ERROR ESTIMATES FOR GAUSS-CHEBYSHEV AND CLENSHAW-CURTIS QUADRATURE FORMULAS

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ABSTRACT - For functions f analytic in a circle we obtain bounds for the Chebyshev-Fourier coefficients of f. These results are then used to obtain bounds for the errors of Gauss-Chebyshev quadratures, and the quadrature formula of Clenshaw and Curtis. Two examples are given to illustrate the error bounds.

1. Introduction.

In Chawla and Jain [1], Chawla [2] and Chawla [3] estimation of errors of Gauss-Chebyshev quadratures has been discussed for functions belonging to $A(\varepsilon_\varrho)$, the class of functions analytic in the closure of an ellipse ε_ϱ with foci at ± 1 and semiaxes $\frac{1}{2}(\varrho + \varrho^{-1})$ and $\frac{1}{2}(\varrho - \varrho^{-1})$, $\varrho > 1$. In Chawla and Jain [1] and Chawla [2] error estimates are also given for functions f which are analytic in the circular domain $|z| \leq r$, for large values of r. Rabinowitz [4] gives two types of estimates for the error in the numerical integration of f. In the first, the error is expressed in terms of a certain derivative of f, and then this derivative is estimated by use of the Cauchy integral formula over the boundary of the domain which is the union of the rectangle $|\mathrm{Re}(z)| \leq 1$, $|\mathrm{Im}(z)| \leq r$ and the circle $|z \pm 1| \leq r$. This estimate is useful only for values of $r > \frac{1}{2}$. The second type of estimates are obtained by expanding f in a Chebyshev-Fourier series in $[-1, 1]$ and then estimating the Chebyshev-Fourier coefficients with the help of the results of Elliott [5].

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Let $C_r : |z| = r$, $r > 1$, and let $A(C_r) = \{f : f$ is analytic in C_r and continuous on C_r, $r > 1\}$. In the present note we obtain bounds for the coefficients in the Chebyshev-Fourier expansion of a function f. These estimates are then used to obtain bounds for the errors of the Gauss-Chebyshev quadrature formulas of the first kind, the second kind and the closed type. We also obtain error estimates for the quadrature formula of Clenshaw and Curtis. An example is given which indicates that the present estimates are better than those given in Chawla [8].

2. Estimates for the Chebyshev-Fourier Coefficients.

Let $T_k(x) = \cos(k \arccos x)$, $k = 0, 1, 2, \ldots$, designate the Chebyshev polynomials of the first kind defined over $[-1, 1]$. If $f(x)$ is continuous and of bounded variation on $[-1, 1]$, then f has a uniformly convergent expansion over $[-1, 1]$ in terms of these polynomials:

$$f(x) = \sum_{k=0}^{\infty}{}' a_k T_k(x) \tag{1}$$

where the prime on the summation sign indicates that the first term is to be multiplied by $\frac{1}{2}$. The Chebyshev-Fourier coefficients a_k are given by

$$a_k = (2/\pi) \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} f(x) T_k(x)\, dx, \quad k = 0, 1, \ldots . \tag{2}$$

If $f \in A(C_r)$, then from (2) with the help of Cauchy integral formula we obtain

$$a_k = (\pi^2 i)^{-1} \int_{C_r} f(z) \left(\int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} (T_k(x)/(z - x))\, dx \right) dz. \tag{3}$$

Let

$$Q_k^*(z) = \frac{1}{2} \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} (T_k(x)/(z - x))\, dx \tag{4}$$

denote the Chebyshev function of the second kind. Then, from (3) and (4) we obtain

$$a_k = 2(\pi^2 i)^{-1} \int_{C_r} f(z)\, Q_k^*(z)\, dz. \tag{5}$$

The following result gives an estimate for $Q_k^*(z)$ on C_r.

LEMMA 1. For $z \in C_r$, $r > 1$,

$$|Q_k^*(z)| \leq (\pi/2)(r^2-1)^{-\frac{1}{2}}(r+(r^2-1)^{\frac{1}{2}})^{-k}, \qquad k = 0, 1, 2, \ldots. \tag{6}$$

PROOF. From Chawla and Jain [1], Lemma 4, for $|z| > 1$,

$$Q_k^*(z) = \sum_{m=k}^{\infty} \sigma_{km}^* z^{-m-1} \tag{7}$$

where

$$\sigma_{km}^* = \begin{cases} \pi\, 2^{-m-1} \binom{m}{(m-k)/2}, & m = k,\ k+2, \ldots, \\ 0, & m = k+1,\ k+3, \ldots,\ k = 0, 1, 2, \ldots. \end{cases} \tag{8}$$

Since all the σ_{km}^* are non-negative, therefore, on C_r,

$$|Q_k^*(z)| \leq \sum_{m=k}^{\infty} \sigma_{km}^* r^{-m-1} = Q_k^*(r). \tag{9}$$

Again, in Chawla and Jain [1] Equation (27′), it has been proved that for $z \in \varepsilon_\varrho$, $\varrho > 1$,

$$Q_k^*(z) = (\pi/2)(z^2-1)^{-\frac{1}{2}}(z+(z^2-1)^{\frac{1}{2}})^{-k}, \tag{10}$$

where the sign of the square root is to be chosen so that $|z+(z^2-1)^{\frac{1}{2}}| > 1$. However, we note that (10) is valid for all $z \notin [-1, 1]$ and substituting for $Q_k^*(r)$ from (10) in (9) we obtain (6).

