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Convex Polytopes Whose Projection Bodies and Difference Sets Are Polars

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Abstract. In a paper by the author and B. Weissbach it was proved that the projection body and the difference set of a d-simplex ($d \geq 2$) are polars. Obviously, for $d = 2$ a convex domain has this property if and only if its difference set is bounded by a so-called Radon curve. A natural question emerges about further classes of convex bodies in R^d ($d \geq 3$) inducing the mentioned polarity. The aim of this paper is to show that a convex d-polytope ($d \geq 3$) is a simplex if and only if its projection body and its difference set are polars.

1. Definitions and Background Material

Let R^d ($d \geqslant 3$) denote the d-dimensional Euclidean vector space with scalar product $\langle \cdot, \cdot \rangle$, norm $\|\cdot\|$, and unit sphere $S^{d-1} := \{u \in R^d \mid \langle u, u \rangle = 1\}$. For basic notation the reader is referred to [4], [5], and [7]. In particular, we write P^d for the set of *convex d-polytopes*, i.e., compact, convex subsets of R^d with nonempty interior and a finite number of extreme points (vertices).

For $P \in P^d$ with $n \geq d+1$ facets let a_i denote the outward normal vector of the ith $(d-1)$-face, where $\|a_i\|$ represents the area of this facet ($i \in \{1, \ldots, n\}$). Then the vector sum of line segments given by

$$\Pi P := \sum_{i=1}^{n} \operatorname{conv}\{o, a_i\} = \tfrac{1}{2} \sum_{i=1}^{n} \operatorname{conv}\{-a_i, a_i\} \tag{1}$$

is called the *projection body* of P (see Section 7 in [4], and [11]). The polytope ΠP is a *d-zonotope* (see [9] and [11]) with center o, i.e., besides the body ΠP itself each face of it has a center of symmetry. Moreover, every segment summand $\operatorname{conv}\{o, a_i\}$ ($i = 1, \ldots, n$) generates a *zone* of facets in $\operatorname{bd} \Pi P$, that is the intersection with the zonotope of supporting hyperplanes with normal vectors orthogonal to conv$\{o, a_i\}$. For that reason ΠP has exactly m zones, where $m \leq n$ is the number of nonoriented facet normals of P. As is well known, a $(d-2)$-flat corresponds to each zone of the zonotope (and vice versa) in the so-called *projective diagram* $\bar{A}_m^{d-1}$ of ΠP, which is a dissection of real projective $(d-1)$-space by $(d-2)$-flats with no common point into cells of various dimensions. More precisely, assuming these cells to be relatively bounded, every $(d-r-1)$-cell represents the intersection of S^{d-1} and the closed normal cones of two opposite r-faces of ΠP $(r = 0, \ldots, d-1)$. Hence $\bar{A}_m^{d-1}$ is the projective representation of the usual *spherical image* (see Section I.5 of [1]) of the centrally symmetric polytope ΠP. Further, by

$$(\Pi P)_\delta^* := \{x \in R^d \mid \langle x, y \rangle \leq \delta^2 \quad \text{for all } y \in \Pi P\} \tag{2}$$

the *polar body* of ΠP with respect to $\delta \cdot S^{d-1}$ $(\delta \in R^+)$ is defined. This polarity implies (see once more Section I.5 of [1]) that $\bar{A}_m^{d-1}$ coincides with the *central projection* of $(\Pi P)_\delta^*$, i.e., the projection of the $(d-2)$-skeleton of $(\Pi P)_\delta^*$ from the center o onto the hyperplane at infinity belonging to the projective augmentation of R^d. (This way of projecting a polytope onto the real projective $(d-1)$-space is briefly discussed in Section 23 of [6]. Notice further that the $(d-2)$-skeleton of a polytope $P \in P^d$ is the union of its faces with dimensions not larger than $d-2$.)

