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Linear Programming Approaches to the Measurement and Analysis of Productive Efficiency

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1. INTRODUCTION

An important and rapidly growing empirical application of operations research techniques involves the measurement and analysis of the efficiency with which goods are produced and services are provided. The production activities whose efficiency has been the subject of investigation have varied widely, from profit-oriented industrial manufacturing enterprises all the way to public and private service providers operating in a not-for-profit environment. A similarly wide variety of operations research techniques has been utilized in the measurement and analysis of productive efficiency, ranging from stochastic parametric regression-based methods to nonstochastic nonparametric mathematical programming methods. Foremost among the latter is a family of linear programming models collectively referred to as Data Envelopment Analysis (DEA). The purpose of this paper is to provide a selective overview of some of the most useful DEA models. I analyze these models in terms of their ability to accurately reflect the structure of the underlying production technology, their ability to accurately measure the productive efficiency of the producers being analyzed, and their data requirements and their sensivity to shortcomings in the data that form the basis of the analysis.

The selectivity of the overview reflects my talents and my interests. Thus the orientation of the overview is toward the user of existing DEA models, not toward the developer of new models. In particular, the review is intended to inform practitioners in the fields of management science, economics and public administration who want to learn and

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apply the techniques of DEA to draw inferences concerning producer performance, for the ultimate purpose of guiding busines, economic or public policy. It is not directed at mathematical programmers and other specialists interested in formal model structure, computational issues, and the like.

The paper unfolds as follows. In Section 2 I begin by describing what I mean by productive efficiency. Efficiency can refer to the proficiency with which inputs to the production process are converted to outputs of the process, in which case it is referred to as *technical* efficiency. Efficiency can also refer to the proficiency with which producers achieve their economic objective, such as production at minimum cost, generation of maximum revenue, or maximization of profit. In this latter case efficiency is referred to as *economic* efficiency, one component of which is technical efficiency. I then provide a brief verbal description of how DEA techniques are used to measure productive efficiency, and how they differ from other techniques commonly used for the same purpose.

In Section 3 I present and discuss the structure of what I consider to be the core family of DEA models. Since they are linear programming models, they have primal and dual representations, and I pay special attention to the role of duality for the application-oriented operations research analyst. In Sections 4-8 I discuss some modifications to the basic DEA models. These modifications entail relaxing assumptions on the structure of the underlying production technology, moving from the measurement of technical efficiency to the measurement of economic efficiency, and relaxing restrictions on the types of data used in the analysis.

DEA models are nonstochastic, and inferences drawn from them may be sensitive to noise in the underlying data. In Section 9 I describe the use of chance-constrained programming techniques in an effort to introduce a stochastic element to DEA.

Section 10 concludes with a summary and evaluation of DEA, a list of omitted topics, and some brief speculation concerning future directions of DEA research.

2. TECHNOLOGY, EFFICIENCY AND DEA

Let producers use variable inputs $x = (x_1, \ldots, x_N) \in \mathbb{R}^N_{++}$ to produce variable outputs $y = (y_1, \ldots, y_M) \in \mathbb{R}^M_{++}$. A treatment of data sets that do not satisfy the strict positivity requirement is deferred to section 8.1.

Fixed, or nondiscretionary, variables can be incorporated into the analysis, but I leave them to section 8.2. Additional variables that are neither inputs nor outputs, but which characterize the operating environment in which production takes place, can also be incorporated into the analysis, and they are considered in section 8.3.

The technology with which inputs are transformed to outputs is represented by the *graph*

$$GR = \{(x, y) : \ x \text{ can produce } y\}. \tag{2.1}$$

GR is assumed to be closed and bounded, and to satisfy strong disposability. GR satisfies strong disposability if $(x, y) \in GR \Rightarrow (x', y') \in GR$ for all $(x', -y') \geq (x, -y)$. The graph contains its *isoquants*

$$Isoq\ GR = \{(x, y) : \ (x, y) \in GR,\ (\delta x, \delta^{-1} y) \notin GR,\ 0 < \delta < 1\}, \tag{2.2}$$

which in turn contain their *efficient subsets*

$$Eff\ GR = \{(x, y) : \ (x, y) \in GR,\ (x', y') \notin GR,\ 0 \leq x' \leq x,\ y' \geq y\}. \tag{2.3}$$

Corresponding to the graph of the technology is a family of *input sets*

$$L(y) = \{x : \ (x, y) \in GR\},\ y \in \mathbb{R}^M_{++}. \tag{2.4}$$

Input sets are assumed to be closed and bounded below, and to satisfy the properties of convexity and strong disposability of inputs. Input sets contain their *isoquants*

$$Isoq\ L(y) = \{x : \ x \in L(y),\ \theta x \notin L(y),\ \theta \in [0, 1)\},\ y \in \mathbb{R}^M_{++}, \tag{2.5}$$

which in turn contain their *efficient subsets*

$$Eff\ L(y) = \{x : \ x \in L(y),\ x' \notin L(y),\ x' \leq x\},\ y \in \mathbb{R}^M_{++}. \tag{2.6}$$

Also corresponding to the graph of the technology is a family of *output sets*

$$P(x) = \{y : \ (y, x) \in GR\},\ x \in \mathbb{R}^N_{++}. \tag{2.7}$$

Output sets are assumed to be closed and bounded above, and to satisfy the properties of convexity and strong disposability of outputs. Output sets contain their *isoquants*

$$Isoq\, P(x) = \{y : \ y \in P(x),\ \phi y \notin P(x),\ \phi \in (1, +\infty)\},\ x \in \mathbb{R}^N_{++}, \quad (2.8)$$

which is turn contain their *efficient subsets*

$$Eff\ P(x) = \{y : \ y \in P(x),\ y' \notin P(x),\ y' \geq y\},\ x \in \mathbb{R}^N_{++}. \quad (2.9)$$

Equations (2.1)-(2.9) characterize the production technology relative to which efficiency is measured. Additional details appear in Färe (1988). I now provide a formal definition of efficient production, which is completely general in the sense that it is neither oriented nor equiproportionate. I then provide three measures of productive efficiency, which are restricted by both orientation and equiproportionality. The three measures are structurally similar, differing only in their orientation toward conserving inputs, expanding outputs, or achieving both objectives simultaneously.

Definition 1. (Koopmans (1951,1957)) *Input-output vector (x,y) is technically efficient if, and only if, $(x,y) \in Eff\ GR$. Input vector x is technically efficient in the production of output vector y if, and only if, $x \in Eff\ L(y)$. Output vector y is technically efficient given input vector x if, and only if, $y \in Eff\ P(x)$.*

Definition 2. (Färe, Grosskopf and Lovell (1985)) *A hyperbolic measure of the technical efficiency of input-output vector (x,y) is given by*

$$TE_G(x,y) = \min\{\delta : \ (\delta x, \delta^{-1} y) \in GR\},$$

with $\delta = 1$ indicating hyperbolic technical efficiency and $\delta < 1$ indicating the degree of hyperbolic technical inefficiency.

Definition 3. (Debreu (1951), Farrell (1957)) *A radial measure of the technical efficiency of input vector x in the production of output vector y is given by*

$$TE_I(x,y) = \min\{\theta : \ \theta x \in L(y)\},$$

with $\theta = 1$ indicating radial technical efficiency and $\theta < 1$ indicating the degree of radial technical inefficiency.

Definition 4. (Debreu (1951), Farrell (1957)) *A radial measure of the technical efficiency of output vector y produced by input vector x is given by*

$$TE_0(y, x) = \max\{\phi : \ \phi y \in P(x)\},$$

with $\phi = 1$ indicating radial technical efficiency and $\phi > 1$ indicating the degree of radial technical inefficiency.

It should be clear that

$$Eff\ GR \subseteq Isoq\ GR = \{(x, y)\} : \ TE_G(x, y) = 1\}, \tag{2.10}$$

that

$$Eff\ L(y) \subseteq Isoq\ L(y) = \{x : \ TE_I(x, y) = 1\}, \ y \in \mathbb{R}^M_{++}, \tag{2.11}$$

and that

$$Eff\ P(x) \subseteq Isoq\ P(x) = \{y : \ TE_0(y, x) = 1\}, \ x \in \mathbb{R}^N_{++}. \tag{2.12}$$

Thus $TE_G(x, y) = 1$ is necessary, but not sufficient, for $(x, y) \in Eff\ GR$. Similarly, $TE_I(x, y) = 1$ is necessary, but not sufficient, for $x \in Eff\ L(y)$, and $TE_0(y, x) = 1$ is necessary, but not sufficient, for $y \in Eff\ P(x)$. Sufficiency fails because the three efficiency measures are equiproportionate (hyperbolic or radial) measures that may leave non-proportional inefficiency undetected. The inability of the three equiproportionate measures of efficiency to match Koopmans' definition of efficiency has been the subject of much debate in the literature. Färe and Lovell (1978) proposed a nonproportionate measure of efficiency that coincides with Koopmans' definition of efficiency, but Russell (1990) noted some flaws in a nonproportionate measure. I am currently working with one of my discussants, Philippe Vanden Eeckaut, on an alternative measure that coincides with Koopmans' definition. This measure is equiproportionate, but relative to a translated origin.

The three measures do not necessarily provide consistent information on the technical efficiency of a producer. Three results, proved in Färe, Grosskopf and Lovell (1985), are particularly useful. Since two of the three results rely on the nature of scale economies which characterize technology, I begin with definitions of two popular restrictions on scale economies.

Definition 5. *Technology is homogenous of degree* $\alpha > 0$ *if* $L(\theta y) = \theta^{1/\alpha} L(y)$, $\theta > 0$, *or, equivalently, if* $P(\lambda x) = \lambda^{\alpha} P(x)$, $\lambda > 0$. *Returns to scale are globally increasing, constant or decreasing according as* $\alpha \gtreqless 1$.

Definition 6. *Technology is sub-homogeneous if* $L(\theta y) \overset{\subseteq}{\underset{\supset}{=}} \theta L(y)$, $\theta \gtreqless 1$, *and super-homogeneous if* $L(\theta y) \overset{\supseteq}{\underset{\subset}{=}} \theta L(y)$, $\theta \gtreqless 1$. *Returns to scale are non-increasing where technology is sub-homogeneous, and non-decreasing where technology is super-homogeneous.*

Proposition 1. $TE_G(x,y) \geq \max\{TE_I(x,y), (TE_0(y,x))^{-1}\}$.

Proposition 2. $(TE_G(x,y))^2 = TE_I(x,y) = (TE_0(y,x))^{-1}$ *if, and only if, technology is homogeneous of degree +1.*

Proposition 3. *If technology is sub- (super-) homogeneous, then* $TE_I(x,y) \leq (\geq)(TE_0(y,x))^{-1}$.

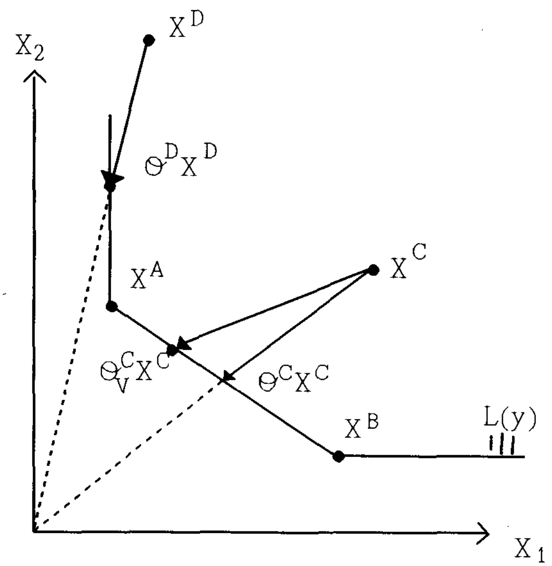

Figure 1. *Input-oriented Technical Efficiency Measurement*

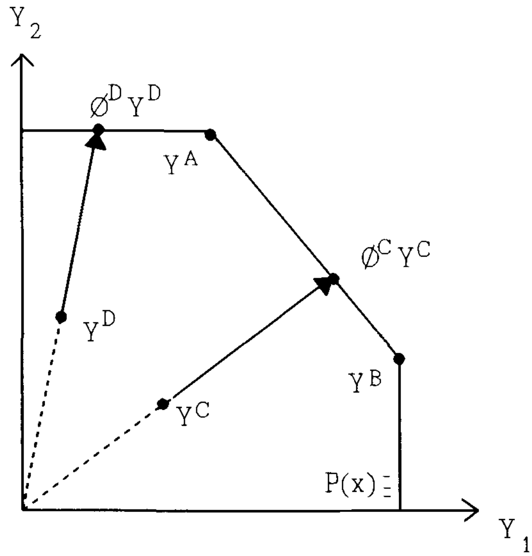

Figure 2. *Output-oriented Technical Efficiency Measurement*

The two radial technical efficiency measures are illustrated in Figures 1 and 2. In Figure 1 $Eff\ L(y)$ is the line segment $x^A x^B$, and $Isoq\ L(y)$ is the line segment $x^A x^B$ plus the vertical extension above x^A and the horizontal extension to the right of x^B. $L(y)$ is the region bounded below by $Isoq\ L(y)$. $TE_I(x^A, y) = TE_I(x^B, y) = 1$ because no radial contraction of x^A or x^B is feasible for y. $TE_I(x^C, y) = \theta^C < 1$, and no nonradial inefficiency remains at $\theta^C x^C$. $TE_I(x^D, y) = \theta^D < 1$, and nonradial slack input x_2 in the amount $(\theta^D x_2^D - x_2^A)$ remains at $\theta^D x^D$. (The nonradial projection from x^C to $\theta_V^C x^C$ is discussed in Section 8.2.) In Figure 2 $Eff\ P(x)$ is the line segment $y^A y^B$, and $Isoq\ P(x)$ is the line segment $y^A y^B$ plus the horizontal extension to the left of y^A and the vertical extension beneath y^B. $P(x)$ is the region bounded above by $Isoq\ P(x)$. $TE_0(y^A, x) = TE_0(y^B, x) = 1$ because no radial expansion of y^A or y^B is feasible with x. $TE_0(y^C, x) = \phi^C > 1$, and nonradial efficiency remains at $\phi^C y^C$. $TE_0(y^D, x) = \phi^D > 1$, and nonradial slack in output y_1, in the amount $(y_1^A - \phi^D y_1^D)$ remains at $\phi^D y^D$.

So far the analysis has concentred on the measurement of the *technical* efficiency of production. If input prices and output prices are not observed, or observed but thought to be unreliable or distorted by regulation or market power, this is all that can be accomplished. However if prices are accurately observed, and if a common behavioral objective can be attributed to all producers, then *economic* efficiency can be measured. Let producers face input prices $w = (w_1, \ldots, w_N) \in \mathbb{R}^N_{++}$ and output prices $p = (p_1, \ldots, p_M) \in \mathbb{R}^M_{++}$. Then the minimum expenditure required to produce output vector y when input prices are w is given by the *cost function*

$$c(y, w) = \min_x \{w^T x : x \in L(y)\}, \ y \in \mathbb{R}^M_{++}. \tag{2.13}$$

The maximum revenue that can be obtained from input vector x when output prices are p is given by the *revenue function*

$$r(x, p) = \max_y \{p^T y : y \in P(x)\}, \ x \in \mathbb{R}^N_{++}. \tag{2.14}$$

The maximum profit that can be obtained when input prices are w and output prices are p is given by the *profit function*

$$\pi(p, w) = \max_{x,y} \{p^T y - w^T x : (x, y) \in GR\}. \tag{2.15}$$

Definition 7. *The cost efficiency of a producer using input vector x to produce output vector y when input prices are w is measured by the ratio of minimum cost to actual cost,*

$$CE(x, y, w) = c(y, w)/w^T x,$$

with $CE(x, y, w) = 1$ indicating cost efficiency and $CE(x, y, w) < 1$ indicating the degree of cost inefficiency.