From (5), with the help of (6), we obtain the following result.

THEOREM 1. If $f \in A(C_r)$, $r > 1$, then

$$|a_k| \leq 2r(r^2-1)^{-\frac{1}{2}}(r+(r^2-1)^{\frac{1}{2}})^{-k} M(r), \quad k = 0, 1, \ldots, \tag{11}$$

where $M(r) = \max_{|z|=r} |f(z)|$.

3. Error Estimates for Gauss-Chebyshev and Clenshaw-Curtis Quadratures.

Let $E_{T_n}(f)$ denote the error of an n-point Gauss-Chebyshev quadrature (first kind):

$$(12) \qquad E_{T_n}(f) = \int_{-1}^{1} (1 - x^2)^{-\frac{1}{2}} f(x)\, dx - \sum_{k=1}^{n} (\pi/n) f(\cos((2k-1)\pi/2n)).$$

Following Chawla [6], p. 107, the error can be expressed in terms of the Chebyshev-Fourier coefficients of the integrand,

$$(13) \qquad E_{T_n}(f) = \pi \sum_{m=1}^{\infty} (-1)^{m+1} a_{2nm}\,.$$

Substituting the estimates (11) in (13) we obtain the following estimate for the error $E_{T_n}(f)$.

THEOREM 2. If $f \in A(C_r)$, $r > 1$, then

$$(14) \qquad |E_{T_n}(f)| \leq 2\pi r (r^2 - 1)^{-\frac{1}{2}} ((r + (r^2 - 1)^{\frac{1}{2}})^{2n} - 1)^{-1} M(r).$$

EXAMPLE 1. Let $f(x) = 1/(4 + x)$. Then, $M(r) = 1/(4 - r)$. If we select $r = 25/7$, then $M(25/7) = 7/3$. For this function, we compare the actual error $E_{T_n}(f)$ with that estimated from (14) in Table 1.

TABLE 1 (2)

n	Error estimate (14)	Actual error
4	0.265 (— 5)	0.110 (— 6)
8	0.460 (— 12)	0.394 (— 12)

(2) The number in the parentheses indicates the power of 10 by which the corresponding number is to be multiplied.

Let $E_{P_{n+1}}(f)$ denote the error of the Gauss-Chebyshev quadrature formula of the closed type based on the "practical abscissas" $\cos(k\pi/n)$, $k = 0, 1, \ldots, n$,

$$(15) \qquad E_{P_{n+1}}(f) = \int_{-1}^{1} (1-x^2)^{-\frac{1}{2}} f(x)\,dx - \sum_{k=0}^{n}{}'' (\pi/n) f(\cos(k\pi/n))$$

where the double prime on the summation sign means that the first and the last terms are to be multiplied by $\frac{1}{2}$. From Chawla [6], p. 108,

$$(16) \qquad E_{P_{n+1}}(f) = -\pi \sum_{m=1}^{\infty} a_{2nm} .$$

Substituting the estimates (11) in (16) we obtain the following estimate for $E_{P_{n+1}}(f)$.

THEOREM 3. If $f \in A(C_r)$, $r > 1$, then

$$(17) \qquad |E_{P_{n+1}}(f)| \le 2\pi r (r^2-1)^{-\frac{1}{2}} ((r + (r^2-1)^{\frac{1}{2}})^{2n} - 1)^{-1} M(r).$$

Note that the estimates (14) and (17) for $E_{T_n}(f)$ and $E_{P_{n+1}}(f)$, respectively, are the same.

Let $E_{U_n}(f)$ denote the error of an n-point Gauss Chebyshev quadrature (second kind):

$$(18) \quad E_{U_n}(f) = \int_{-1}^{1} (1-x^2)^{\frac{1}{2}} f(x)\,dx - (\pi/(n+1)) \sum_{k=1}^{n} \sin^2(k\pi/(n+1)) f\left(\cos \frac{k\pi}{n+1}\right).$$

Now, it has been shown in Chawla [6] that

$$(19) \qquad E_{U_n}(f) = E_{P_{n+2}}((1-x^2) f(x)).$$

Since, on C_r,

$$(20) \qquad |(1-z^2) f(z)| \le (1+r^2) M(r),$$

with the help of (17) and (20), from (19) we obtain the following estimate for $E_{U_n}(f)$.