The intersection of the $(d-2)$-skeleton of $P \in P^d$ and some hyperplane is called a $(d-2)$-*circle* of P, if it is homeomorphic to a $(d-2)$-sphere. A polytope $P \in P^d$ with center o is said to be $(d-2)$-equatorial if, for each $(d-1)$-subspace H through a $(d-2)$-face, the intersection of H and the $(d-2)$-skeleton of P is a $(d-2)$-circle. Moreover, such an intersection is called a $(d-2)$-*equator* of P. From the described correspondence between $\bar{A}_m^{d-1}$ and the central projection of $(\Pi P)_\delta^*$ it follows immediately that $(\Pi P)_\delta^*$ is a $(d-2)$-equatorial polytope. (Clearly, the $(d-2)$-equatoriality is only a necessary property of polars of d-zonotopes (see [2]). For necessary and sufficient conditions we can consult [14], whereas polars of $(d-2)$-equatorial polytopes are investigated in [3].)

Finally, the *difference set* of $P \in P^d$ is defined by

$$\begin{aligned} DP &:= P + (-1)P \\ &= \{x \in R^d \mid x = x_1 - x_2 \text{ with } x_1, x_2 \in P\} \end{aligned} \tag{3}$$

(see Section 7 of [4]). It should be noticed that we use DM in an analogous manner if $M \subset R^d$ is an arbitrary point set.

2. Proof of the Result

We shall show the announced characteristic property of d-simplices $(d \geq 3)$ by proving the more general statement that a convex d-polytope P is a simplex if and only if the central projection of DP and the projective diagram of ΠP coincide. To see this, four lemmas are taken into consideration.

Lemma 1. *Let $P \in P^d$ $(d \geq 3)$ be a polytope with coincidence between the central projection of DP and the projective diagram of ΠP. Then P is simplicial, i.e., each facet of the polytope is a $(d-1)$-simplex.*

Proof. If $F(Q, u)$ is written for the intersection of a polytope $Q \in P^d$ and its supporting hyperplane with outward normal direction $u \in S^{d-1}$, then for $P \in P^d$ and each $u \in S^{d-1}$ the relation

$$F(DP, u) = F(P, u) + F((-1)P, u) \tag{4}$$

holds (see Section 15 of [5]). Further, it is known that every simplex (of dimension $d-1$) is indecomposable in the sense of vector addition, i.e., it has only positively homothetic summands [5, Section 15]. By the supposition of this lemma the central projection of DP is a projective $(d-1)$-arrangement of m $(d-2)$-flats with no common point. As is shown in [12], such an arrangement has (at least) m $(d-1)$-cells which are projective simplices. Hence DP has (at least) $2m$ simplices as facets. By their indecomposability they cannot be vector sums of r-faces $(1 \leq r \leq d-2)$ of P and $(-1)P$ in the sense of (4). Thus each of their m nonoriented facet normals is also a facet normal of P, corresponding to simplex facets of this polytope in each case. Therefore nonsimplices as $(d-1)$-faces of P are excluded. □

Lemma 2. *Let P be a convex d-polytope $(d \geq 3)$. For every $(d-2)$-equator $\bar{M} \subset$ bd DP there exists exactly one $(d-2)$-circle $M \subset$ bd P with*

$$(x_1 - x_2) \in \bar{M} \quad \Rightarrow \quad x_1, x_2 \in M.$$

Proof. For generating $\bar{M}$ it suffices to use the $(d-2)$-skeleton of P (see (4) and the definition of $\bar{M}$). Let us consider the union of all difference sets generated from the intersections of this skeleton with hyperplanes whose normals are orthogonal to lin $\bar{M}$. Clearly, $\bar{M}$ is exactly the relative boundary of this union. We assume that more than one such difference set meets $\bar{M}$. Then there would exist a point $x \in \bar{M}$ belonging to the relative boundaries of two such difference sets DP_1 and DP_2, whose generators $P_1, P_2 \subset P$ lie in different parallel hyperplanes. Hence

$$x = x_1 - x_2 = y_1 - y_2, \qquad x_1, x_2 \in P_1, \quad y_1, y_2 \in P_2,$$

would hold. Since conv$\{x_1, x_2, y_1, y_2\}$ is a nondegenerate parallelogram, x would be a relatively interior point of conv$\{y_1 - x_2, x_1 - y_2\} \subset DP$.