Definition 8. *The input allocative efficiency of a producer using input vector x to produce output vector y when input prices are w is measured by the ratio of cost efficiency to input technical efficiency,*

$$AE_I(x, y, w) = CE(x, y, w)/TE_I(x, y),$$

with $AE_I(x, y, w) = 1$ indicating input allocative efficiency and

$AE_I(x, y, w) < 1$ *indicating the degree of input allocative inefficiency.*

Definition 9. *The revenue efficiency of a producer producing output vector y with input vector x when output prices are p is measured by the ratio of maximum revenue to actual revenue,*

$$RE(y, x, p) = r(x, p)/p^T y,$$

with $RE(y, x, p) = 1$ indicating revenue efficiency and $RE(y, x, p) > 1$ indicating the degree of revenue inefficiency.

Definition 10. *The output allocative efficiency of a producer producing output vector y with input vector x when output prices are p is measured by the ratio of revenue efficiency to output technical efficiency,*

$$AE_0(y, x, p) = RE(y, x, p)/TE_0(y, x),$$

with $AE_0(y, x, p) = 1$ indicating output allocative efficiency and $AE_0(y, x, p) > 1$ indicating the degree of output allocative inefficiency.

Definition 11. *The profit efficiency of a producer facing input prices w and output prices p is measured by the ratio of maximum profit to actual profit,*

$$\pi E(y, x, p, w) = \pi(p, w)/(p^T y - w^T x),$$

provided $(p^T y - w^T x) > 0$. $\pi E(y, x, p, w) = 1$ indicates profit efficiency and $\pi E(y, x, p, w) > 1$ indicates the degree of profit inefficiency.

Definition 12. *The graph allocative efficiency of a producer using input vector x to produce output vector y when input prices are w and output prices are p is measured by the ratio of profit efficiency to the reciprocal of graph technical efficiency,*

$$AE_G(y, x, p, w) = \pi E(y, x, p, w)/(TE_G(x, y))^{-1},$$

with $AE_G(y, x, p, w) = 1$ indicating graph allocative efficiency and $AE_G(y, x, p, w) > 1$ indicating the degree of graph allocative inefficiency.

Figures 3 and 4 illustrate the concepts of cost efficiency and revenue efficiency. In Figure 3, $TE_I(x^B, y) = AE_I(x^B, y, w) = CE(x^B, y, w) = 1$ because it is not possible to produce output vector y at lower cost, given input prices w. However $TE_I(x^A, y) = 1$ but $AE_I(x^A, y, w) = CE(x^A, y, w) < 1$ because x^A is not an allocatively efficient mix of inputs, given input prices w. In Figure 4 $TE_0(y^A, x) = AE_0(y^A, x, p) = RE(y^A, x, p) = 1$ because it is not possible to generate more revenue from input vector x, given output prices p. However $TE_0(y^B, x) = 1$ but $AE_0(y^B, x, p) = RE(y^B, x, p) > 1$ because y^B is not an allocatively efficient mix of outputs, given output prices p.

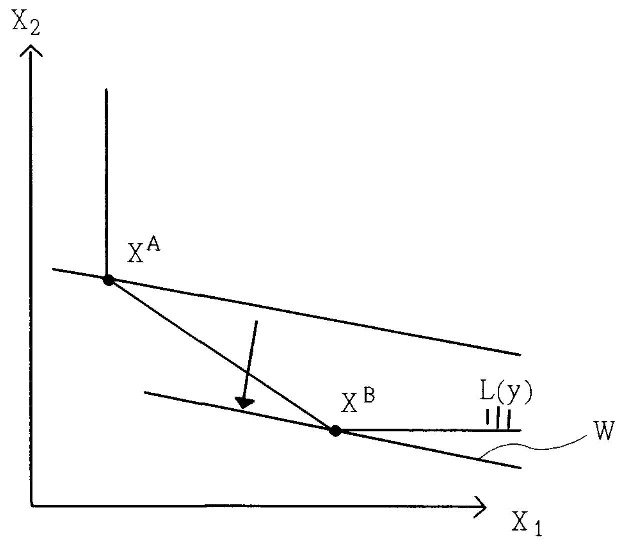

Figure 3. *Cost Efficiency Measurement*

The type of efficiency to be measured depends on what data are available, and on what behavioral assumptions are appropriate. If only input quantity and output quantity data are available, only *technical efficiency* can be measured, regardless of what behavioral assumption is appropriate. Orientation, i.e., the use of $TE_G(x, y)$, $TE_I(x, y)$ or $TE_0(y, x)$, is at the discretion of the analyst. If input price data are also available, and if cost minimization is a tenable behavioral objective, then cost *efficiency* can be measured and decomposed into its technical and input allocative components. If output price data are also available, and if revenue maximization is thought to be an appropriate behavioral objective, then *revenue efficiency* can be measured and decomposed into its technical and output allocative components. Finally if all four types of data are available, and if profit maximization is considered to be an appropriate behavioral objective, *profit efficiency* can be measured and decomposed into its technical and allocative components.

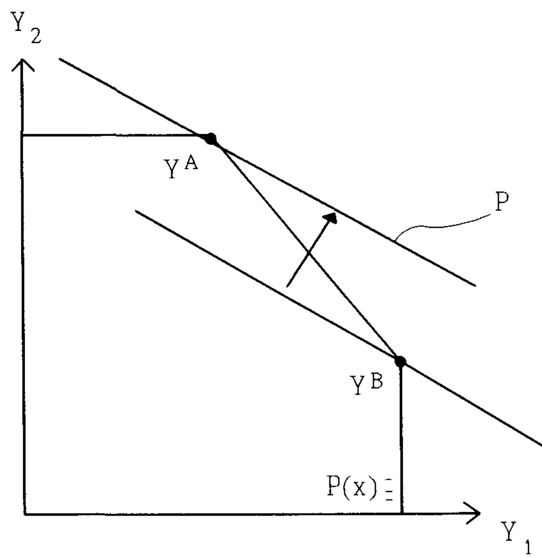

Figure 4. *Revenue Efficiency Measurement*

The actual measurement of productive efficiency proceeds as follows. First, collect data on (y, x), and perhaps also on (p, w), for a collection of producers whose performance is to be evaluated. If technical efficiency is to be measured, use the data (x, y) to construct the graph GR, the input sets $L(y)$ or the output sets $P(x)$. Then measure $TE_G(x, y)$, $TE_I(x, y)$ or $TE_0(y, x)$ for each producer. If cost efficiency is to be measured, use the data (x, y, w) to construct the cost function $c(y, w)$. Then measure $CE(x, y, w)$, $AE_I(x, y, w)$ and $TE_I(x, y)$ for each producer. If revenue efficiency is to be measured, use the data (y, x, p) to construct the revenue function $r(x, p)$. Then measure $RE(y, x, p)$, $AE_0(y, x, p)$ and $TE_0(y, x)$. Finally, if profit efficiency is to be measured, use the data (x, y, w, p) to construct the profit function $\pi(p, w)$. Then measure $\pi E(y, x, p, w)$, $AE_G(y, x, p, w)$ and $TE_G(x, y)$ for each producer.

For details and extensions, see Färe, Grosskopf and Lovell (1985, 1994).

There remains the practical problem of calculating these, and other, efficiency measures. Three general approaches have been developed. Econometricians have developed *parametric stochastic* frontiers, because they believe in allowing for statistical noise. Estimation is by maximum likelihood techniques, and efficiency is inferred from a component of the skewed regression residual for each producer. Stochastic frontiers were developed simultaneously by three groups on three continents: Aigner, Lovell and Schmidt (1977), Battese and Corra (1977) and Meeusen and van den Broeck (1977). Details are available in Lovell (1992) and Greene (1993).

Both economists and management scientists have developed *parametric nonstochastic* frontiers, calculated using either maximum likelihood or goal programming techniques, after which efficiency is inferred from the one-sided deviations from the calculated frontier. Parametric nonstochastic frontiers were developed by Aigner and Chu (1968), and by Førsund and his colleagues, whose early work is summarized in Førsund and Hjalmarsson (1987). An illustrative example is provided by Charnes, Cooper and Sueyoshi (1988).

Management scientists have developed *nonparametric nonstochastic* frontiers, because they prefer not to impose possibly unwarranted parametric structure on technology, preferring instead to let the data reveal the structure of technology. The frontier is calculated so that it envelops the data as tightly as possible, subject to various constraints such as monotonicity and convexity. Calculation is by linear programming methods, which collectively have come to be known as Data Envelopment Analysis (DEA). The DEA literature began at about the same time as the stochastic frontier literature, with important early contributions by Charnes, Cooper and Rhodes (1978,1981) and Banker, Charnes and Cooper (1984). Analysis, illustrative applications and an extensive bibliography are provided in Charnes et al. (1994). Interestingly enough, DEA was anticipated over a decade earlier in the agricultural economics literature. Building on activity analysis models of production developed by Koopmans (1951,1957), Boles (1966), Bressler (1966), Seitz (1966) and Sitorus (1966) developed and calculated remarkably sophisticated DEA-type models. Their work in turn was extended by Afriat (1972) and Shephard (1974). Indeed, reading these early contributions provides a good introduction to the DEA literature.

The three approaches to frontier construction and efficiency measurement are surveyed, compared and applied empirically in Lewin and Lovell (1990), Gulledge and Lovell (1992) and Fried, Lovell and Schmidt (1993).

3. BASIC DEA MODELS

Producers use inputs $x^i \in \mathbb{R}^N_{++}$ to produce outputs $y^i \in \mathbb{R}^M_{++}$, i = 1,...,I with I indicating the number of producers in the sample. All inputs and all outputs are discretionary, in the sense that they are freely variable and under the control of management. The analyst's objective is to evaluate the technical efficiency of each producer relative to the best observed practice in the sample. Designate the producer being evaluated as having data (x^0, y^0), and consider the problem

$$\min_{\mu,\nu} \nu^T x^0 / \mu^T y^0 \tag{3.1}$$

subject to

$$\nu^T x^i / \mu^T y^i \geq 1, \qquad i = 1, \ldots, 0, \ldots, I$$
$$\mu, \nu \geq 0.$$

The problem seeks a set of nonnegative weights (ν, μ) which, when applied to the inputs and outputs of the producer being evaluated, minimizes the ratio of weighted (or "virtual") input to weighted (or "virtual") output, subject to the normalizing constraint that no producer in the sample, including the producer being evaluated, have a ratio less than unity when weights of the producer being evaluated are applied.

This nonlinear ratio model can be converted to the linear programming "multiplier" or "pricing" problem through the Charnes and Cooper (1962) change of variables $u = t\mu$, $v = t\nu$, where $t = (\mu^T y^0)^{-1}$, to obtain the problem

$$\min_{u,v} v^T x^0 \tag{3.2}$$

subject to

$$u^T y^0 = 1$$
$$v^T X - u^T Y \geq 0$$
$$u, v \geq 0,$$

the dual to which is the output-oriented linear programming "envelopment" or "projection" problem

$$TE_0(y^0, x^0) = \max_{\phi,\lambda} \phi \tag{3.3}$$

subject to

$$\begin{aligned} -x^0 + X\lambda &\leq 0 \\ \phi y^0 - Y\lambda &\leq 0 \\ \lambda &\geq 0 \end{aligned}$$

where X is an $N \times I$ input matrix with columns x^i, Y is an $M \times I$ output matrix with columns y^i, and λ is an $I \times 1$ intensity vector. The dual linear programming problems (3.2) and (3.3) are a slightly simplified output-oriented version of the DEA problem introduced by Charnes, Cooper and Rhodes (1978). Consequently they have come to be known as the output-oriented CCR DEA model. The multiplier problem (3.2) has $(M + N)$ variables and $(I + 1)$ constraints, and the envelopment problem (3.3) has $(I + 1)$ variables and $(M + N)$ constraints. Since $(M + N) < (I + 1)$, the multiplier problem is computationally simpler. It must be solved I times, once for each producer in the sample.

The technology implied by the constraints to the envelopment problem (3.3) is

$$P(x) = \{y : \ x \geq X\ \lambda,\ y \leq Y\ \lambda,\ \lambda \geq 0\},\ y \in \mathbb{R}^{M}_{++}, \tag{3.4}$$

which satisfies strong disposability of inputs and outputs, convexity, and homogeneity of degree +1 (or constant returns to scale). Thus problem (3.3) provides a linear programming representation of the radial efficiency measure given in Definition 4. The corresponding graph of the technology is thus a convex free disposal polyhedral cone. The interpretation given to the output-oriented envelopment problem (3.3) is that it seeks the maximum feasible radial expansion in all outputs of the producer being evaluated, consistent with the technology generated by the sample data. The envelopment problem is illustrated in Figure 5 for the $M = N = 1$ case, and in Figure 6 for the $M = 2$ case. In both figures (x^0, y^0) is projected to $(x^0, \phi y^0)$, where $\phi y^0 \in Isoq\ P(x^0)$. The optimal value of ϕ is the technical efficiency measure for the producer being evaluated. The optimal value of λ indicates the linear combination of technically efficient producers to which the producer being evaluated is compared. Technically efficient role models have a positive entries in the optimal value of λ, and other producers, whether or not they are technically efficient, have zero entries in the optimal value of λ. The optimal value of ϕ is unique, although alternative optimal values of λ can exist.

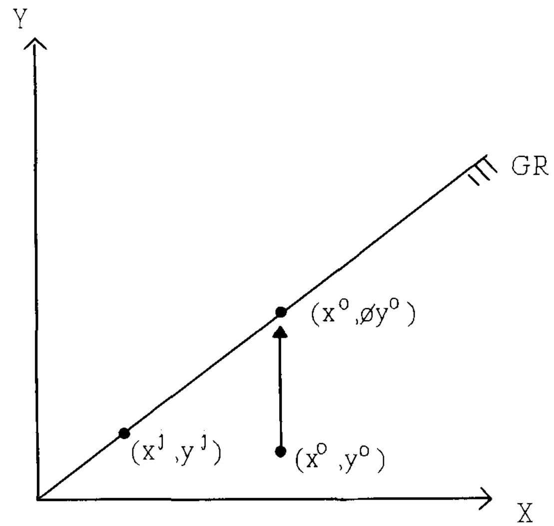

Figure 5. *The Output-Oriented CCR DEA Model (M=N=1)*

The interpretation given to the dual multiplier problem (3.2) is that it seeks optimal multipliers, or normalized shadow prices, for the outputs and inputs of the producer being evaluated. These multipliers are the coefficients of the hyperplane $v^T x - u^T y = v^T x^0 - u^T y^0$. In Figures 5 and 6 ratios of the optimal multipliers describe the slope of a supporting hyperplane to the frontier at $(x^0, \phi y^0)$. The interpretation of the optimal multipliers as normalized shadow prices should be clear. It should also be clear from Figure 6 that optimal multipliers are not necessarily unique. For example, producer B, located at a vertex of *Isoq* $P(x^0)$, has optimal multipliers bounded by the multipliers that describe the slopes of the adjacent facets.

