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Dispersed Points and Geometric Embedding of Complete Bipartite Graphs

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Abstract. The minimum dimension needed to represent $K(m, n)$ as a "unit neighborhood graph" in Euclidean space is considered. Some upper and lower bounds on this dimension are given, and the exact values of the dimension are calculated for $m \leq 3$, $n \leq 10$.

1. Introduction

Let X be a nonempty subset of Euclidean n-space E^n. Then the graph having vertex set X and edge set

$$\{xy: |x-y| \leq 1, x, y \in X, x \neq y\}$$

(where $| \ |$ denotes the Euclidean norm) is called the *unit neighborhood graph* on X. For a (finite simple) graph G, the *sphericity* of G, sph G, is the minimum dimension n such that G is isomorphic to a unit neighborhood graph in E^n. Concerning sphericity and some other geometric graph dimensions, see, e.g., [1]–[4] and [6]–[12]. In this paper we give some bounds on the sphericity of the complete bipartite graph $K(m, n)$ and derive exact values of sph $K(m, n)$, $m \leq 3$, $n \leq 10$. Since sphericity is a hereditary property (that is, if H is an induced subgraph of G, then sph $H \leq$ sph G), these results will be useful to estimate sph G for other graphs G.

A nonempty subset X of a Euclidean space is said to be *dispersed* if the unit neighborhood graph on X is an edgeless graph, that is, $|x-y| > 1$ for any two distinct points x, y of X. The maximum cardinality of a dispersed subset of a set Y is called the *dispersed point number* of Y and denoted by dpn Y. For example, the dispersed point number of a unit circle is 5. Let us denote by $S^k(r)$ a hypersphere of radius r in k-space E^k, and put

$$N(k, m) = \text{dpn } S^k(s(m)),$$

where $s(m) = \sqrt{(m+1)/(2m)}$.

Theorem 1. sph $K(m, N(k, m)) \leq m-1+k$.

Since the "equator" of the sphere $S^{k+1}(s(m))$ contains $N(k, m)$ dispersed points, by adding the two "poles," we see that $N(k+1, m) \geq N(k, m)+2$. And since $N(1, m)=2$, we have

$$N(k, m) \geq 2k.$$

Hence the next corollary follows.

Corollary 1 [6]. sph $K(m, 2n) \leq m-1+n$.

Let us define

$$d(m, n) = m-1+\min\{k: N(k, m) \geq n\}.$$

Then Theorem 1 is written as

$$\text{sph } K(m, n) \leq d(m, n).$$

We conjecture that, for $m \leq n$, sph $K(m, n) = d(m, n)$. In fact, this is true for infinitely many m, n (see Corollary 3 and Theorem 4 below).

Using the results of Schütte and Van der Waerden [13], the values of $d(m, n)$ for $m \leq 4$, $n \leq 10$, are calculated in [8], which are given in Table 1.

Theorem 2. *For $n \geq 3$, sph $K(n, n) > n$.*

This and Theorem 1 give

$$\text{sph } K(4, 4) = 5.$$

Let $\mathrm{R}(m, n)$ denote the "Ramsey number."

Theorem 3. *Suppose that*

$$m+k \geq \max\{n, \mathrm{R}(m-1, n)\}+1, \qquad M \geq \max\{m+k, N(k, m)-n+m+k\}.$$

Table 1. The values of $d(m, n)$ for $m \leq 4$, $n \leq 10$.

	n								
m	2	3	4	5	6	7	8	9	10
1	1	2	2	2	3	3	3	3	3
2	2	3	3	3	4	4	4	5	5
3	3	4	4	5	5	5	6	6	6
4	4	5	5	6	6	7	7	7	7

Then sph $K(m, M) \geq m+k$. *For* $m=5$, *we can replace* $\mathrm{R}(m-1, n)$ *by* $\mathrm{R}(3, n)$ *in the condition.*

Letting $n=2$, $M=N(k, m)+m+k-2$, $k \geq 1$, we have Corollary 2.

Corollary 2. sph $K(m, N(k, m)+m+k-2) \geq m+k$ *for* $k \geq 1$.

For $m \leq 3$, $k \geq m-1$, then letting $n=m+k-1$ and $M=N(k, m)+1$, the conditions of Theorem 3 are satisfied (because $\mathrm{R}(2, n)=n$, and $N(k, m) \geq 2k$). Hence sph $K(m, N(k, m)+1) \geq m+k$. Combining this result with Theorem 1, we have the following.