THEOREM 4. If $f \in A(C_r)$, $r > 1$, then

(21) $$|E_{U_n}(f)| \leq 3\pi r (r^2-1)^{-\frac{1}{2}} (1+r^2)((r+(r^2-1)^{\frac{1}{2}})^{2n+2}-1)^{-1} M(r).$$

If ε_ϱ is the largest ellipse wich lies in a circle C_r, $r > 1$, then $\varrho = r + (r^2-1)^{\frac{1}{2}}$. With this ϱ, the above estimates are close to those obtained in Chawla [3] for functions $f \in A(\varepsilon_\varrho)$, $\varrho > 1$.

Finally we point out that the estimates of the above type can be obtained for the error $E_n(f)$ of any quadrature formula over $[-1, 1]$ once the error has been expressed in terms of the Chebyshev-Fourier coefficients of f and estimates for $E_n(T_k)$ are available.

For example, consider the error of the Clenshaw-Curtis quadrature formula given, for n even, by

(22) $$E_n(f) = \int_{-1}^{1} f(x)\,dx - \sum_{i=0}^{n}{}'' \lambda_i f(x_i)$$

where $x_i = \cos(\pi i/n)$, $i = 0, 1, \ldots, n$, and

(23) $$\lambda_i = -(4/n) \sum_{j=0}^{n/2}{}'' T_{2j}(x_i)/(4j^2-1), \quad i = 0, 1, \ldots, n.$$

It has been shown by O'Hara and Smith [7] that, if f has the expansion (1) and if the Chebyshev-Fourier coefficients a_k of f are neglected for $k \geq 3n$, then

(24) $$E_n(f) = \frac{16 \cdot 1 \cdot n}{(n^2-1^2)(n^2-3^2)} a_{n+2} + \frac{16 \cdot 2 \cdot n}{(n^2-3^2)(n^2-5^2)} a_{n+4} + \cdots$$
$$+ \frac{16 \cdot (n/2-1) \cdot n}{3(2n-1)(2n-3)} a_{2n-2} - (2 + 2/(4n^2-1)) a_{2n} +$$
$$+ (2/3 - 2/((2n+1)(2n+3))) a_{2n+2} \cdots .$$

They have further pointed out that if f is analytic with singularities not too close to the interval of integration, the first term in (24) dominates over the sum of the remaining terms, and then

(25) $$|E_n(f)| \leq 32n (n^2-1)^{-1} (n^2-9)^{-1} |a_{n+2}|.$$

Assuming that $f \in A(C_r)$, $r > 1$, and substituting the estimate (11) for $|a_{n+2}|$, we obtain

$$(26)\qquad |E_n(f)| \leq 64n(n^2-1)^{-1}(n^2-9)^{-1}(r^2-1)^{-\frac{1}{2}}(r+(r^2-1)^{\frac{1}{2}})^{-n-2}rM(r).$$

Thus, if r is not too close to unity, (26) gives a reasonable estimate for the error of the Clenshaw-Curtis quadrature formula. This estimate should be compared with the following obtained in Chawla [8]. If $f \in A(\varepsilon_\varrho)$, $\varrho > 1$, then for n even,

$$(27)\qquad |E_n(f)| \leq (16n^2/(4n^2-1))(\varrho^2-1)^{-1}(\varrho^n-\varrho^{-n})^{-1}M(\varepsilon_\varrho),$$

where $M(\varepsilon_\varrho) = \max|f(z)|$ on ε_ϱ. For functions $f \in A(C_r)$, $r > 1$, and taking ε_ϱ as the largest ellipse in C_r, i. e., with $\varrho = r + (r^2-1)^{\frac{1}{2}}$, the estimate (26) agrees with (27) as far as the order of r is concerned. However the first factor on the right of (26) is of order n^{-3}, for large n, while that in (27) is of order 1. Thus, for r not too close to unity, and for large n, say $n > 8$, our estimate (26) will be better than (27). We illustrate this by means of the following example.

Example 2. Let $f(x) = 1/(4+x)$. For the estimate (26), we select $r = 25/7$; then for the estimate (27), the largest ellipse in $C_{25/7}$ will be ε_7. We compare, in Table 2, the estimates obtained for $C_{25/7}$

Table

n	Our estimate (26)	Actual error	Chawla's estimate (27)
4	0.504 (— 4)	0.125 (— 5)	0.823 (— 4)
8	0.127 (— 8)	0.247 (— 10)	0.339 (— 7)
10	0.125 (— 10)	0.905 (— 12)	0.690 (— 9)

with those obtained by Chawla [8] for ε_7. We find that our estimates are better than those of Chawla [8].

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