From linear programming duality theory, $v^T x^0 \geq \phi$ and, at optimum, $v^T x^0 = \phi \geq 1$. Output-oriented radial efficiency requires $v^T x^0 = \phi = 1$. By complementary slackness, $v_n > 0$, $n = 1, \ldots, N$ and

$u_M > 0$, $m = 1, \ldots, M \Rightarrow x_n^0 = \sum_{i=1}^{I} \lambda_i x_n^i$, $n = 1, \ldots, N$ and $\phi y_m^0 = \sum_{i=1}^{I} \lambda_i y_m^i$, $m = 1, \ldots, M$. Also by complementary slackness, $x_n^0 > \sum_{i=1}^{I} \lambda_i x_n^i$, $n = 1, \ldots, N$ and $\phi y_m^0 < \sum_{i=1}^{I} \lambda_i y_m^i$, $m = 1, \ldots, M \Rightarrow v_n = 0$, $n = 1, \ldots, N$ and $u_m = 0$, $m = 1, \ldots, M$. Thus slack in any of $M + N$ variables at the optimal projection is associated with zero normalized shadow prices, and positive normalized shadow prices are associated with zero slack. To summarize, a producer is judged to be technically inefficient if at optimum $\phi > 1$, and technically efficient if at optimum $\phi = 1$, even though slack in at most $(M + N - 1)$ dimensions may be present. The allowance for zero multipliers means that this interpretation of technical efficiency correponds to the Debreu-Farrell measure given in Definition 4 rather than the more demanding Koopmans definition given in Definition 1. I return to this issue in Section 5.2.

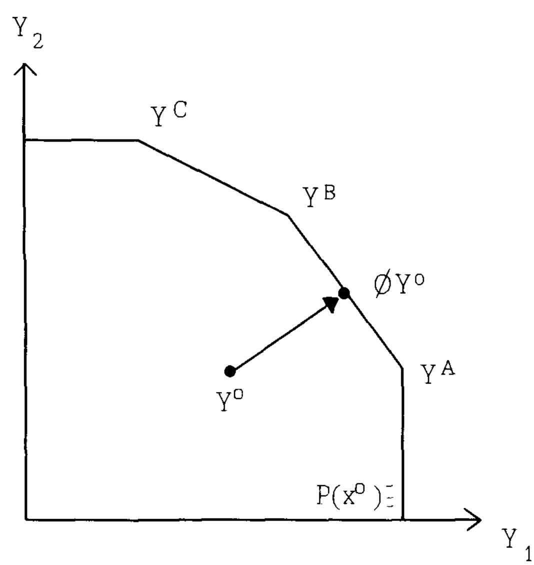

Figure 6. *The Output-Oriented CCR DEA Model (M=2)*

The dual linear programming problems (3.2) and (3.3) have an output orientation to the measurement of technical efficiency. If in the analyst's judgement an input conservation approach is more appropriate, it is possible to adopt an input orientation. The ratio model (3.1) can equally well be converted to the linear programming multiplier problem

$$\max_{u,v} u^T y^0 \tag{3.5}$$

subject to

$$\begin{aligned} v^T x^0 &= 1 \\ u^T Y - v^T X &\leq 0 \\ u, v &\geq 0, \end{aligned}$$

the dual to which is the input-oriented linear programming envelopment problem

$$TE_I(x^0, y^0) = \min_{\theta,\lambda} \theta \tag{3.6}$$

subject to

$$\begin{aligned} \theta x^0 - X\lambda &\geq 0 \\ -y^0 + Y\lambda &\geq 0 \\ \lambda &\geq 0. \end{aligned}$$

The dual linear programming problems (3.5) and (3.6) are slightly simplified version of the input-oriented CCR DEA Model.

The technology implied by the constraints to the envelopment problem (3.6) is

$$L(y) = \{x : \ x \geq X\lambda, \ y \leq Y\lambda, \ \lambda \geq 0\}, \ x \in \mathbb{R}^N_{++}, \tag{3.7}$$

which is precisely the same as that described in (3.4). Thus problem (3.6) provides a linear programming representation of the radial efficiency measure given in Definition 3. However the input orientation of problem (3.6) leads to a different optimal projection. The interpretation of the input-oriented envelopment problem (3.6) is that it seeks the minimum feasible radial scaling of all inputs of the producer being evaluated, consistent with the technology generated by the sample data. The envelopment problem is illustrated in Figure 7 for the $M = N = 1$ case, and in Figure 8 for the $N = 2$ case. In both figures (x^0, y^0) is projected to $(\theta x^0, y^0)$, where $\theta x^0 \in Isoq\ L(y^0)$. The optimal value of θ provides a technical efficiency measure for the producer being evaluated. The non-zero entries in the optimal value of λ identify the technically efficient role models for the producer being evaluated. As in the output-oriented envelopment model (3.3), the optimal value of θ is unique, although alternate optimal values of λ can exist. The interpretation given to the dual multiplier problem is that it seeks optimal multipliers, or normalized shadow prices, for the inputs and outputs of the producer being evaluated. These multipliers are the coefficients of the hyperplane $u^T y - v^T x = u^T y^0 - v^T x^0$, and their ratios describe the slope of a supporting hyperplane to the frontier at $(\theta x^0, y^0)$. As in the CCR multiplier problem (3.2), the optimal multipliers in problem (3.5) are not necessarily unique, as should be clear form Figure 8.

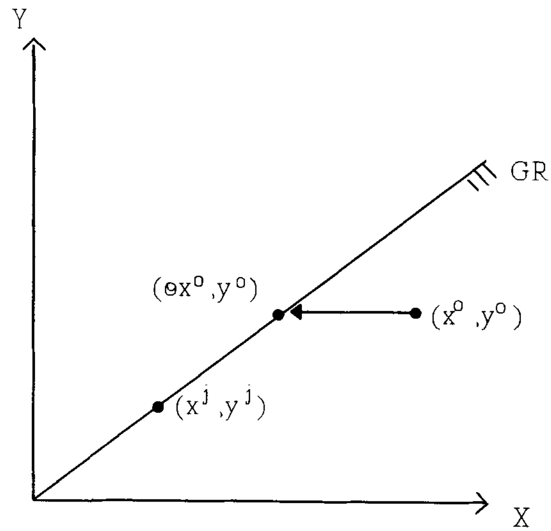

Figure 7. *The Input-Oriented CCR DEA Model (M=N=1)*

Again from linear programming duality theory, $u^T y^0 \leq \theta$ and, at optimum, $u^T y^0 = \theta \leq 1$. Input-oriented radial efficiency requires $u^T y^0 = \theta = 1$. By complementary slackness, $u_m > 0$, $m = 1, \ldots, M$ and $v_n > 0$, $n = 1, \ldots, N \Rightarrow y_m^0 = \sum_{i=1}^{I} \lambda_i y_m^i$, $m = 1, \ldots, M$ and $\theta x_n^0 = \sum_{i=1}^{I} \lambda_i x_n^i$, $n = 1, \ldots, N$. Also $y_m^0 < \sum_{i=1}^{I} \lambda_i y_m^i$, $m = 1, \ldots, M$ and $\theta x_n^0 > \sum_{i=1}^{I} \lambda_i x_n^i$, $n = 1, \ldots, N \Rightarrow u_m = 0$, $m = 1, \ldots, M$ and $v_n = 0$, $n = 1, \ldots, N$. A producer is judged to be technically inefficient if at optimum $\theta < 1$, and technically efficient if at optimum $\theta = 1$, even though positive slack may be present in at most $(M + N - 1)$ dimensions. As before, positive slack is associated with zero normalized shadow prices, which are allowed by the nonnegativity constraints on (u, v) in problem (3.5).

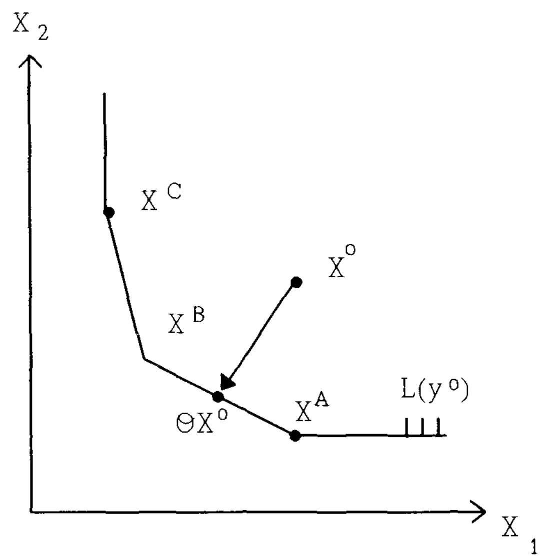

Figure 8. *The Input-Oriented CCR DEA Model (N=2)*

The dual problems (3.2) and (3.3) have an output orientation, while the dual problems (3.5) and (3.6) have an input orientation. It is also possible to convert the ratio problem (3.1) to a pair of dual problems that have both an output-expanding and an input-conserving orientation. The ratio problem (3.1) can be converted to the multiplier problem.

$$\max_{u,v} u^T y^0 - v^T x^0 \tag{3.8}$$

subject to

$$\begin{aligned} -v^T x^0 &\leq -1 \\ u^T y^0 &\leq 1 \\ u^T Y - v^T X &\leq 0 \\ u, v &\geq 0, \end{aligned}$$

the dual to which is the nonlinear programming envelopment problem

$$TE_G(x^0, y^0) = \min_{\delta, \lambda} \delta \tag{3.9}$$

subject to

$$\begin{aligned} \delta x^0 - X\lambda &\geq 0 \\ -\delta^{-1}y^0 + Y\lambda &\geq 0 \\ \lambda &\geq 0. \end{aligned}$$

The technology implied by the constraints to the envelopment problem (3.9) is

$$GR = \{(x,y): \ x \geq X\lambda, \ y \leq Y\lambda, \ \lambda \geq 0\}, \tag{3.10}$$

which is the same as those in (3.4) and (3.7). Thus problem (3.9) provides a nonlinear programming representation of the hyperbolic efficiency measure given in Definition 2. The interpretation given to the envelopment problem (3.9) is that it seeks the maximum equiproportionate contraction in inputs and expansion of outputs for the producer being evaluated, consistent with the technology generated by the sample data as described in (3.10). The hyperbolic technical efficiency score is given by the optimal value of δ, and the technically efficient role models are identified by the non-zero entries in the optimal value of λ. The optimal values of the multipliers (u,v) in the multiplier problem (3.8) describe normalized shadow prices of outputs and inputs at the optimal projection $(\delta x^0, \delta^{-1}y^0)$.

The envelopment problem (3.9) is a nonlinear programming problem. However Färe, Grosskopf and Lovell (1985) showed that it can be converted to a linear programming problem by means of the transformations $\Delta = \delta^2$, $\Lambda = \delta\lambda$. The transformed linear programming envelopment problem is identical to the input-oriented envelopment problem (3.6), although $TE_G(x^0,y^0) = \delta = \Delta^{1/2} \geq \Delta$ in keeping with Proposition 1, and $\lambda = \Lambda/\delta \geq \Lambda$.

How are the solutions to the output-oriented model (3.2)-(3.3), the input-oriented model (3.5)-(3.6) and the graph model (3.8)-(3.9) related? First, optimal values of the objectives in the three envelopmet problems satisfy $\phi^{-1} = \theta = \delta^2$ by virtue of Proposition 2. Second, optimal values of λ in the three envelopment problems can differ, both in terms of the identity of the technically efficient role models and in terms of the magnitudes of the non-zero elements of λ. This is because the different orientation of the three models generates role models of different size. Third, optimal values of (u,v) in the three multiplier models can also differ, again due to the different optimal projections to the frontier.

4. RELAXING CONSTANT RETURNS TO SCALE

The output-oriented CCR DEA model (3.2)-(3.3), the input-oriented CCR DEA model (3.5)-(3.6) and the graph DEA model (3.8)-(3.9) each incorporate the assumption of constant returns to scale in production. In many instances this assumption may be unwarranted. Consequently Banker, Charnes and Cooper (BCC) (1984) generalized the CCR formulation to allow for variable returns to scale. The BCC model envelops the data al least as lightly as does the CCR model, and more tightly if returns to scale are not everywhere constant.

The constant returns to scale ratio problem (3.1) can be converted to a variable returns to scale ratio problem by adding a free variable ν_* to obtain

$$\min_{\mu,\nu,\nu_*} (\nu^T x^0 + \nu_*)/\mu^T y^0 \tag{4.1}$$

subject to

$$\begin{aligned} (\nu^T x^i + \nu_*)/\mu^T y^i \geq 1 \qquad & i = 1,\ldots,0,\ldots,I \\ \nu,\mu \geq 0 \\ \nu_* \text{ free.} \end{aligned}$$

The same change of variables that was used to convert (3.1) to (3.2) can be applied to the variable returns to scale ratio problem (4.1) to obtain the BCC multiplier problem

$$\min_{u,v,v_*} v^T x^0 + v_* \tag{4.2}$$

subject to

$$\begin{aligned} u^T y^0 &= 1 \\ v^T X - u^T Y + v_* &\geq 0 \\ u, v &\geq 0 \\ v_* &\text{ free,} \end{aligned}$$

the dual to which is the BCC variable returns to scale output-oriented linear programming envelopment problem

$$TE_0(y^0, x^0) = \max_{\phi,\lambda} \phi \tag{4.3}$$

subject to

$$\begin{aligned} -x^0 + X\lambda &\leq 0 \\ \phi y^0 - Y\lambda &\leq 0 \\ e^T\lambda &= 1 \\ \lambda &\geq 0, \end{aligned}$$

where e^T is an $I \times 1$ row vector of ones. The technology implied by the constraints to the envelopment problem (4.3) is

$$P(x) = \{y : \ x \geq X\lambda, \ y \leq Y\lambda, \ \lambda \geq 0, \ e^T\lambda = 1\}, \tag{4.4}$$

which satisfies strong disposability and convexity, but not homogeneity. The corresponding graph of the technology is thus a polyhedral convex free disposal hull.

The BCC models (4.2)-(4.3) are structurally similar to the CCR models (3.2)-(3.3). However the BCC envelopment problem contains an additional equality constraint ($e^T\lambda = 1$), which restricts the elements of the intensity vector to sum to one. Thus only convex combinations of sample producers are allowed to be created in forming the production frontier; radial expansions and contractions of sample producers are no longer allowed. This convexification process shrinks the set of feasible production possibilities, and converts a constant returns to scale technology to a variable returns to scale technology.

Adding an equality constraint to the envelopment problem requires adding a free variable (v_*) to the dual multiplier problem. The additional variable allows the supporting hyperplane $v^Tx - u^Ty = v^Tx^0 - u^Ty^0 + v_*$ to the production frontier to have non-zero intercept. The multipliers (u, v) are interpreted as before, as normalized shadow prices at the optimal projection. The additional variable v_* in the multiplier problem provides information on whether returns to scale are increasing, constant, or decreasing at the optimal projection $(x^0, \phi y^0)$. Figure 9 illustrates, for the case $M = N = 1$. Inefficient observation (x^0, y^0) is projected to $(x^0, \phi y^0)$ on the variable returns to scale frontier. At that projection the supporting hyperplane has slope (u/v) and output intercept $v_* > 0$, and decreasing returns prevail at $(x^0, \phi y^0)$.