By $(y_1 - x_2)$, $(x_1 - y_2) \notin$ lin $\bar{M}$, x cannot lie in a face of DP which fully belongs to lin $\bar{M}$. But this contradicts the definition of $\bar{M}$. Thus $\bar{M}$ is generated by the difference set of exactly one linear $(d-1)$-cut of the used $(d-2)$-skeleton. The additivity of faces of convex bodies under vector addition (see (4)) implies that this linear cut is homeomorphic to a $(d-2)$-sphere, i.e., it is a $(d-2)$-circle. □

Lemma 3. *Let* conv M_i $(i=1,2)$ *be two* $(d-1)$*-simplices whose relative boundaries* M_i *are* $(d-2)$*-circles of* $P \in P^d$. *If each* M_i *generates a* $(d-2)$*-equator* $\bar{M}_i \subset \text{bd } DP$ *(in the sense of the preceding lemma), then the simplices* conv M_i *coincide in* $d-1$ *pairs of extreme points.*

Proof. Since every edge of these simplices is also an edge of P, conv M_i lies in a closed half-space with respect to aff M_j, $\{i,j\}=\{1,2\}$. Let $v_1, v_2 \in \text{vert } P$ be two extreme points of conv M_2 with $v_1, v_2 \notin \text{conv } M_1$. We now distinguish two subcases regarding the relative position of conv M_1 and conv$\{v_1, v_2\}$.

1. There exists no parallel projection of R^d onto aff M_1 with the property that the image of conv$\{v_1, v_2\}$ is contained in conv M_1. Then we choose a parallel projection such that the image of conv$\{v_1, v_2\}$ covers the longest chord of the same direction in conv M_1. The following considerations confirm the existence of such a projection.

For aff$\{v_1, v_2\} \parallel$ aff M_1 all possible image lines of aff$\{v_1, v_2\}$ in aff M_1 are parallel. Thus one of them contains the longest chord of conv M_1 with the same direction. By the supposition this chord is contained in an image of conv$\{v_1, v_2\}$ on this line. For aff$\{v_1, v_2\} \nparallel$ aff M_1 we use the point $\{p_0\} := \text{aff}\{v_1, v_2\} \cap \text{aff } M_1$. If $\{v_3, \ldots, v_{d+2}\}$ denotes the vertex set of conv M_1, then

$$p_0 = \lambda_3 v_3 + \cdots + \lambda_{d+2} v_{d+2}, \qquad \sum_{i=3}^{d+2} \lambda_i = 1,$$

is obtained. The assumed projection property implies $p_0 \notin \text{conv } M_1$. Therefore at least one λ_i above has to be negative and at least one positive. Without loss of generality we can write

$$\lambda_3 < 0, \ldots, \lambda_k < 0, \qquad \lambda_{k+1} \geq 0, \ldots, \lambda_{d+2} \geq 0 \qquad (3 < k < d+2)$$

and define

$$\tau_1 := \sum_{i=3}^{k} \lambda_i \neq 0, \qquad \tau_2 := \sum_{i=k+1}^{d+2} \lambda_i \neq 0, \qquad \tau_1 + \tau_2 = 1.$$

Introducing

$$p_1 := \frac{1}{\tau_1} \sum_{i=3}^{k} \lambda_i v_i \quad \text{and} \quad p_2 := \frac{1}{\tau_2} \sum_{i=k+1}^{d+2} \lambda_i v_i, \tag{5}$$

we see by $p_0 = \tau_1 p_1 + \tau_2 p_2$ $(\tau_1 + \tau_2 = 1)$ that p_0 is a point from aff$\{p_1, p_2\}$. On the other hand, by (5) we have

$$p_1 \in \text{conv}\{v_3, \ldots, v_k\}, \qquad p_2 \in \text{conv}\{v_{k+1}, \ldots, v_{d+2}\}.$$

Hence

$$\text{aff}\{p_1, p_2\} \cap \text{conv } M_1 = \text{conv}\{p_1, p_2\}.$$

Obviously, the sets $\{v_3, \ldots, v_k\}$ and $\{v_{k+1}, \ldots, v_{d+2}\}$ represent a dissection of ext conv M_1. Since conv M_1 is a $(d-1)$-simplex, there exist two uniquely determined parallel $(d-2)$-flats H_1, H_2 in aff M_1 with

$$\{v_3, \ldots, v_k\} \subset H_1, \qquad \{v_{k+1}, \ldots, v_{d+2}\} \subset H_2,$$

and $p_1 \in H_1$, $p_2 \in H_2$. Because further conv M_1 is contained in $\operatorname{conv}(H_1 \cup H_2)$, no chord of conv M_1 parallel to $\operatorname{conv}\{p_1, p_2\}$ can be longer than this segment. Thus, the existence of a parallel projection with the demanded property is confirmed.

