$$

Then proceeding in the same manner as in (3.26) and (3.32), we immediately obtain (5.5). At the same time, by virtue of (5.2) and (2.9), we get

$$\|\gamma_\varepsilon(u_\varepsilon)_t\|_{L^\infty(0,T;L^1(\Omega))} \leq C \text{ (independent of } \varepsilon).$$

Using again a duality argument (between L^1 and L^∞) and the approximation property (5.3), it is not difficult to show (5.6) (see [15, p. 401] for the details)//.

Now we state and prove the quasioptimal L^2-error estimate.

(5.8) THEOREM. *Let* u_ε *and* $u_{\varepsilon,h,\tau}$ *be the solutions of* (P_ε) *and* $(P_{\varepsilon,h,\tau})$ *respectively. Then there exists a constant* $C>0$ *independent of* ε, h *and* τ, *such that*

$$\|u_\varepsilon-u_{\varepsilon,h,\tau}\|_{L^2(Q)}+\|\gamma_\varepsilon(u_\varepsilon)-\gamma_\varepsilon(u_{\varepsilon,h,\tau})\|_{L^\infty(0,T;V^*)} \leqslant C(h|\log h|+\tau^{1/2}).$$

Proof: Recalling (4.11) and using (3.34) and (5.5), we get the relation

$$(5.9)\qquad \tau\sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u^n_\varepsilon),U_n-u^n_\varepsilon)_{L^2(\Omega)}$$

$$+\Big\|\tau\sum_{n=1}^{n_0} K^{1/2}\cdot\nabla(U_n-Y_n)\Big\|^2_{L^2(\Omega)}+\Big\|\tau\sum_{n=1}^{n_0} p^{1/2}(U_n-Y_n)\Big\|^2_{L^2(\Gamma_1)}$$

$$\leqslant C(h+\tau)^2+\tau\sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u^n_\varepsilon),Y_n-u^n_\varepsilon)_{L^2(\Omega)}.$$

Our aim is to analyse the second term on the right hand side of (5.9) in a different manner from that one in (4.1), employing now the additional information (5.6). In fact we have

$$\tau\sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u^n_\varepsilon),Y_n-u^n_\varepsilon)_{L^2(\Omega)}$$

$$=\tau\sum_{n=1}^{n_0} (c(U_n)-c(u^n_\varepsilon),Y_n-u^n_\varepsilon)_{L^2(\Omega)}+\tau\sum_{n=1}^{n_0} (\chi_\varepsilon(U_n)-\chi_\varepsilon(u^n_\varepsilon),U_n-u^n_\varepsilon)_{L^2(\Omega)}$$

$$\leqslant \frac{1}{2\lambda_2}\tau\sum_{n=1}^{n_0} \|c(U_n)-c(u^n_\varepsilon)\|^2_{L^2(\Omega)}+C\tau\sum_{n=1}^{n_0} \|u^n_\varepsilon-Y_n\|^2_{L^2(\Omega)}$$

$$+\Big(\tau\sum_{n=1}^{n_0} \|\chi_\varepsilon(U_n)-\chi_\varepsilon(u^n_\varepsilon)\|_{L^\infty(\Omega)}\Big)\cdot\Big(\max_{1\leqslant n\leqslant n_0}\|u^n_\varepsilon-Y_n\|_{L^1(\Omega)}\Big)$$

$$\leqslant \frac{1}{2}\tau\sum_{n=1}^{n_0} (\gamma_\varepsilon(U_n)-\gamma_\varepsilon(u^n_\varepsilon),U_n-u^n_\varepsilon)_{L^2(\Omega)}+C(h^2|\log h|^2+\tau).$$

Now combining this expression and (5.9) we obtain the L^2-error estimate for the temperature and two remaining terms that, proceeding as in the third step of (4.1), yield the enthalpy error estimate. Then we have finished the proof//.

The previous theorem establishes a rate of convergence of order $h|\log h|+\tau^{1/2}$ for discrete problems $(P_{\varepsilon,h,\tau})$. Then for general two-phase Stefan problems the suitable relationship between the parameters would be

$$(5.10)\qquad h^2 \sim \varepsilon \sim \tau,$$

in order to obtain a *global L^2-rate of convergence of order* $h|\log h|$. In this case we have an almost linear order of convergence, improving the order $h^{2/3}$ given by (4.17). Besides, (4.17) is more restrictive than (5.10).

For non-degenerate cases the adequate relationship is

$$(5.11)\qquad h \sim \varepsilon \sim \tau^{1/2},$$

which yields a *global L^2-rate of convergence of order* $h|\log h|$. Actually (5.11) is more restrictive than (4.18) and provides almost the same order.

Finally we point out that, in general, *the choice* $U_0=u^I_0$ reduces the order of convergence to $h^{1/2}$, but the essentially linear order is preserved under the assumptions stated in (3.8).

(5.12) *Remark*: Call $u_{h,\tau}$ the solution of the discrete problems $(P_{h,\tau})$ without regularization ($\varepsilon=0$). It is possible, using the same technique as in (2.14), to obtain an L^2-error estimate of order $\varepsilon^{1/2}$ between $u_{h,\tau}$ and $u_{\varepsilon,h,\tau}$. Thus, taking $\varepsilon\to 0$, we get the estimate

$$\|u-u_{h,\tau}\|_{L^2(Q)}+\|\gamma(u)-\gamma(u_{h,\tau})\|_{L^\infty(0,T;V^*)} \leq C(h|\log h|+\tau^{1/2}).$$

Then the advantage of a regularization procedure is that it makes the effective calculation of discrete solutions easier, for instance using standard iterative methods, and allows us to relate the parameters ε, h and τ in order to preserve the order of convergence.

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