Optimal solutions to the multiplier problem are not necessarily unique, however, as for producers (x^A, y^A) and (x^C, y^C). However since all optimal solutions for (x^A, y^A) have $v_* > 0$, production is characterized by decreasing returns to scale at (x^A, y^A). Since all optimal solutions for (x^C, y^C) have $v_* < 0$, production is characterized by increasing returns to scale at (x^C, y^C). Production is characterized by constant returns to scale at (x^B, y^B) since there exists at least one optimal solution having $v_* = 0$. I return to this issue below.

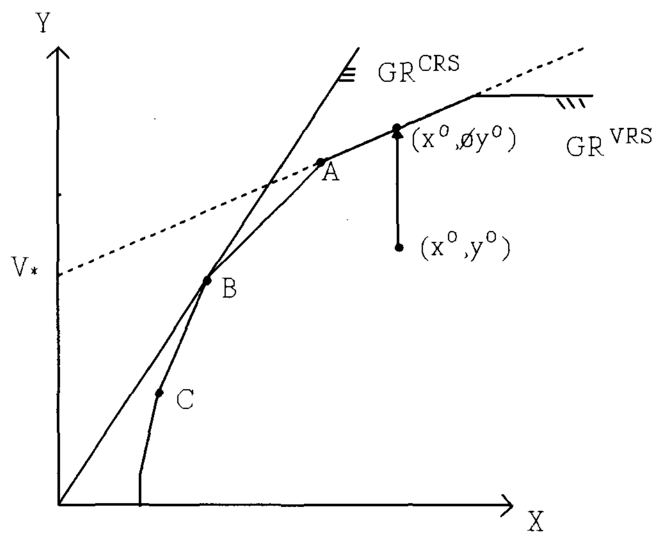

Figure 9. *The Output-Oriented BCC DEA Model (M=N=1)*

A different BCC multiplier problem can also be obtained from the ratio model (4.1), and is expressed as

$$\max_{u,v,u_*} u^T y^0 + u_* \tag{4.5}$$

subject to

$$\begin{aligned} v^T x^0 &= 1 \\ u^T Y - v^T X + u_* &\leq 0 \\ u, v &\geq 0 \\ u_* &\text{ free,} \end{aligned}$$

the dual to which is the input-oriented BCC linear programming envelopment problem

$$TE_I(x^0, y^0) = \min_{\theta,\lambda} \theta \tag{4.6}$$

subject to

$$\begin{aligned} \theta x^0 - X\lambda &\geq 0 \\ -y^0 + Y\lambda &\geq 0 \\ e^T\lambda &= 1 \\ \lambda &\geq 0. \end{aligned}$$

The BCC models (4.5)-(4.6) are structurally similar to the CCR models (3.5)-(3.6), with an equality restriction on λ added in (4.6) and an additional free variable (u_*) added to (4.5).

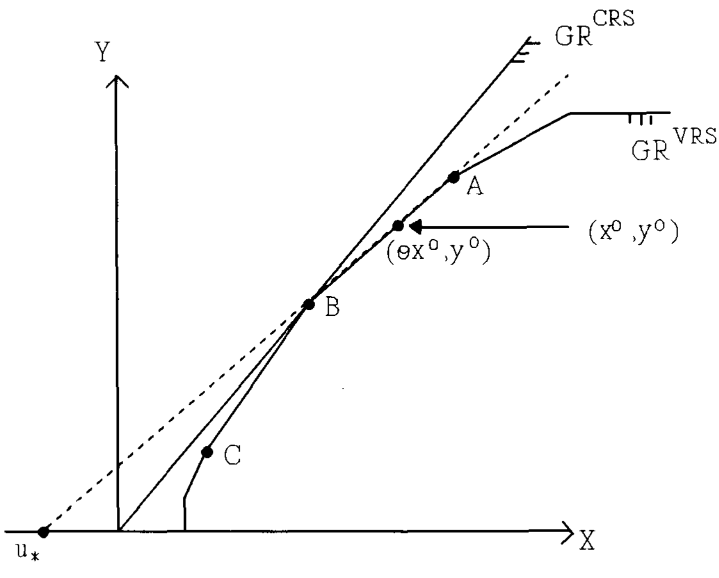

Figure 10. *The Input-Oriented BCC DEA Model (M=N=1)*

The technology implied by (4.6) is the same as the technology implied by (4.3), but the orientation is different. Hence the optimal projection is different. Figure 10 illustrates, for the $M = N = 1$ case. Inefficient producer (x^0, y^0) is projected to $(\theta x^0, y^0)$. Since the input intercept $u_* < 0$, decreasing returns prevail at $(\theta x^0, y^0)$. The remarks concerning non-uniqueness of optimal multipliers made above hold for the input-oriented model as well.

Problems (4.2), (4.3) are output-oriented, and problems (4.5), (4.6) are input-oriented. A problem that seeks to expand outputs and contract inputs by the same proportion can be formulated from problems (3.8), (3.9) by adding the free variable u_* to the maximand of (3.8), adding the equality constraint ($e^T\lambda = 1$) to the constraint set of (3.9), and converting to a linear programming problem as before.

For future reference, I also note that there exists a non-increasing returns to scale DEA model. To form a non-increasing returns to scale output-oriented DEA model, add to the minimand of (3.2) the constrained variable $v_* \geq 0$, and add to the constraint set of (3.3) the condition $e^T\lambda \leq 1$. To form the same non-increasing returns to scale input-oriented DEA model, add to the maximand of (3.5) the constrained variable $u_* \leq 0$, and add to the constraint set of (3.6) the condition $e^T\lambda \leq 1$. Solving all three DEA models (the CCR constant returns to scale model, the BCC variable returns to scale model and the non-increasing returns to scale model) with either output orientation or input orientation yields the following result.

Proposition 4. (Färe, Grosskopf and Lovell (1985)) *For the output-oriented models, technology exhibits the following scale characterictics at the optimal projection:*

$$\begin{aligned}&\textit{increasing returns to scale} \Leftrightarrow 1 \leq \phi^{VRS} < \phi^{NIRS} = \phi^{CRS}\\&\textit{constant returns to scale} \Leftrightarrow 1 \leq \phi^{VRS} = \phi^{NIRS} = \phi^{CRS}\\&\textit{decreasing returns to scale} \Leftrightarrow 1 \leq \phi^{VRS} = \phi^{NIRS} < \phi^{CRS}\end{aligned}$$

For the input-oriented models, technology exhibits the following scale characteristics at the optimal projection:

$$\begin{aligned}&\textit{increasing returns to scale} \Leftrightarrow 1 \geq \theta^{VRS} > \theta^{NIRS} = \theta^{CRS}\\&\textit{constant returns to scale} \Leftrightarrow 1 \geq \theta^{VRS} = \theta^{NIRS} = \theta^{CRS}\\&\textit{decreasing returns to scale} \Leftrightarrow 1 \geq \theta^{VRS} = \theta^{NIRS} > \theta^{CRS}\end{aligned}$$

The above proposition requires solving three DEA problems in order to determine the value of scale economies at the optimal projection. The next proposition requires solving only the variable returns to scale BCC model.

Proposition 5. (Banker and Thrall (1992)) *For the output-oriented BCC model, technology exhibits the following scale characteristics at the optimal projection:*

$$\textit{increasing returns to scale} \Leftrightarrow \max\{v_*\} < 0$$

constant returns to scale $\Leftrightarrow \min\{v_*\} < 0 < \max\{v_*\}$
decreasing returns to scale $\Leftrightarrow \min\{v_*\} > 0$.

For the input-oriented BCC model, technology exhibits the following scale characteristics at the optimal projection:

increasing returns to scale $\Leftrightarrow \min\{u_*\} > 0$
constant returns to scale $\Leftrightarrow \min\{u_*\} < 0 < \max\{u_*\}$
decreasing returns to scale $\Leftrightarrow \max\{u_*\} < 0$.

I conclude this section with two observations on the use of DEA to draw inferences on scale economies. First, orientation matters. An output-oriented model generates a different optimal projection than an input-oriented model does, and different optimal projections can generate qualitatively different inferences on the nature of scale economies. For example, output-oriented projections are more likely to occur on the decreasing returns to scale portion of the production frontier, whereas input-oriented projections are more likely to accur in the increasing returns to scale portion of the production frontier. Second, Propositions 4 and 5 can lead to different inferences concerning the nature of scale economies. In particular, Proposition 5 typically generates a larger region of the production set for which constant returns to scale are inferred than does Proposition 4.

5. MODIFYING STRONG DISPOSABILITY

Thus far I have maintained the assumption of strong, or free, disposability of inputs and outputs. This assumption can be relaxated to one of weak, or costly, disposability of a subvector of inputs and/or outputs. Weak disposability is appropriate when, for example, it is costly to dispose of, i.e., abate, undesirable outputs such as pollutants generated as by-products of the production of desirable outputs such as electricity. Alternatively, the assumption can be strengthened so as to eliminate the possibility of slacks in envelopment problems. Strengthening strong disposability is appropriate when, for example, expert judgment suggests that normalized shadow prices should be positive. I consider these modifications in turn, using for simplicity the output-oriented variable returns to scale BCC model.

5.1. Weak Disposability

Partition the output vector $y \in \mathbb{R}^M_{++}$ into a strongly disposable subvector $y_S \in \mathbb{R}^S_{++}$ and its complement, a weakly disposable subvector

$y_W \in \mathbb{R}_{++}^{M-S}$. Then the variable returns to scale output-oriented BCC envelopment problem is expressed as

$$TE_0(y_S^0, y_W^0, x^0) = \max_{\phi,\lambda,\mu} \phi \tag{5.1}$$

subject to

$$\begin{aligned} -x^0 + X\lambda &\leq 0 \\ \phi y_S^0 - Y_S\lambda &\leq 0 \\ \phi y_W^0 - \mu Y_W \lambda &= 0 \\ \lambda &\geq 0 \\ e^T\lambda &= 1 \\ 1 \geq \mu &\geq 0. \end{aligned}$$

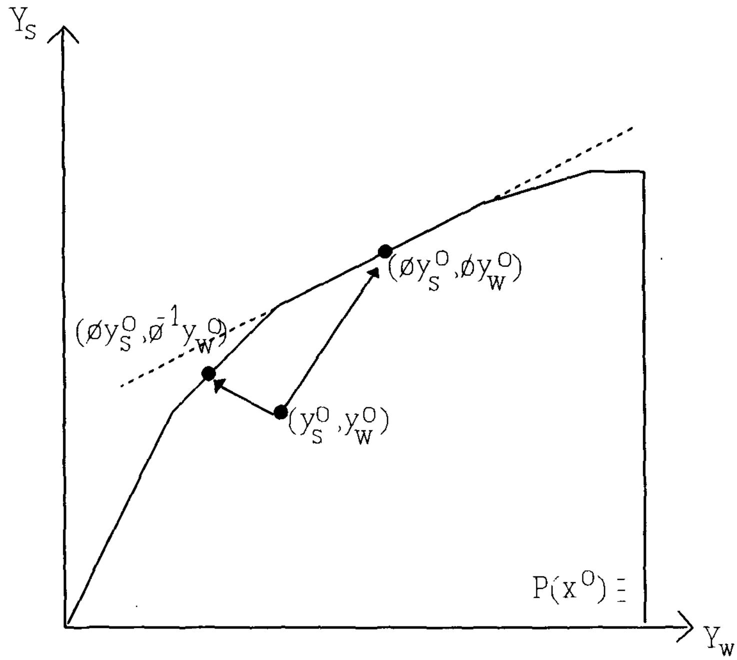

Figure 11. *The Output-Oriented BCC DEA Model with Weak Disposability of a Subset of Outputs (M=2)*

The equality constraints ($\phi y_W^0 - \mu Y_W \lambda = 0$), together with ($1 \geq \mu \geq 0$), guarantee that y_W is weakly, but not strongly, disposable. These constraints make problem (5.1) a nonlinear programming problem. However it can be converted to a linear program by setting $\mu = 1$; such a transformation leaves the optimal values of (ϕ, λ) unchanged. The problem is illustrated in Figure 11. Notice that at the optimal projection $(\phi y_S^0, \phi y_W^0, x^0)$ the normalized shadow price of y_W is negative, indicating the cost of disposing of, or abating, the undesirable output y_W.

Problem (5.1) generates technical efficiency scores and identifies technically efficient role models, and the dual multiplier problem generates normalized shadow prices of all variables, including the weakly disposable outputs. However when the weakly disposable outputs are undesirable, the orientation of problem (5.1) toward expanding all outputs, desirable and undesirable, may be considered inappropriate. At least three alternative orientations are available. Perhaps a more suitable measure of performance is provided by a technical efficiency measure that measures the ability to expand all desirable outputs and reduce all undesirable outputs. In this case a modified version of the hyperbolic technical efficiency measure given in Definition 2 is appropriate, and the envelopment problem (5.1) is converted to

$$TE_0(y_S^0, y_W^0, x^0) = \max_{\phi,\lambda,\mu} \phi \tag{5.2}$$

subject to

$$\begin{aligned} -x^0 + X\lambda &\leq 0 \\ \phi y_S^0 - Y_S\lambda &\leq 0 \\ \phi^{-1} y_W^0 - \mu Y_W\lambda &= 0 \\ \lambda &\geq 0 \\ e^T\lambda &= 1 \\ 1 \geq \mu &\geq 0. \end{aligned}$$

This is also nonlinear programming problem, although it can be converted to a linear programming problem using the same techniques as used in problems (5.1) and (3.9). The problem seeks the maximum equiproportinate expansion of desirable outputs, and contraction of undesirable outputs, consistent with best practice technology. As Figure 11 illustrates, the optimal projection $(\phi y_S, \phi^{-1} y_W^0, x^0)$ differs from the optimal projection $(\phi y_S^0, \phi y_W^0, x^0)$ obtained from problem (5.1), and the optimal normalized shadow prices differ as well.

The degree of the deviation of the weakly disposable technology from the strongly disposable technology provides an indication of the costliness of disposing of the undesirable outputs. Abatement costs can be inferred in either of two ways: (i) from the magnitudes of the normalized shadow prices associated with y_W in the multiplier dual to problem (5.1) or problem (5.2), or (ii) by comparing the solutions to the pair of envelopment problems (5.1), (4.3) or (5.2) and the corresponding problem with strong disposability of all variables. Other orientations are easily obtained; see Färe et al. (1989) for details. The entire analysis can also be applied to the case in which a subvector of inputs is weakly, but not strongly, disposable.

5.2. Constraining Normalized Shadow Prices

In Figure 1 producer x^D receives a radial efficiency score of $\theta^D < 1$, which overstates its true efficiency since slack in input x_2 remains at the optimal projection $\theta^D x^D$. In the dual multiplier program, the optimal multiplier $v_2 = 0$. A similar result appears in Figure 2, where ϕ^D understates the true inefficiency of producer y^D, since slack in output y_1 remains at the optimal projection $\phi^D y^D$. In both cases the possibility of positive slack at an optimal solution to the envelopment problem can be eliminated by enforcing strict positivity of the multipliers in the dual multiplier problem. In either case the property of strong disposability is strengthened, by an amount to be determined by the analyst, perhaps guided by expert judgment.