Now let us fix the position of the origin relative to conv M_1 and $\operatorname{conv}\{v_1, v_2\}$. (Clearly, the presented relations are translation invariant. This determination of o is only taken for the sake of convenience.) We denote by v_1', v_2' the images of v_1, v_2 with respect to the introduced parallel projection. Let v_1' coincide with p_1. Then p_2 (by the supposition a relatively interior point of $\operatorname{conv}\{v_1', v_2'\}$) will be the origin.

Our next step is the confirmation of

$$\operatorname{conv}\{v_1' - \delta v_2', \delta v_2' - v_1'\} \subset DP \tag{6}$$

for a suitable δ with $0 < \delta < 1$ (and $\delta \neq \frac{1}{2}$). To see this, we consider the parallelogram $\operatorname{conv}\{v_1, v_2 - v_1', -v_1, v_1' - v_2\}$. By (3) the vertices of this 4-gon belong to DP, and with $v_1, v_2 \notin \operatorname{aff} M_1$ they do not lie in aff M_1.

We introduce a Cartesian coordinate system in the 2-plane of the parallelogram by using an $\bar{x}$- and a $\bar{y}$-axis. By

$$v_1 = (\bar{x}_1, \bar{y}_1), \quad \bar{x}_1, \bar{y}_1 > 0, \qquad v_1' = (\bar{x}_1, 0),$$
$$v_2 = (-\bar{x}_2, \bar{y}_2), \qquad \bar{x}_2, \bar{y}_2 > 0, \quad \bar{y}_1 \neq \bar{y}_2, \qquad v_2' = (-\bar{x}_2, 0),$$

the relations

$$v_2 - v_1' = (-\bar{x}_1 - \bar{x}_2, \bar{y}_2) \quad \text{and} \quad -v_1 = (-\bar{x}_1, -\bar{y}_1)$$

hold. For $p = (\bar{x}, \bar{y})$ from $\operatorname{aff}\{v_2 - v_1', -v_1\}$ the equation

$$-\bar{x}_2(\bar{y} + \bar{y}_1) = (\bar{y}_1 + \bar{y}_2)(\bar{x} + \bar{x}_1)$$

is observed. The intersection of this line and the $\bar{x}$-axis is a point $q(\bar{q}, 0)$ with

$$\bar{q} = -\bar{x}_1 - \frac{\bar{y}_1 \bar{x}_2}{\bar{y}_1 + \bar{y}_2}, \quad \text{i.e.,} \quad q = -v_1' + \delta v_2', \qquad \delta = \frac{\bar{y}_1}{\bar{y}_1 + \bar{y}_2}.$$

(Because $\bar{y}_1, \bar{y}_2 > 0$ the relation $0 < \delta < 1$ is clear, and $\bar{y} \neq \bar{y}_2$ implies $\delta \neq \frac{1}{2}$.)

Thus, by the symmetry of DP with respect to o the inclusion (6) is proved. Since $\|p_1 - p_2\|$ is the maximal chord length of conv M_1 in the direction $v_1'/\|v_1'\| \in S^{d-1}$, we have $(v_1' - \delta v_2') \notin DM_1$ $(DM_1 = M_1 + (-1)M_1)$, although the relation $(v_1' - \delta v_2') \in (DP \cap \operatorname{aff} \bar{M}_1)$ holds. Thus, by Lemma 2 the $(d-2)$-circle M_1 cannot generate a $(d-2)$-equator of DP.

2. There exists a parallel projection of the line segment $\operatorname{conv}\{v_1, v_2\}$ into the set $\operatorname{conv} M_1$. We now use the oppositely oriented projection, namely from $\operatorname{conv}\{v_1', v_2'\}$ onto $\operatorname{conv}\{v_1, v_2\}$. Since $\|v_2 - v_1\|$ is an edge length of $\operatorname{conv} M_2$, we get $\pm(v_1 - v_2) \in \operatorname{ext} DM_2$ (see [8]). By (3) the points $\pm(v_1 - v_2')$, $\pm(v_2 - v_1')$ belong to DP. On the other hand, the inclusions

$$\pm(v_1 - v_2) \in \operatorname{relint} \operatorname{conv}\{\pm(v_1' - v_2), \pm(v_1 - v_2')\}$$

hold. Since these line segments are not fully contained in $\operatorname{aff} M_2$, the relative boundary of $M_2 + (-1)M_2$ cannot be a $(d-2)$-equator of DP. □

Lemma 4. *For $P \in P^3$, let the central projection of DP and the projective diagram $\bar{A}_m^2$ of ΠP coincide. Then P is a tetrahedron.*

Proof. Clearly, this coincidence implies that P is simplicial (see Lemma 1) and that to each facet of P, there exists a parallel 1-equator in $\operatorname{bd} DP$. Here we distinguish two subcases with respect to 1-circles of P that are generators of 1-equators in $\operatorname{bd} DP$.

1. Let us assume that every such 1-circle is the relative boundary of a facet of P. Since P has only triangles as facets, we can identify the sets $\operatorname{conv} M_i$ and the points v_i $(i = 1, 2)$ from the proof of Lemma 3 with facets and vertices of P. Each nonsimplex $P \in P^3$ has two vertices not contained in one of its facets. (For the dual version of this statement see [13].) Thus, assuming that nonsimplices P have no parallel 2-faces, Lemma 3 implies the existence of facets of these polytopes not parallel to 1-equators in $\operatorname{bd} DP$. On the other hand, if a nonsimplex $P \in P^3$ has a pair $\{\operatorname{conv} M_1, \operatorname{conv} M_2\}$ of parallel 2-faces, then by Lemma 2 (and suitable notation)

$$(M_1 + (-1)M_1) \subseteq \operatorname{relint}(M_2 + (-1)M_2) \tag{7}$$

must hold. Then let us consider the three facets having edges in common with $\operatorname{conv} M_1$. Obviously, their longest chords in the directions of these edges are the edges themselves. Hence by (7) the relative boundaries of these three triangles cannot generate 1-equators in $\operatorname{bd} DP$, too. We can continue this process of crossing over to further neighboring facets through all the boundary of P. But this contradicts the assumed existence of 2-faces in $\operatorname{bd} P$ whose relative boundaries generate 1-equators in $\operatorname{bd} DP$. Thus a nonsimplex $P \in P^3$ cannot have only relative facet boundaries as 1-circles which generate the 1-equators of DP.

2. Therefore we have to assume the existence of such a generating 1-circle $L \subset \operatorname{bd} P$ which is not a relative facet boundary of the nonsimplex $P \in P^3$. We denote by $\bar{G}$ the projective line representing $L + (-1)L$ in the central projection $\bar{A}_m^2$ of DP. Then each 0-cell from $\bar{G}$ is contained in (at least) three lines of this projective 2-arrangement. Namely, by the connection of $\bar{A}_m^2$ with ΠP the line $\bar{G}$ must represent a zone of this zonohedron, i.e., a facet normal of P which is orthogonal to $\operatorname{aff} L$. Further, every edge of P in L belongs to two facets of the

polyhedron whose normals are different and, additionally, different from the facet normal described above. Thus, arbitrarily oriented directions of these three normals form a linearly dependent system, whose representatives are pairwise linearly independent. Such a system corresponds to a triplet of projective lines through one 0-cell from $\bar{G}$ in $\bar{A}^2_m$. In general, for a projective 2-arrangement $\bar{A}^2_m$ of m lines with no common point the following relations are known or can be derived without difficulty (see Section 18 of [5]).