Consider, for example, the output-oriented variable returns to scale BCC problems (4.2)-(4.3), and suppose that it is desired to restrict multipliers to be strictly positive. Then, following Cook, Kazakov and Roll (1989), the BCC multiplier problem (4.2) becomes

$$\min_{u,v,v_*} v^T x^0 + v_* \tag{5.3}$$

subject to

$$\begin{aligned} u^T y^0 &= 1 \\ v^T X - u^T Y + v_* &\geq 0 \\ v - v_L &\geq 0 \\ u - u_L &\geq 0 \\ u, v &\geq 0 \\ v_* &\text{ free,} \end{aligned}$$

where v_L and u_L are $N \times 1$ and $M \times 1$ column vectors of strictly positive lower bounds on input and output multipliers, respectively. Adding

$N + M$ constraints to the multiplier problem requires adding $N + M$ variables (z_n, z_m) to the dual envelopment problem, which becomes

$$TE_0(y^0, x^0) = \max_{\phi,\lambda,z_N,z_M} \phi \tag{5.4}$$

subject to

$$\begin{aligned} -x^0 + X\lambda - z_N v_L &\leq 0 \\ \phi y^0 - Y\lambda - z_M u_L &\leq 0 \\ \lambda &\geq 0 \\ e^T\lambda &= 1, \end{aligned}$$

where z_N and z_M are diagonal matrices with nonzero elements z_n and z_m, respectively.

Constraining the multipliers in problem (5.3) to be strictly positive leads to an expansion of the graph of the technology described by the constraints in problem (5.4). Consequently technical efficiency scores obtained from problem (5.4) are no better than those obtained from problem (4.2).

The strictly positive multiplier restrictions imposed on problem (5.3) are intended merely to eliminate slacks from the optimal solution to the envelopment problem (5.4). However multiplier restrictions can take many forms to serve many purposes. A much more flexible formulation of problems (5.3) and (5.4) is provided by the dual programs

$$\min_{u,v,v_*} v^T x^0 + v_* \tag{5.5}$$

subject to

$$\begin{aligned} u^T y^0 &= 1 \\ v^T X - u^T Y + v_* &\geq 0 \\ v^T A + u^T B &\geq 0 \\ u, v &\geq 0 \\ v_* &\text{ free} \end{aligned}$$

and

$$TE_0(y^0, x^0) = \max_{\phi,\lambda,z} \phi \tag{5.6}$$

subject to

$$
\begin{aligned}
-x^0 + X\lambda - Az &\leq 0 \\
\phi y^0 - Y\lambda - Bz &\leq 0 \\
\lambda &\geq 0 \\
e^T\lambda &= 1,
\end{aligned}
$$

where A is an $N \times K$ matrix of coefficients for the multipliers v and B is an $M \times K$ matrix of coefficients for the multipliers u, there being K constraints imposed on the multipliers. Adding K inequality constraints to the multiplier problem (5.5) requires adding the same number of variables z to the envelopment problem (5.6). The constraints can take the form of restrictions on multipliers or on their ratios. Extensions are provided by Thompson et al. (1986,1990), who develop assurance regions for multipliers, and by Charnes et al. (1990), who require multipliers to belong to closed cones.

6. RELAXING CONVEXITY

Thus far all DEA models have had convexity as a maintained hypothesis. This means that inefficient producers are compared not to efficient producers, but to non-existent convex (or linear) combinations of efficient producers. Moreover, an inefficient producer can be compared to a convex combination of efficient producers, none of which dominate it. Both possibilities can be avoided if the convexity assumption is relaxed. I now show how to build a non-convex variable returns to scale DEA model that imposes only strong disposability. The model was introduced by Deprins, Simar and Tulkens (1984), who coined the model FDH, because the production frontier is the (non-convex) free disposal hull of the data generated by the sample producers.

The output-oriented FDH envelopment problem is expressed as

$$
TE_0(y^0, x^0) = \max_{\phi,\lambda} \phi \qquad (6.1)
$$

subject to

$$
\begin{aligned}
-x^0 + X\lambda &\leq 0 \\
\phi y^0 - Y\lambda &\leq 0 \\
e^T\lambda &= 1 \\
\lambda &\geq 0 \\
\lambda &\in \{0, 1\}.
\end{aligned}
$$

This problem is identical to the BCC variable returns to scale envelopment problem (4.3), with the additional constraint $\lambda \in \{0,1\}$ on the intensity vector. Together with the restriction $e^T\lambda = 1$, the added restriction implies that exactly one element of λ has a value of unity, the remaining $(I - 1)$ elements being zero. This in turn implies that the technical efficiency of the producer being eveluated is calculated relative to exactly one undominated producer, that producer being the one assigned the only non-zero value of λ in the envelopment problem.

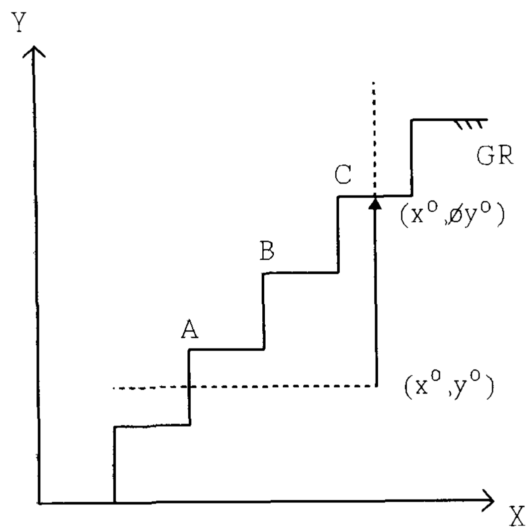

Figure 12. *The Output-Oriented FDH Model (M=N=1)*

The output-oriented FDH problem is illustrated in Figures 12 and 13. In Figure 12 producer (y^0, x^0) is dominated by producers A, B and C, and perhaps by others as well, but the output orientation of the problem identifies producer C as the role model for producer (y^0, x^0). The optimal projection $(\phi y^0, x^0)$ leaves input slack in the amount $(x^0 - x^C)$. In Figure 13 producer (y^0, x^0) is dominated by producers A, B and C, and producer C remains the role model for producer (y^0, x^0).

Here the optimal projection $(\phi y^0, x^0)$ leaves output slack in the amount $(y_1^C - \phi y_1^0)$.

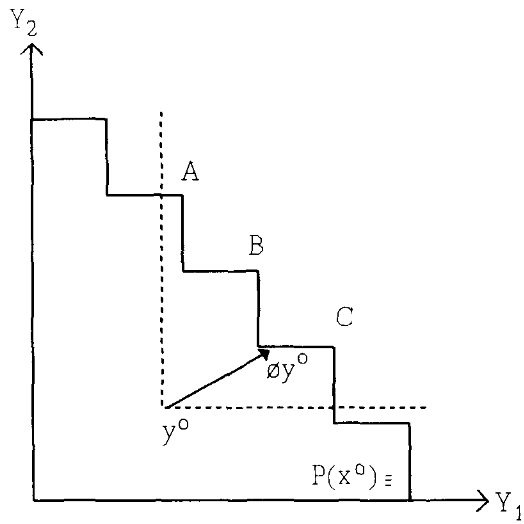

Figure 13. *The Output-Oriented FDH Model (M=2)*

The FDH envelopment problem (6.1) has a dual multiplier problem, but it is uninformative, since it is clear from Figures 12 and 13 that all multipliers are either zero or infinite. A statement of the FDH multiplier problem is given in Lovell and Vanden Eeckaut (1993). The FDH envelopment problem is a mixed integer programming problem, unlike DEA problems, which are (or can be converted to) linear programming problems. Nonetheless the FDH problem is easier to solve, since it involves only a series of vector comparisons. For the producer (y^0, x^0) being evaluated, it is a straightforward exercise to identify the set D_0 of dominating producers, that is, producers for which $\binom{y}{-x} \geq \binom{y^0}{-x^0}$. If this set is empty, the producer being evaluated is undominated, and so $\phi = 1$, no slacks exist, and the exercise is over. If D_0 is nonempty, the second set of functional constraints in the FDH envelopment program is transformed to

$$\phi \leq \frac{\sum_{k \in D_0} \lambda_k y_m^k}{y_m^0}, \quad m = 1, \ldots, M, \tag{6.2}$$

from which it follows that, at optimum,

$$\phi = \max_{k \in D_0} \left\{ \min_{m=1,\dots,M} \left\{ \frac{y_m^k}{y_m^0} \right\} \right\}, \quad k = 1, \dots, I. \tag{6.3}$$

From this maximin procedure an optimal radial efficiency score ϕ is obtained for each producer. The "max" part of the algorithm identifies the most dominant producer relative to which producer (y^0, x^0) is compared. Once the most dominant producer is identified, slacks are calculated from the "min" part of the algorithm.

An extensive discussion of FDH, a comparison of FDH and DEA, and a detailed analysis of the use of FDH with time-series or panel data, is provided by Tulkens and Vanden Eeckaut (1993).

7. ECONOMIC EFFICIENCY

Thus far all models considered have been directed toward the measurement of technical efficiency. However if output prices $p \in \mathbb{R}^M_{++}$ and/or imput prices $w \in \mathbb{R}^N_{++}$ are available, economic efficiency can be calculated, and decomposed into its technical and allocative components. In this section I show how to use the linear programming techniques of DEA to calculate and decompose cost efficiency.

Cost efficiency is defined in Definition 7 as the ratio of minimum cost to actual cost. It can be calculated for producer (y^0, x^0, w^0) by solving the linear programming problem

$$c(y^0, w^0) = \min_x w^{0T} x \tag{7.1}$$

subject to

$$\begin{aligned} x - X\lambda &\geq 0 \\ -y^0 + Y\lambda &\geq 0 \\ e^T\lambda &= 1 \\ \lambda &\geq 0. \end{aligned}$$

The constraint set in this problem is the same as that of the strong disposal variable returns to scale BCC envelopment problem given in (4.6). Solving this problem identifies the cost minimizing input vector, say x^*, for the producer being evaluated. Once x^* is known, Definition 7 can be used to calculate the cost efficiency of the producer being evaluated as $CE(x^0, y^0, w^0) = c(y^0, w^0)/w^{0T}x^0 = w^{0T}x^*/w^{0T}x^0$. Then solving problem (4.6) provides the input-oriented technical efficiency of the producer being evaluated as $TE_I(x^0, y^0) = w^{0T}\theta x^0/w^{0T}x^0 = \theta$. Finally, using Definition 8, the input allocative efficiency of the producer being evaluated is obtained as $AE_I(x^0, y^0, w^0) = w^{0T}x^*/w^{0T}\theta x^0$.

Similar exercises can be undertaken to measure and decompose revenue efficiency and profit efficiency. See Färe, Grosskopf and Lovell (1994) for details.

8. DATA ISSUES

In this section I consider three issues concerning the input-output data used in DEA. The first issue conserns the ability of DEA to accommodate nonnegative, rather than strictly positive, input vectors and output vectors when measuring technical or economic efficiency. The second issue concerns the possibility that some inputs or outputs may be nondiscretionary, beyond the control of management. The third issue concerns the ability of DEA to accommodate environmental variables that categorize producers, but which are neither inputs nor outputs. I address each of these data issues in turn.

8.1. Relaxing Strict Positivity of Variables

Thus far every producer has been assumed to use strictly positive amounts of every input to produce strictly positive amounts of every output. This rules out specialization in the use of inputs and in the production of outputs. Under what conditions can this assumption be relaxed to $x \in \mathbb{R}^N_+$, $y \in \mathbb{R}^M_+$, where every producer uses a positive amount of at least one input to produce a positive amount of at least one output?

Karlin (1959) required that the data matrices X and Y each have strictly positive row sums and strictly positive column sums, in a linear programming problem similar to DEA. Karlin's condition thus allows $x \in \mathbb{R}^N_+$, $y \in \mathbb{R}^M_+$. More recently Charnes, Cooper and Thrall (1986,1991) provided a rigorous demonstration that DEA models can be applied to data matrices satisfying Karlin's condition.

In the vast majority of empirical applications the data matrices satisfy Karlin's condition, and it is appropriate to use any of the DEA models discussed above, and others as well. But in some applications some producers may use negative amounts of at least one input, or may produce negative amounts of at least one output. Examples of the latter case include Lovell (1994) and Pastor (1993c). What can be done in these circumstances?

To answer this question I use the notion of *translation invariance*, which I use to summarize the results of Ali and Seiford (1990) and Pastor (1993a,b). By translation I refer to affine displacements of the input and output vectors by means of $\bar{x}^i = x^i + \alpha,\ \alpha \geq 0$, and $\bar{y}^i = y^i + \beta,\ \beta \geq 0,\ i = 1, \ldots, I$, so as to eliminate zero or negative values that may exist in x^i and y^i. Thus $\bar{x}^i \in \mathbb{R}^N_{++}$, $\bar{y}^i \in \mathbb{R}^M_{++}$, $i = 1, \ldots, I$.

Proposition 6. *The constant returns to scale CCR envelopment models (3.3) and (3.6) are not translation invariant. Producers judged technically efficient using (x, y) data can be judged technically inefficient using $(\bar{x}, \bar{y})$ data, and vice versa. Rankings of producers are not invariant to the transformation.*

Proof. The input-oriented CCR envelopment model (3.6) with translated data $(\bar{x}, \bar{y})$ has functional constraints

$$\begin{aligned} \theta x^0 - X\lambda - (e^T\lambda - \theta)\alpha \geq 0 \\ -y^0 + Y\lambda + (e^T\lambda - 1)\beta \geq 0, \end{aligned}$$

which differs from the constraint set in (3.6) unless $e^T\lambda = 1$ and $\theta = 1$. The same procedure can be used to show that the output-oriented CCR envelopment model (3.3) with data $(\bar{x}, \bar{y})$ has a different constraint set than that in (3.3). ■

Proposition 7. *The variable returns to scale BCC and FDH envelopment models (4.3), (4.6) and (6.1) are translation invariant in a limited sense. Input- (output-) oriented models are invariant to output (input) translation. Input (output) translation in an input- (output-) oriented model generates the same classification of producers as efficient or inefficient, but the ranking of inefficient producers is not invariant to the translation. The first part of the second condition also requires nonnegative inputs (outputs) and at least one positive input (output).*

Proof. The input-oriented BCC and FDH envelopment models with translated data $(\bar{x}, \bar{y})$ have functional constraints

$$\begin{aligned} \theta x^0 - X\lambda - (1 - \theta)\alpha \geq 0 \\ -y^0 + Y\lambda \geq 0. \end{aligned}$$

Thus these models are output translation-invariant, but not input translation-invariant. The classification of producers as efficient or inefficient is invariant to input translation, but the ranking of inefficient producers is sensitive to the translation. The proof for output-oriented models is similar. ■

The conclusion is that if data must be translated to eliminate zeros or negative values in the data matrices, the constant returns to scale CCR models are inappropriate. The variable returns to scale BCC and FDH models are invariant to translation of either inputs or outputs, depending on the orientation, but if both inputs and outputs are translated the invariance property is restricted to the classification of the producers as efficient or inefficient. The ranking of inefficient producers is invariant to output (but not input) translation in input-oriented models, and to input (but not output) translation in output-oriented models.

8.2. Incorporating Non-Discretionary Variables

Thus far I have assumed that all N inputs and all M outputs are freely variable, and I have measured technical efficiency in terms of the ability of producers to adjust all inputs, or all outputs, or all variables, equiproportionately. However in practice some inputs or some outputs may be temporarily fixed, or non-discretionary. I now follow Banker and Morey (1986) to show how to incorporate non-discretionary variables into a DEA analysis. Extensions of this approach are provided by Golany and Roll (1993).

Suppose that all M outputs are freely variable, and partition the N inputs into V variable inputs $x_V \in \mathbb{R}^{V}_{++}$ and $N-V=F$ fixed inputs $x_F \in \mathbb{R}^{N-V}_{++}$. Then the input-oriented variable returns to scale BCC envelopment problem becomes

$$TE_I(x_V^0, x_F^0, y^0) = \min_{\theta,\lambda} \theta \tag{8.1}$$

subject to

$$\begin{aligned} \theta x_V^0 - X_V\lambda &\geq 0 \\ x_F^0 - X_F\lambda &\geq 0 \\ -y^0 + Y\lambda &\geq 0 \\ \lambda &\geq 0 \\ e^T\lambda &= 1, \end{aligned}$$

which should be compared to problem (4.6), in which all N inputs are discretionary. In problem (8.1) the objective is to radially contract only discretionary inputs, while maintaining the constraints that fixed inputs not fall short of a convex combination of fixed input usage in the sample. Thus in the envelopment problem (8.1) fixed inputs continue to play a role in the analysis, although a somewhat diminished role since they are no longer discretionary.

Since the optimal solution for problem (4.6) is necessarily feasible for problem (8.1), but the optimal solution for problem (8.1) is not necessarily feasible for problem (4.6), it follows that at optimum $\theta(8.1) \leq \theta(4.6) \leq 1$. This can be seen in Figure 1, where $\theta_V^C < \theta^C < 1$ if input x_1 is treated as discretionary and input x_2 is treated as non-discretionary.

The inclusion of non-discretionary outputs in the input-oriented BCC envelopment model (4.6) is straightforward. The problem requires no alteration whatsoever, although any slacks in nondiscretionary outputs (or inputs, for that matter) must be interpreted differently since they no longer constitute a component of overall inefficiency.

8.3. Environmental Variables

I now consider how to incorporate variables that are neither inputs to, nor outputs of, the production process. Nonetheless they are thought to influence performance, and should be incorporated into the analysis. Such variables are exogenous, and they can be either continuous or categorical. An example of the former is population density. An example of the latter is type of ownership, public or private. There are several ways of incorporating these environmental variables into a DEA analysis.

Let $z^i \in \mathbb{R}_+$, i = 1, ..., I be a discrete or continuous environmental variable, and assume that larger values of z^i are preferred to smaller values. If it is desired to evaluate the productive efficiency of a producer relative to the subset of producers having no more (no less) favorable environments, then all that is required is to restrict the comparison set to those producers having no more (no less) favorable environments. The comparison set then consists of the set of producers indexed by $J^0 = \{i = 1, \ldots, I : \; z^i \leq (\geq) z^0\}$. The size of the comparison set J^0 generally differs for each producer being evaluated. This procedure can be applied to any of the DEA models proviously discussed. For example, the output-oriented variable returns to scale BCC envelopment problem becomes

$$TE_0(y^0, x^0, z^0) = \max_{\phi, \lambda_J} \phi \tag{8.2}$$

subject to

$$\begin{aligned} -x^0 + X_J\lambda_J &\leq 0 \\ \phi y^0 - Y_J\lambda_J &\leq 0 \\ \lambda_J &\geq 0 \\ e^T\lambda_J &= 1, \end{aligned}$$

where λ_J is $(J^0 \times 1)$, J^0 being the number of producers with no more (no less) favorable environments than that of the producer being evaluated. Similarly, X_J is $(N \times J^0)$ and Y_J is $(M \times J^0)$. The reduced dimensionality of the problem reflects the restricted comparison sets relative to which producers are evaluated.

In the above formulation there is a natural ordering of the variable characterizing the operating environment. This is not always the case. Suppose that z^i = (private, public), $i = 1, \ldots, I$. Since it is not clear which type of ownership is more conducive to operating efficiency, the approach outlined above cannot be implemented. The following approach, developed by Charnes, Cooper and Rhodes (1981), is appropriate when the effect of a categorical environmental variable is to be determined rather than known in advance. (1) Partition the data set into two mutually exclusive and exhaustive subsets by type of ownership. (2) Solve two DEA problems separately, one for private producers and the other for public producers. (3) Project all private producers to the private production frontier, and do likewise for all public producers. This step eliminates managerial inefficiency in both sectors. (4) Solve a DEA problem on the merged, managerially efficient, data set consisting of all producers, public and private. The effect of ownership on performance is determined by the efficiency scores of private and public producers in this second stage.

An alternative approach to the incorporation of environmental variables, the direction of whose impact is unknown in advance, has been developed by Grifell, Prior and Salas (1992). If $z^i \in \mathbb{R}_+$, i = 1, ..., I, is an environmental variable whose influence on productive efficiency is to be determined, the output-oriented variable returns to scale BCC envelopment problem becomes

$$TE_0(y^0, x^0, z^0) = \max_{\phi,\lambda} \phi \tag{8.3}$$

subject to

$$\begin{aligned} -x^0 + X\lambda &\leq 0 \\ \phi y^0 - Y\lambda &\leq 0 \\ z^0 - z^T\lambda &= 0 \\ \lambda &\geq 0 \\ e^T\lambda &= 1, \end{aligned}$$

the multiplier dual to which is

$$\min_{u,v,w,v_*} \; v^T x^0 + w z^0 + v_* \tag{8.4}$$

subject to

$$\begin{aligned} u^T y^0 &= 1 \\ v^T X - u^T Y + w^T Z + v_* &\geq 0 \\ u, v &\geq 0 \\ w, v_* &\text{free.} \end{aligned}$$

The envelopment problem (8.3) has $(N + M + 2)$ constraints and $(I + 1)$ variables, while the multiplier problem (8.4) has $(I + 1)$ constraints and $(N + M + 2)$ variables. The sign of the free multiplier w associated with the added equality constraint $(z^0 - z^T\lambda = 0)$ determines whether the environmental variable enhances or inhibits the performance of the producer being evaluated. The impact may be positive for some producers and negative for others, suggesting that the magnitude of z may be too small for the former and too large for the latter.

A potential difficulty with the approach embodied in the envelopment problem (8.2) is that restricting the comparison set may generate an undesirably large number of producers being judged efficient. This problem is likely to be much more serious in the approach embodied in the envelopment problem (8.3). One way of circumventing this problem is to adopt a two-stage approach. In the first stage efficiencies are calculated using a DEA model in which the environmental variables are ignored. In the second stage variation in calculated efficiencies is attributed to variation in operating environments by means of a regression model of general form $\theta^i = f(z^i) + e^i$, $i = 1, \ldots, I$. Dimensionality is not reduced in this approach, and estimated regression coefficients provide information on the direction and magnitude of the effects of the environmental variables on productive efficiency. Details and extensions of this two-stage approach appear in Fried, Lovell and Vanden Eeckaut (1993).

9. CHANCE-CONSTRAINED DEA

DEA is non-stochastic, and relies heavily on the accuracy of the underlying data. Although conventional linear programming sensitivity analysis is available for testing the sensitivity of DEA results to perturbations in the data, it is desirable to have a stochastic DEA technique. The application of chance-constrained programming techniques to DEA models represents one strand of research directed to this end. Initial efforts to build a chance-constrained DEA are reported in Land, Lovell and Thore (1993) and Olesen and Petersen (1993). Here I outline the former model, but it should be noted that the Olesen and Petersen model is quite different, and joint work is underway seeking to reconcile the two different formulations.

The following developments are based on the input-oriented variable returns to scale BCC model, but they can equally well be based on the other DEA models. Suppose that the evaluator is uncertain about the accuracy of the data used to measure the efficiency of the producers in the sample. This uncertainty suggests the use of chance-constrained programming techniques, in which case the envelopment problem (4.6) can be rewritten as

$$TE_I(x^0, y^0) = \min_{\theta,\lambda} \theta \tag{9.1}$$

subject to

$$\begin{aligned} Pr((\theta x^0 - X\lambda) \geq 0) &\geq P_N \\ Pr((-y^0 + Y\lambda) \geq 0) &\geq P_M \\ \lambda &\geq 0 \\ e^T\lambda &= 1, \end{aligned}$$

where P_M and P_N are $(M \times 1)$ and $(N \times 1)$ vectors of probability levels. The interpretation of problem (9.1) is as follows. The evaluator's objective is to radially contract x^0 as much as possible, subject to the constraint that the technically efficient projection $(\theta x^0, y^0)$ "probably" be feasible relative to the best practice frontier constructed from the sample data. Because of uncertainty about the accuracy of the sample data, however, the evaluator is uncertain about the exact placement of the frontier. Consequently the prudent evaluator allows for the possibility that the best practice feasibility constraints might be violated some (small) percentage of the time.

Next, assume that y_m^i is a randomly distributed normal variable with expectation Ey_m^i and covariance matrix $\mathrm{COV}(y_m^i, y_m^j)$, and that x_n^i is a randomly distributed normal variable with expectation Ex_m^i and covariance matrix $\mathrm{COV}(x_n^i, x_n^j)$. Note that covariances are among producers. Following Charnes and Cooper (1963), problem (9.1) can be converted to certainty-equivalent form

$$TE_I(x^0, y^0) = \min_{\theta,\lambda} \theta \tag{9.2}$$

subject to

$$\begin{aligned} \theta x^0 - X\lambda &\geq (EX - X)\lambda + F^{-1}(P_N)\Sigma_X \\ -y^0 + Y\lambda &\geq -(EY - Y)\lambda + F^{-1}(P_M)\Sigma_Y \\ \lambda &\geq 0 \\ e^T\lambda &= 1, \end{aligned}$$

where $F(\cdot)$ is the distribution function of the standard normal distribution and

$$\Sigma_X = \left[\sum_{i=1}^{I}\sum_{j=1}^{I} \alpha_i\alpha_j \mathrm{COV}\left(x_n^i, x_n^j\right)\right]^{1/2}$$

$$\Sigma_Y = \left[\sum_{i=1}^{I}\sum_{j=1}^{I} \alpha_i\alpha_j \mathrm{COV}\left(y_m^i, y_m^j\right)\right]^{1/2}$$

with $\alpha_i = \lambda_i$ for all $i \neq 0$, and $\alpha_i = (\lambda_0 - 1)$ for $i = 0$. The constraint set in (9.2) is convex only if $P_N \geq 0.5,\ P_M \geq 0.5$.

Problem (9.2) is a chance-constrained formulation of the input-oriented BCC envelopment problem. Problem (9.2) modifies best practice technology in two ways. It *adjusts* best practice standards by introducing $N+M$ *two-sided* terms, $(EX-X)$ and $-(EY-Y)$ respectively, to reflect the fact that observed values of X and Y might depart from their respective expected values. These terms vanish if $X = EX$ and $Y = EY$. It also *relaxes* best practice standards by introducing $N + M$ *one-sided* contingency terms, $F^{-1}(P_N)\Sigma_X$ and $F^{-1}(P_M)\Sigma_Y$ respectively. These terms reflect the variation, and the covariation across producers, of inputs and outputs. They vanish if either $\Sigma_X = 0$, $\Sigma_Y = 0$ or $P_N = 0.5,\ P_M = 0.5$. Otherwise, for $P_N > 0.5,\ P_M > 0.5,\ F^{-1}(P_N)\Sigma_X \geq 0$

and $F^{-1}(P_M)\Sigma_Y \geq 0$, and these two contingency terms relax best practice standards. Thus the chance-constrained formulation (9.2) collapses to the BCC DEA model (4.6) under either of two sets of conditions: (i) $X = EX$, $Y = EY$ and $\Sigma_X = 0$, $\Sigma Y = 0$, or (ii) $X = EX$, $Y = EY$ and $P_N = 0.5$, $P_M = 0.5$.

The chance-constrained problem (9.2) is a nonlinear programming problem; that is part of the price to be paid for introducing a stochastic element into the efficiency measurement exercise. Another price to be paid is the extra information that must be supplied by the evaluator. In addition to information on (X, Y), the evaluator must provide information on EX, EY, Σ_X, Σ_Y, P_N and P_M. This information can be deduced from previous data generated by the producers in the sample, or from expert judgement supplied by inside sources.

10. CONCLUDING REMARKS

As public and private organizations face increasingly competitive environments, and as all levels of government encounter growing fiscal difficulties, the attainment of high degrees of productive efficiency becomes increasingly imperative. The objective of this paper has been to discuss the measurement of productive efficiency, and to survey one of several approaches to the empirical implementation of efficiency measurement. The basic ideas are laid out in section 2, where I analyzed technical, allocative and economic efficiency, and the conditions under which each type of efficiency is an appropriate yardstick against which to measure producer performance.

One operations research technique that has been developed for this purpose is DEA, which since its inception in 1978 has envolved from a single linear programming model to a large and still growing family of mathematical programming models. In section 3 I analyzed the basic DEA model, which consists of a dual pair of linear programs, a multiplier problem and an envelopment problem. The solutions to these problems provide, for each producer, a measure of technical efficiency, the identity of all technically efficient role models, and normalized shadow prices of all inputs and outputs.

The basic DEA model of section 3 imposes restrictive conditions on the structure of technology. In sections 4-6 I showed how the basic DEA model can be generalized to allow for variable returns to scale, weak disposability of some inputs or some outputs, restrictions on normalized shadow prices, and non-convexities in production. In section 7

I analyzed translation invariance of some DEA models, and I showed how DEA models can be modified to incorporate non-discretionary inputs and outputs and environmental variables that are neither inputs nor outputs, but which may influence producer performance. Finally, in section 9 I outlined a chance-constrained DEA model whose objective is to incorporate a stochastic element into what is essentially a deterministic evaluation methodology.

My overview of DEA has been brief, and selective. I have attempted to offset some of the brevity with references to the relevant literature, where the topics I have explored are covered in more detail. I now address the selectivity of my survey, by way of reference to some omitted topics.

I have not discussed additive DEA models, preferring instead to concentrate on oriented models. Additive DEA models were introduced by Charnes et al. (1985a), and are discussed in detail in Charnes et al. (1994). I have ignored approaches other than chance-constrained programming to the problem of noise in the underlying data. The use of sensitivity analysis is discussed by Charnes et al. (1985b), and by Charnes and Neralic (1989a, 1989b,1992). The use of bootstrapping techniques to develop confidence regions for efficiency scores is discussed by Simar (1992) and Wilson (1994). I have not discussed the use of DEA and FDH in a panel data context. One approach to the use of DEA in this context is "window analysis", which is discussed in Charnes et al. (1994). Another approach to the use of DEA in this context is to construct a Malmquist productivity index, which can be decomposed into separate measures of technical change, technical efficiency change, the bias of technical change, and scale economies. This literature began with Färe et al. (1989), and has been extended by Grifell and Lovell (1993a,1993b). I have not mentioned any of the myriad of empirical applications of DEA that have appeared in management science, economics and other fields. DEA is a family of techniques whose use has spread to virtually all applied areas in these disciplines and any effort to summarize the empirical DEA literature would require a separate survey. An extensive but incomplete bibliography appears in Charnes et al. (1994); complementary bibliographies appear in Färe, Grosskopf and Lovell (1994) and Fried, Lovell and Schmidt (1993). Finally, I have not discussed software. In addition to such general purpose software as LINDO, SAS and GAMS, several specialized DEA software packages have appeared in recent years.