If f_i ($i \in \{0, 1, 2\}$) denotes the number of i-cells, p_j ($j \geq 3$) the number of j-sided 2-cells, and w_k ($k \geq 3$) the number of 0-cells which belong to k projective lines of $\bar{A}^2_m$, then with $c := \sum_{k \geq 3} (k-2) w_k$ we obtain $f_0 - f_1 + f_2 = 1$ (Euler's relation for $\bar{A}^2_m$), $\sum_{j \geq 3} p_j = f_2$, $\sum_{j \geq 3} j p_j = 2 f_1$, and $f_1 = 2 f_0 + c$. It is obvious, that the first lemma implies $p_3 = m$ and $p_j = 0$ ($j > 4$). A suitable combination of these equations leads to $2c = m - 4$. But since (at least) three projective lines of $\bar{A}^2_m$ are incident with every 0-cell from $\bar{G}$, $2c$ cannot be smaller than $m-1$. Hence we finish, as in the first case, with a contradiction. □

These lemmas allow the formulation of

Theorem 1. *For a convex d-polytope P ($d \geq 3$) the projective diagram of ΠP and the central projection of DP coincide if and only if P is a simplex.*

Proof. Because each simplex has this property (see [8]), only the converse implication remains to be verified. By means of Lemma 4 we may assume that in P^{d-1} ($d \geq 4$) the simplices are characterized by such a coincidence. Further, we also assume that for $P \in P^d$ the projective arrangement $\bar{A}^{d-1}_m$ has this double meaning. We shall show that P is then a d-simplex. Let $\bar{Q}$ be an arbitrary $(d-2)$-equator from bd DP. Obviously, the $(d-1)$-polytope conv $\bar{Q}$ is then $(d-3)$-equatorial. By Lemma 2 there exists a uniquely determined $(d-2)$-circle $Q \subset$ bd P with conv $\bar{Q} = Q + (-1)Q$. We write $\bar{A}^{d-2}_p$ ($p < m$) for the relative central projection of $DQ := Q + (-1)Q$ in the $(d-2)$-plane at infinity with respect to the projective augmentation of lin DQ. Now let us show that the assumed correspondence of $\bar{A}^{d-1}_m$, ΠP, and DP implies an analogous connection between $\bar{A}^{d-2}_p$, DQ, and the relative projection body $\Pi \hat{Q}$ of conv $Q =: \hat{Q}$. Clearly, an arbitrary relative facet of $\hat{Q}$ equals just a $(d-2)$-face of P. Thus such a face R is the intersection of exactly two facets of P, whose normals span a 2-subspace totally orthogonal to aff R. Therefore the set of the relative facet normals of $\hat{Q}$ is contained in the set of orthogonal images of facet normals of P in lin DQ. If q is the number of the relative facet normals of $\hat{Q}$, this means $q \leq p$. On the other hand, we have $p \leq q$. This follows by Shannon's result that $\bar{A}^{d-2}_p$ contains (at least) p projective $(d-1)$-simplices, necessarily corresponding here to (at least) p relative facet normals of $\hat{Q}$ (see Lemma 1). Hence $\Pi \hat{Q}$ and DQ have the same $(d-2)$-arrangement as the relative projective diagram (resp. relative central projection). Hence, by the induction hypothesis $\hat{Q} \in P^{d-1}$ is a $(d-1)$-simplex.

Now let C denote the complex of all $(d-1)$-simplices $\hat{Q}_i$ which generate $(d-2)$-equators of DP ($i = 1, \ldots, m$). From a statement in [13] it follows that each nonsimplex $P \in P^d$ has a facet not containing two vertices of P. This and

Lemma 3 imply that conv C is a d-simplex, say S. If ΠP has less than $d+1$ zones, then P is necessarily a d-parallelotope, whose difference set is not $(d-2)$-equatorial. Thus $m=d+1$ and $DS=DP$. Finally, we confirm that this equality implies $S=P$.

We have $S\subseteq P$. For $\{e_0,\ldots,e_d\}:=\text{vert } S$ in [8] the relation

$$\text{vert } DS=\{e_i-e_j\}, \qquad i,j\in\{0,\ldots,d\}, \quad i\neq j, \tag{8}$$

was shown. If there exists an extreme point z of P with $z\notin S$, then $\pm(z-z')\in$ vert DS holds, where z' denotes a suitable point from vert P. For $z'\in$ vert S this relation contradicts (8), and $z'\notin$ vert S implies the existence of a pair k, l from $\{0,\ldots,d\}$ with $\pm(e_k-e_l)=\pm(z-z')$. But then e_k-e_l would be from the relative interior of $\text{conv}\{e_k-z', z-e_l\}\subset DP$, contradictory to $(e_k-e_l)\in\text{vert } DP$. □