I conclude this survey with some brief conjectures on where future research effort might pay large dividends. In my judgement three areas are relatively underdeveloped, and warrant greater attention. One problem to which I have already alluded is the development of a stochastic DEA, which can incorporate measurement error and other sources of noise that inevitably contaminate the data used to implement DEA. Until a stochastic DEA is developed, statisticians and econometricians will remain skeptical of the managerial and policy implications drawn from DEA. A second problem concerns the development of a second-stage regression model designed to associate DEA efficiency scores with other measurable variables that are either under the control of management or exogenous at the level of observation. Better yet would be a simultaneous system of second-stage regression equations designed to associate DEA slacks (radial plus non-radial) with explanatory variables. Initial efforts in this direction seem promising: see Fried, Lovell and Vanden Eeckaut (1993). A third area in which additional effort would pay dividends concerns the nature of the relationship between DEA practitioners and their subjects. Too often producers are narrowly viewed as sources of the data upon which their performance is to be evaluated. Too rarely are producers viewed more broadly as sources of post-analysis information, which might lead to a modified DEA model and to a more reliable performance analysis. There are examples of such post-analysis interaction, of course, but an increase in the utilization of such interaction would surely enhance the acceptance of DEA as a valuable policy and management performance evaluation technique.

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DISCUSSION

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Knox Lovell's overview of linear programming approaches to measuring and analyzing performance was intentionally selective, focusing on those models which have proved most useful in operations research applications. These were then presented in light of their (1) accuracy in representing the structure of the underlying technology, (2) accuracy in guaging performance, (3) sensitivity to imperfect data, and (4) data requirements. He has succeeded admirably in making accesible what has become a very large, and increasingly technically demanding literature.

Since the author has succeeded so admirably at his task, it leaves the discussant with little to discuss. That being the case, I have decided to take a different slant. Since I came to the efficiency measurement world as an economist rather than from the world of operations research, I propose to add a brief selective guide to some of the economics literature that relates to efficiency measurement[1]. In fact, one could argue that the intellectual property rights to this topic could very well be shared between economics and operations research.

Obviously, economists have been concerned with various notions of efficiency for a very long time. As mentioned in the overview, Koopmans provides one widely accepted definition, which is also attributed to Pareto. Those associated with the operations research approach to efficiency measurement using DEA have spent a great deal of effort trying to modify the standard "radial" problems as summarized in (3.2) and (3.3) to identify as efficient only those observations that satisfy Pareto-Koopmans efficiency. This has proved to be difficult in the linear programming framework because of the likelihood of projections to "flat spots", i.e., segments of the reference technology in which there remains output or input "slack". Economists can provide several solutions to this problem. One is to ignore the technical efficiency notion and introduce (strictly positive) prices and compute minimum cost (or maximum revenue). Here prices serve the same purpose as the restrictions on normalized shadow prices discussed in section 5.2. Another alternative is to calculate a "Russell efficiency measure", due to Färe and Lovell (1978), which, in contrast to a radial technical efficiency measure, allows for nonradial contraction (expansion) of individual inputs (outputs), yet maintains independence of unit of measurement[2].

An alternative to the Pareto-Koopmans definition of efficiency was credited to Debreu (1951) and Farrell (1957). These latter names were associated with the "radial" definition of technical efficiency which is

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[1] This brief summary borrows from several jointly authored works including Färe, Grosskopf and Lovell (1985,1994), Färe and Grosskopf (1993), and Chambers, Färe and Grosskopf (1994)

[2] The input-oriented Russell measure may be written as $R_i(y,x) = \min\{\Sigma_{n=1}^{N} \lambda_n/N : (\lambda_1 x_1, \lambda_2 x_2, \ldots, \lambda_N x_N) \in L(y)\}$, where x is positive.

most commonly used in DEA. We could also add several more names here, including Shephard (1953) and Malmquist (1953), Debreu, Shephard and Malmquist all employed "distance functions": Debreu to measure welfare loss of economic waste, Shephard to represent technology in an axiomatic production theoretic framework, and Malmquist to develop quantly indexes. As it turns out, these distance functions are equivalent to Farrell's measure of technical efficiency[3]. To see this, consider the definition of Shephard's input distance function:

$$D_i(y,x) = \sup\{\lambda : \; x/\lambda \in L(y)\}, \tag{1}$$

which is the reciprocal of $TE_I(x,y)$ in Definition 3.[4]

Since

$$D_i(y,x) \geq 1 \Leftrightarrow x \in L(y) \tag{2}$$
$$D_i(y,x) = 1 \Leftrightarrow x \in Isoq\ L(y), \tag{3}$$

it follows that the distance function completely describes technology, and simultaneously provides a very useful measure of deviations from frontier performance or technical efficiency.[5]

The distance function has even more to offer the analyst. One of Shephard's fundamental contributions to economics and operations research is his explication of the duality between cost and input distance functions.[6] This provides theoretical underpinning for the Farrell decomposition of cost efficiency. The various derivative properties (Shephard-type lemmas) also provide convenient ways of retrieving shadow prices, input demands, etc., when such information is not available directly.

In his 1970 book, Shephard also showed how activity analysis could be used as a computational tool. Following von Neumann (1938,1945)

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[3]Distance functions have long proved useful in mathematics, where they are known also as gauge functions, see Newman (1987).

[4]The Shephard (1974) output distance function is similarly reciprocal to $TE_0(y,x)$ in Definition 4.

[5]Distance functions have also proved useful in applications to consumer behavior, and public finance, see Deaton (1979).

[6]He also later developed the duality between revenue and output distance functions, as well as introducing restricted (cost and revenue constrained) variations on these functions.

and Karlin (1959) he specified "piecewise linear" representations of technology, which are identical to the "envelopment" problem constraints used in DEA today.[7]

Others recognized the usefulness of these approaches. To my knowledge, Boles (1966) was the first to use linear programming to solve the Farrell technical efficiency problem. Afriat (1972) showed how to allow for variable returns to scale in the activity analysis approach. He also pioneered what I would call the "regularity test" literature, which employs linear programming tests of consistency of data which classical production and demand functions. Afriat clearly recognized the link to efficiency measurement. Related work on the production side includes Varian (1984), Diewert and Parkan (1983), and Hanoch and Rothschild (1972). More recent applications include work by Chavas and Cox (1990). Of course, the link between the fractional programming formulation of performance to the linear programming problem a la Farrell was provided by Charnes, Cooper and Rhodes (1978).

Returning to the distance function we note that the mathematical properties of the distance function make it extremely useful as an aggregator function and therefore building block for index numbers.[8] This brings us back to Malmquist, who recognized that the distance function could serve to construct "nice" quantity indexes. This idea was later exploited by Caves, Christensen and Diewert (1982) to construct the "theoretical" Malmquist productivity index as ratios of distance functions.[9] Färe, Grosskopf, Lindgren and Roos (1992) recognized that the "theoretical" index proposed by Caves, Christensen and Diewert could be calculated by exploiting the equivalence between distance functions and Farrell technical efficiency measures, i.e. they used linear programming techniques to calculate the component distance functions.[10] Since distance functions require only data on input and output quantities, the

[7] The primal and dual linear programming formulations of the revenue maximization and cost minimization problems were included in his 1970 book. He also exploited results by Morgenstern and Thompson which allowed for inclusion of zeros in the data. Zeros are allowed as long as there is at least one nonzero in each row and column of the data matrix. For the envelopment problem this implies that each input (output) must be used (produced) by at least one decisionmaking unit, and each decisionmaking unit uses at least one input to produce at least one output.

Malmquist index allows calculation of productivity even in the absence of information on input and output price or share data.

I have ended up back where those in operations research began-with the problem of evaluating performance even without the economist's measuring sticks derived from prices: profit, cost, revenue. That is probably the most fundamental contribution of the efficiency measurement literature. Nevertheless, it is useful to keep in mind the notions of optimization to inform us in choosing a sensible setup of the efficiency measurement problem, whether input or output oriented, primal or dual. Also, we owe a great debt to Shephard (whom I claim as an economist at heart, but who spent many years leading Berkeley's operations research department), who provided us with a rigorous axiomatic framework to production theory which informs us as to the properties we are imposing on technology and also allows us to exploit his duality theory to reveal the structure of technology.

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[8] For the input distance function some of these properties include homogeneity of degree plus one in inputs, quasiconvexity in outputs, concavity in inputs. See Färe (1988) for a discussion of properties of both input and output distance functions. See Färe and Grosskopf (1994) for a discussion of indirect distance functions and their properties.

[9] Following the terminology of Diewert (1976), distance functions are perfect aggregator functions yielding index numbers that are "exact".

[10] Some progress has been made on employing econometric techniques to estimate distance functions. Färe, Fukuyama and Primont (1985) were the first to my knowledge to show how one may exploit the homogeneity of the distance function in order to easily estimate it using OLS. Others, including Grosskopf and Hayes (1993) have used composed error models to estimate distance functions.

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It is both a great honour and pleasure to discuss this excellent survey of linear-programming efficiency measurement methods by one of the significant contributors to the literature. The author provides us with an enjoyable guided tour of the different methods and techniques used to assess the relative efficiency of decision units. This is a most important paper which I hope that will prompt many rigorous theoretical and applied work in this area in Southern Europe.

Little can be added to the masterful review of the linear-programming methods. Therefore, I will take here the *Econometrician's* stand

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and focus on two issues which I believe are of crucial importance and should attract more research effort in the future. Namely, (i) accomodating stochastic error, and (ii) the relationship between virtual multipliers and prices.

1. Stochastic Error

As Professor Lovell notes, the parametric approach to efficiency estimation assumes particular parametric form for, e.g., the production function, adds a onesided random error reflecting the presence of technical inefficiency in the production process, and, in the case of a stochastic frontier, a second random component reflecting the effects of noise, measurement error, model misspecification and exogenous shocks.[1] On the other hand, the nonparametric approach -which is the focus of Lovell's paper- doesn't assume any specific form for the production function but envelops the sample data by the smallest convex strong-disposal hull that satisfies a list of conditions. Since very little structure is imposed on the data this approach is unable to accommodate stochastic elements in a satisfactory way. This inability to accommodate random noise makes its results very sensitive to measurement error and model misspecification. There have been various attempts to combine the functional flexibility of the nonparametric approach with the ability of handling statistical noise.

Banker and Maindiratta (1992) and Banker (1993) have attempted to lay some statistical foundations for DEA. Assuming a monotonically decreasing density function for the deviation between actual and efficient level of output, Banker proves that the DEA estimates of the best-practice monotone-increasing and concave production function are also maximum likelihood estimates. Since the number of *incidental* parameters to be estimated by DEA methods grows with the sample size, the usual statistical properties of the maximum likelihood estimators do not apply there. However, Banker manages to prove the consistency of the DEA estimates from first principles. Nevertheless, other properties of these estimators are uncertain and no estimates for their standard errors are derived. Land, Lovell and Thore (1988,1993) use chance-constrained programing techniques to allow for uncertainty about the structure of the efficient production technology. They appeng the methods of chance-constrained programminng to the nonparametric deterministic frontier

[1]This is an active field of research today (e.g., Lee (1993)) with a very important recent contribution from a Bayesian perspective by Broeck et al. (1994).

model. In addition to the usual input-output data, evaluator-supplied information concerning accuracy of the data and willingness to take risk are required. Furthermore, chance-constrained efficiency measurement continues to be deterministic: efficiency is calculated by means of nonlinear programming techniques and no parameters are actually estimated in the process. Ley (1992) presents an approach based on a stochastic specification in a linear activities context which allows the use of statistical techniques to estimate technological parameters. Minimal functional constraints are imposed, as in the DEA approach. At the same time, a composed-error specification is borrowed from the econometric approach to frontier estimation. While Ley's model might be useful to better understand the relationship between DEA and econometric stochastic frontier models it doesn't seem to offer yet a feasible alternative to practitioners.

More research is needed in this area -the recent developments surveyed by Lovell where bootstrapping are used seem especially promising.

2. Economic Efficiency

It is well known that one of the advantages of DEA is that no price data is needed. Should price data be available to the researcher, then appropriate frontiers (profit or cost) can be specified. For example, cost effiency for unit 0 is defined as[2]

$$CE(x^0, y^0, w^0) = \frac{c(y^0, w^0)}{w^0 x^0} \tag{1}$$

that is, minimum feasible cost over actual cost (see section 7 in Lovell's paper). The problem is what to do when no price data is available.

It is *economic* efficiency, in my opinion, which is the ultimate objective of any policy-oriented efficiency-measurement study. I am using the term *economic* in a broad general sense, without restricting to profit-maximizing situations where market prices accurately reflect social opportunity costs. I am aware that in many situations of interest there are no markets for some of the inputs or outputs, or when the markets exist, they might be distorted and the prices do not reflect social opportunity costs. Nonetheless, should the study be of any help in the design of

[2] I will drop "T" superscript indicating transpose in Lovell's paper. I will instead assume that the vectors are of the required dimensions -i.e., row or column- as needed.

public policy then the analyst must make an effort to provide answers with economic content.

A firm might seem to be very inefficient as measured by, say, a non-radial input-reducing measure of technical efficiency but it might be in fact close to being economically efficient when prices are taken into account (for a similar point see Varian (1990)). In practical situations, it is unlikely that we will be able to discover such "missclassifications". However, one should pay more attention to the values of the virtual multipliers. When an economist sees equation (3.1),

$$\begin{aligned} &\min_{\mu,\nu} \frac{\nu x^0}{\mu y^0} \\ &\text{s.t. } \nu x^i / \mu y^i \geq 1, \qquad \forall i \\ &\quad \mu, \nu \geq 0; \end{aligned}$$

the first reaction is to relate ν to the vector of input prices and μ to the vector of output prices. Even when these are not observable, still the analyst might be ready to postulate admissible regions for those vectors -e.g., the prices of one unit of input i shouldn't be more than twice the price of input j- or even a joint statistical distribution which would allow the analyst to get a distribution of the efficiency measures.

Suppose that you rank a number of firms using one of the measures of technical efficiency discussed in Lovell's survey. Assume further, that price data are not available or prices don't reflect social opportunity costs. The analyst could specify a joint statistical distribution on *social prices* and sampling from it then re-evaluate the firms' ranking in the light of the economic efficiency measure. (For example, we might not be able to observe medical doctor's salaries at each individual hospital but we might be able to consult some national statistics, probably even broken by geographical region. We might also be willing to postulate that nurses' salaries are, on average, say, 45% lower than MD's.) Ley and Varian are exploring along these lines with Spanish hospital data. Equation (1) becomes

$$CE(x^0, y^0, w^{d_0}) = \frac{c(y^0, w^{d_0})}{w^{d_0} x^0} \qquad (2)$$

where the d_0 superscript means that w has been drawn from the distribution of input prices specified for firm 0. One could look at the distribution of $CE(x^0, y^0, w^{d_0})$ or simply focus on some descriptive statistics.

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When I asked Prof. Knox Lovell, on behalf of the Spanish Statistical and Operations Research Society (SEIO), to write an invited paper for our new born "Top" journal I was almost sure that he was going to accept our invitation. And he did. (My guess was an easy one; just have a look at his passport.) I am very much obliged to Prof. Lovell not only for undertaking our proposal but for the high standard of his paper. And this is not only mine but a general opinion: I am writing these lines after receiving the remaining "discussions". The quality of his paper is reflected both in his capacity of synthesis and in his clarity for transmitting us his huge knowledge. In this way he has been able to expose the guidelines and the future trends of DEA, an operations research technique mainly based on linear programming with which the productive efficiency of decision making units can be measured.

As host of Prof. Lovell in Alicante (Spain), where he presented the first version of his paper, I had the opportunity to discuss it directly with him. I have to recognize, with pleasure, that most of my suggestions are now incorporated in his final version. Some definitions have been added for making easier the reading for non-economists (such as the notions of strong disposability or of technologies homogeneous of degree α) as well as some needed references.

It was delightful to hear Prof. Lovell talking about the origins of DEA, anticipating the seminal paper of Charnes, Cooper and Rhodes (1978)* by more than a decade in a paper written by an agricultural economist (Boles (1966)*), and this too now appears in his paper.
Note. The * accompanying a paper indicates that it will not be referenced here because it is already referenced in Lovell's paper or in our introductory paper.

The paper presents some nonlinear programming problems, such as the one numbered (3.9) for measuring the hyperbolic technical efficiency or the one numbered (5.2) for reducing undesirable outputs. In each case Prof. Lovell proposes a change of variables in order to transform the nonlinear program to a linear one. The reader must be aware that the transformed problems are not equivalent to the nonlinear ones, in the sense that they afford different optimal solutions. Generally speaking we have to be very cautious when changing variables. On the other hand, problem (3.9) can easily be solved if we resort to Proposition 2 of Lovell's paper: first evaluate θ^* by means of the CCR-input oriented model; then calculate δ^* as the square root of θ^* and substitute it in problem (3.9); finally, replace its objective function by the sum of slack and excess variables after changing their sign, and solve this new problem. The solution gives us the projected point as well as the values of excess and slack variables. (By the way, the last built up problem is an additive DEA model which has been mentioned in Section 10.) Although, for the sake of brevity, Lovell does not mention it, the corresponding evaluation of the hyperbolic measure for the BCC model is not as easy as for the CCR model, and the corresponding nonlinear problem can only be replaced by an iterative procedure which solves several linear programming problems.

My last point is about a general comment I made to Prof. Lovell in Alicante and which, due to the scope of his paper, remains to be satisfied. I will try to do it now. As Prof. Lovell writes in his introduction "...the review is intended to inform practitioners in the field of management science, economics and public administration ...to draw inferences concerning producer performance, for the ultimate purpose of guiding business, economic or public policy". My feeling was (and remains to be) that a brief look at some published applications could shed some light to the non-specialist reader about the ability of DEA to handle rather different problems. Here are some interesting applications I have selected; they are supposed to be among the initial contributions in their respective fields.

1. Public and Private Services

1.1. Education

The first paper which appeared within the DEA framework was the one by Bessent and Bessent (1980), in which they examined the relative efficiency of schools in the school district of Houston, Texas, resorting to the CCR model. It is worth mentioning the paper by Charnes et al. (1981)* in which they succeeded in comparing, at the secondary school level, two types of educational programs. (See Subsection 8.3 of Lovell's paper, devoted to environmental variables, for further comments.) Many more papers have appeared since then related to all educational levels.

1.2. Health Care

Sherman (1984) was the first to apply DEA for evaluating the performance of hospitals. The recent papers in this area deal not only with hospitals but with related health care areas (nursing homes, child-care, etc) and attempt to incorporate quality factors.

1.3. Banking

Sherman and Gold (1985) applied DEA for evaluating the operating efficiency of bank branches. The first paper comparing banks (Rangan et al. (1988)) and resorting to DEA appeared three years later. This area is nowadays a very active research field, where nonparametric techniques compete with parametric ones.

1.4. Administrative Units

Lewin et al. (1982) were the first to study the efficiency of administrative units within courts in the U.S.. Marchand et al. (1984) published a book in which the first municipalities efficiency evaluation appeared (Belgium), resorting to non-convex DEA. Three years later, Thanassoulis et al. (1987) did the efficiency evaluation at rates departments in Great Britain.

1.5. Transport and Related

In the above mentioned Belgian book, the first paper devoted to urban transit can be found. Several years later, Adolphson et al. (1989) made a railroad property evaluation in the U.S., while Cook et al. (1990) studied the performance of highway maintenence patrols in Canada. Very recently Førsund (1992) studied the case of Norwegian ferries and Chang and Kao (1992) studied the five bus companies competing in Taipei city.

1.6. Electric Utilities

Thomas et al. (1985)*, studied the efficiency of regulated electric distribution utilities in Texas, U.S.; a subsequent study is due to Charnes et al. (1989)*. For further references, see [Lovell and Pastor (1994)].

1.7. Location

A startling study was developed by Thompson et al. (1986) for evaluating six possible locations for building a high energy supercollider. Recently, Kao and Yang (1992) applied DEA in order to reorganize forest districts in Taiwan. Finally, Desai et al. (1994) evaluated the performance of the public liquor outlets in the state of Ohio, U.S..

2. Industrial Applications

Byrnes et al. (1984) were the first to study the productive efficiency of the Illinois strip mines. A second study (Byrnes et al. (1988)) evaluated the U.S. surface mining of coal and the effects of unions on it. Recently, Ray and Kim (1991) evaluated the cost efficiency of the U.S. steel industry.

Ferrantino and Ferrier (1991) were the first to study the activity of Indian sugar plants, while Thompson et al. (1992) evaluated the U.S. independent oil/gas producers over time.

In Great Britain, Smith (1990) evaluated the financial performance of a sample of pharmaceutical firms, while Norman et al. (1991) studied the branch performance of the Anglia Building Society.

An interesting study from the consumer point of view is the one by Doyle and Green (1991), where they evaluate the quality of 27 computer printers. Another interesting study is the one by Thompson and Criswell (1993) where they compared five electric power technologies (considering undesirable outputs).

3. Macroeconomic Performance

Lovell (1994)* was the first to make a non-convex DEA macroeconomic analysis, comparing 10 Asian countries. Färe et al. (1994) evaluate the productivity growth and the efficiency change over a set of 17 OECD countries with panel data for a 10 year period.

4. Spanish Contributions

The still modest Spanish contribution in the DEA field is experiencing a considerable growth in the last few years. The paper by Ley (1991), devoted to the evaluation of the Spanish hospitals during a five year period, may be considered as our starting point. Other papers, ones by Grifell-Tatjé and others by myself are referenced in Lovell's paper. Of course, there are more papers coming up; my wish is that we are able to make an effort in order to reinforce our presence within the international scientific community.

The above list of papers is by no means exhaustive, nor does it cover all the fields where DEA has been applied. For instance, a considerable number of papers already have been published in Education, Health Care and Banking. Nevertheless, we hope our list is illustrative enough of the power of this new technique. The interested reader may consult the books of Charnes et al. (1994)* and of Fried et al. (1993)* where many more references are listed and where several applications are shown.

Last I would like to acknowledge the moral and financial support I received both from the SEIO and from the Editors of Top, which made possible the invitation to such an outstanding scholar and the publication of his DEA paper.

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Over the last fifty years or so, considerable intellectual resources have been invested in the measurement and analysis of efficiency. In this field, both the methodology of efficiency measurement and the numerous applications that have been made using Data Envelopment Analysis (DEA) includes approximately four hundred items in a bibliography compiled by Seiford (1992); Professor Lovell accounts the existence of a myriad of empirical applications of DEA. This is because it is possible to generalize and expand the traditional partial productivity measures (Output/Input), adding the multidimensional aspect of both the outputs and inputs of different firms or other productive organizations. On the other hand, it can be said that DEA is really becoming a useful tool integrated in a managerial system for periodic assessment.

The existent literature requires periodically an important task of compilation **the state of the art** of this field. At the moment, we have suggestive landmarks such as the following works: Färe, Grosskopf and Lovell (1985)*, Lewin and Lovell (1990)* and more recently Fried, Lovell and Schmidt (1993)* and Färe, Grosskopf and Lovell (1994)*. Obviously, Professor Lovell plays an important role in the advancement of production efficiency analysis.

Note. The * accompanying a paper indicates that it will not be referenced here because it is already referenced in Lovell's paper.

In this paper, Professor Lovell presents a stimulating overview of DEA models, as well as the different notions of efficiency and the decomposition into their allocative and technical components. The reader will find a comprehensive explanation of linear programming models and their primal dual representations; after this he offers some extensions of basic formulations: weak disposability, constrained multiplier weights, non-convex technology, translation invariance, non-discretionary and environmental variables, concluding with the introduction of stochastic

DEA models (a fertile area of future research). Professor Lovell says that his overview is brief and selective, and it can be added that it is correctly and adequately organized.

It is obvious that research in frontier analysis has been fruitful and has raised the technical level of evaluation in public and private organizations, but the needs of future research in the field of management science and economics are evident. Qualified opinions have indicated, among others, the following future objectives:

"Today, efficiency measures do not yet adequately account for quality changes, nor are they well suited to evaluate effectiveness (i.e. the adequacy between the output achieved and the needs to be satisfied). These shortcomings introduce biases either in the efficiency measurement themselves or in their interpretation, that need to be corrected. I am confident that they will gradually be circumvented" **Henry Tulkens (1992)**.

"The efficiency literature contains two broad themes. On the one hand, there is a focus on measurement, in which some enormous advances in technique have been made recently. On the other hand, there is the explanation of cause and effect for which Harvey Leibenstein was the pioneering spirit. What is missing at the present, is a serious effort by investigators of efficiency measurement to relate their choice of sample or experimental design to tackling the issues raised by Leibenstein. We have measurement and we have theory; but at present the two are not being related systematically. We have discovered weak but suggestive evidence that this would be a fruitful exercise. Perhaps this will be a major challenge for the next 25 years of X-efficiency theory" **Kenneth J.Button and Thomas G.Weyman-Jones (1992).**

We can sure that Professor Lovell will continue exploring the production economics with the same level of efficiency that he has until now.

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This short note is intended for practitioners interested in applying these linear programming techniques. The objective of this note is not to complete the presentation proposed by Lovell (1994). This overview is already an exhaustive catalogue of all the techniques currently used in this field. This note emphasizes the flexibility and also the constraints inherent in an empirical non parametric deterministic analysis.

Flexibility means that any objectives can be easily modelled and that very few restrictions are imposed on the choice of inputs and outputs. Furthermore, these techniques are simple to use and to explain, and this allows a real interaction between on one hand experts in the field and on the other hand the analyst. This advantage of relying on data is also the major constraint of these models. Indeed, when we are using econometric techniques we have simple tools or tests that allow us to validate or not a model or an assumption. In this non parametric framework such tools either do not exist or are difficult to use. The analyst has to decide if the data used allow a valid interpretation or not. This is not an easy task. What are the factors influencing the efficiency score? Suppose we have an efficiency score for a unit k on the interval $[0,\ldots,1]$, a value of 1 representing an efficient observation, then these factors defining efficiency may be given by the following representation:

$$Eff(k) = (N(-), V(+), ip, rt, ef)$$

The efficiency score is negatively related to the number of observations (N), positively related the number of input(s) and output(s) (V), related to the improvement path (*ip*) or stated differently the choice of the measurement (radial or non radial), the reference technology (all the assumptions about convexity, returns to scale, congestion,...) and finally all the factors leading to inefficiency (lack of ability, lack of effort,...).

We are mainly interested in capturing the last set of factors, namely the inefficiency (*ef*). We are able to control for the improvement path and the reference technology but are uncertain about the influence of dimensionality.

a. The Dilemma of Dimensionality

If we add a dimension (an additional input or output), the impact on the efficiency measure is unambiguous, the efficiency score cannot decrease. This consequence, already pointed out by Nunamaker (1985) and Thrall (1989), is often neglected or hidden. It raises two comments.

First, the inclusion or not of inputs or outputs should be motivated by the economic analysis and not driven (ex-post!) by the plausibility of the results obtained. Secondly, this problem is not a feature of the non parametric technique but instead a feature of the choice of radial measurement of efficiency.[1] It is possible to obtain an unpredicted effect by using non radial measurement. A discussion about the interest of non radial measurement is presented below at section c.

If we need to test for the inclusion of a variable or the aggregation of a set of variable, a procedure has been developed. A test has been proposed by Banker (1989) and later extended by Kittelsen (1993). The null hypothesis tests if two models (with or without the added variable) are drawn from an identical distribution. If this hypothesis is rejected, then the variable is included in the model. This test is close to a sensitivity test.

b. Exogenous or not?

The objective of the analysis is clearly to provide a "fair" representation of the performance of a unit. By using input(output) efficiency, we implicity assume that outputs(inputs) are non-discretionary. This is very often a simplified view of the reality. Labor is for instance nearly never considered as non-discretionary, but it is well known that any change in the demand for labor is most of the time closely restricted (due to regulation, union,...). There is no technical difficulty in computing this kind of model. (see section 8.2.).[2]

The use of environmental variables (see section 8.3.) is often needed in empirical analysis. Adding these variables is equivalent to the creation of a partition of the sample (we reduce N). The consequence for the value of the efficiency score may be dramatic.[3] There are two fundamental criticisms about the use of these variables. First, we need to decide on an orientation for these variables[4] (the environment is favorable or not) and this choice is not an easy one. The approach proposed by Grifell, Prior and Salas (1992) is already a step forward by looking at the incidence of this environment variable on the efficiency score before imposing an orientation. Secondly, we have no idea if this variable has "real" impact on efficiency (the ef component). A partial solution to this problem is to use a combined one step and two step approach. We first regress (in Tobit setting) the efficiency score obtained without the environmental variables: next we incorporate the variable in the model depending on the results of the first step. McCarty and Yaisawarng (1993) propose a simple illustration of his procedure.

This remark is also indirectly an encouragement to use graph efficiency or more generally any measure that allows variation in a subset of inputs and outputs.

c. The equiproportionate standard

Most of the empirical literature on efficiency measurement is based on the radial or equiproportionate measure developed by Debreu (1951) and Farrell (1957). The merit of the radial measurement is its simplicity and its straightforward cost interpretation.

The non radial measurement goes back to the definition of efficiency proposed by Koopmans.[5] This definition of efficiency does not indicate *how* to measure efficiency but instead *when* a unit will be efficient. The condition for being efficient is to belong on the efficient subset (see section 2). Different definitions of non radial measurement have been proposed in the literature.[6] Note also that non radial measurement may also be a practical solution to the problem of slacks mentioned by Lovell (section 5.2.). [7]

d. About reference technologies and economic assumptions

When estimating efficiency, we have to make economic assumptions. These assumptions will define the observations that we consider as "feasible" or realizable. We start with the assumption that only an observed unit is realizable.

We add three sorts of assumptions. First, we have assumptions about the disposability (see section 5.1.). Next, we have assumptions about returns to scale (see section 4). Finally, we have an assumption on convexity (section 6).

The choice between weak and strong disposability is important for explaining the origin of inefficiency. Indeed, in the two cases, the efficient subset is identical but the use of weak disposability may reveal congestion in one of the variables. This test is not regularly done in empirical application and should not be forgotten.

The convexity assumed by DEA models is in most cases too rigid. Furthermore, this assumption is not independent of the assumption about returns to scale. In order, for example, to allow for global increasing returns to scale we need to relax the convexity assumption. This can be done by dropping the convexity assumption as with FDH or by dropping the convexity on the graph (GR in section 2) while keeping this assumption for the input and output sets ($L(y)$ and $P(x)$ in section 2). This approach has been initiated by Bogetoft (1992).

Nearly all these assumptions can be combined and tested. In the case when we have no strong prior on the reference technology, it may be worthwhile to test different assumptions and to report the results.

This short note has emphasized some aspects of the use of linear programming approaches for efficiency measurement. These methods rely strongly on the choice of data and the justification of the model. A good knowledge of the consequence of the model for the efficiency score may help in the interpretation. If possible, efficiency scores should be confronted with reality[8] for validation. Another way to add consistency to a model is to propose different alternatives (models or techniques).[9]

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