3. Formulation of the Result and Its Equivalents

For assertions which are equivalent to the announced polarity property we introduce some additional notions. Let $\underline{V}_1(P,u)$ denote the *inner* 1-*quermass* of $P\in P^d$ at $u\in S^{d-1}$, i.e., the length of the longest chord of this polytope in direction u. On the other hand, by $\bar{V}_{d-1}(P,u)$ we define the *outer* $(d-1)$-*quermass* (*brightness*) of P at u, i.e., the area of the orthogonal image of P in $\{x\in R^d\,|\,\langle x,u\rangle=0\}$ (see Section 7 of [4]). A (compact) *d-prism* D_u with generators in direction u is, by definition, the nondegenerate vector sum of a polytope from P^{d-1} and a line segment with this direction. If under the condition $P\subseteq D_u$ the volume $V(D_u)$ is minimal, then this prism is said to be *optimally circumscribed about* $P\in P^d$. Now we are ready for

Theorem 2. *The following properties of a convex d-polytope P* $(d\geq 3)$ *are equivalent*:

(A) *The polytope P is a d-simplex.*
(B) *The difference set DP and the projection body* ΠP *are polars with respect to a sphere* $\delta\cdot S^{d-1}$ *for some* $\delta\in R^+$.
(C) *The inner* 1-*quermass and the brightness of P satisfy*

$$\underline{V}_1(P,u)\cdot\bar{V}_{d-1}(P,u)=\delta^2$$

for each $u\in S^{d-1}$ *and some* $\delta\in R^+$.
(D) *For P all optimally circumscribed d-prisms have the same volume.*

Proof. Since the introduced correspondence of ΠP, $\bar{A}_m^{d-1}$, and DP is a necessary condition for the polarity of ΠP and DP with respect to $\delta\cdot S^{d-1}$, by [8] and the first theorem, (A)$\Leftrightarrow$(B) is confirmed. From the definitions of ΠP and DP it follows that $\underline{V}_1(P,u)$, $u\in S^{d-1}$, is reciprocal to the restriction $g(DK,u):=\min\{\rho>0\,|\,u\in\rho DK\}$ to S^{d-1} of the distance function of DP, whereas $\bar{V}_{d-1}(P,u)$

represents the correspondingly restricted support function $h(\Pi P, u) := \max\{\langle u, y\rangle \mid y \in \Pi P\}$ of ΠP. Since polarity of ΠP and DP with respect to $\delta \cdot S^{d-1}$ ($\delta \in R^+$) is equivalent to $h(\Pi P, u)/g(DK, u) = \delta^2$ for all $u \in S^{d-1}$ [7, Section 12], (B)$\Leftrightarrow$(C) is verified. Let T_u denote a chord from $P \in P^d$ of maximal length in direction u. Then both endpoints of T_u lie in parallel supporting hyperplanes of P [4, Section 7]. Therefore $\min\{V(D_u) \mid P \subseteq D_u\} = \bar{V}_{d-1}(P, u) \cdot \underline{V}_1(P, u)$ for each $u \in S^{d-1}$ and (C)$\Leftrightarrow$(D) are obtained. □

4. Concluding Remarks

Obviously, by Theorem 2 the polarity of ΠP and the central symmetrization $\frac{1}{2}DP$ [7, Section 20], [4, Section 9] is also a characteristic property of simplices in P^d ($d \geq 3$). We might extend the investigations to the set K^d of convex bodies (compact, convex sets with interior points) in R^d and ask whether the polarity of ΠK and DK implies that $K \in (K^d \setminus P^d)$ is a d-ellipsoid ($d \geq 3$). For $d = 2$, Theorem 2 remains true if K^d is written instead of P^d and if simplices are replaced by convex domains whose difference sets are bounded by Radon curves [7, Section 13]. Assuming polarity of ΠK and DK, we can estimate the constant δ^2 (see Theorem 2) in terms of the volume $V(K)$ of $K \in K^d$ for $d \geq 2$. Namely, by [8] and a result from [10] we have

$$\frac{2\omega_{d-1}}{\omega_d} V(K) \leq \delta^2 \leq dV(K), \qquad \omega_d := \frac{\pi^{d/2}}{\Gamma(1+d/2)}, \quad d \geq 2.$$

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