Corollary 3. *If* $m \leq 3$, *then* sph $K(m, n)=d(m, n)$.

Furthermore,

Theorem 4. *For any fixed integer* $m>1$, sph $K(m, n)=d(m, n)$ *holds for infinitely many* n.

Since $\mathrm{R}(3,3)=6$ and $N(2, m)=4$, $N(3, m)=6$ for $m \geq 4$, letting $n=2$, $m=4$, and $k=2$ in Theorem 3, we have sph $K(4,8) \geq 6$. Similarly, by letting $n=3$, $m=4$, $k=3$; $n=3$, $m=5$, $k=2$; and $n=3$, $m=5$, $k=3$, we have sph $K(4,10) \geq 7$, sph $K(5,8) \geq 7$, and sph $K(5,11) \geq 8$, respectively. Combining these results with the upper bounds obtained from Corollary 1 and Table 1, we have

$$6 \leq \text{sph } K(4,8) \leq 7 \leq \text{sph } K(5,8) \leq 8 \leq \text{sph } K(5,11),$$

and another exact value

$$\text{sph } K(4,10)=7.$$

If a graph G is isomorphic to a unit neighborhood graph in which any adjacent points are closer than a prescribed distance $\delta<1$, then G is said to be δ-*embeddable.* Then there arises a question: is there a $\delta<1$ such that every finite simple graph is δ-embeddable? The answer is NO.

Theorem 5. *If* $0<\delta<1$ *and* $n>1/(1-\delta^2)$, *then* $K(n, n)$ *is not* δ*-embeddable.*

2. Dispersed Set

In this section we recall some results on dispersed sets and then prove Theorem 5 first. The *circumsphere* (c-sphere) of a nonempty compact set X in Euclidean n-space E^n is the sphere of minimum radius that encloses X; its radius, denoted by $c(X)$, is called the *c-radius* of X; its center is the *c-center* of X. If X is the vertex set of a regular $(m-1)$-dimensional simplex of unit side length, then the c-radius of X, $\mathrm{c}(X)$, is denoted by $c(m)$. Then it is not difficult to see

$$c(m)=\sqrt{(m-1)/(2m)}.$$

Recall the definition of $s(m)$:

$$s(m) = \sqrt{(m+1)/(2m)}.$$

Thus we have $c(m)^2 + s(m)^2 = 1$. Similarly, for any nonempty compact set X with $c(X) < 1$, we define $s(X) > 0$ by

$$c(X)^2 + s(X)^2 = 1.$$

The following two theorems are proved in [8].

Theorem A. *If X is a dispersed set of size m, then $c(X) > c(m)$.*

A point set X is said to be *c-spherical* if all points of X lie on the c-sphere of the set X. For example, the vertex set of a regular simplex is c-spherical, but the vertex set of an obtuse triangle is not c-spherical. A point set X is said to be *affinely independent* if X is the vertex set of nondegenerate simplex.

Theorem B. *If X is a dispersed set with $c(X) \leq \sqrt{\frac{1}{2}}$, then X is affinely independent and c-spherical.*

Now we proceed to the proof of Theorem 4. First we prepare a lemma. For two nonempty finite sets X and Y in a Euclidean space, we define

$$\text{dis}(X, Y) = \max\{|x-y| : x \in X, y \in Y\}.$$

Lemma 1. $c(X)^2 + c(Y)^2 \leq \text{dis}(X, Y)^2$.

Proof. We may assume that the c-center of X is at the origin O. Let $\{x_1, \ldots, x_m\} = \{x \in X : |x| = c(X)\}$. Then the origin O belongs to the convex hull of $\{x_1, \ldots, x_m\}$, as is easily seen. Hence O is expressed as a *convex combination* of $x_1, \ldots, x_m$:

$$O = a_1 x_1 + \cdots + a_m x_m \qquad (a_1 + \cdots + a_m = 1,\ a_i \geq 0).$$

For any y of Y,

$$\text{dis}(X, Y)^2 \geq |x_i - y|^2 = |y|^2 + c(X)^2 - 2\langle y, x_i\rangle,$$

where $\langle\ ,\ \rangle$ denotes the inner product. Multiplying both sides by a_i and summing on i, we have

$$\text{dis}(X, Y)^2 \geq |y|^2 + c(X)^2.$$

Hence $\text{dis}(X, Y)^2 \geq \max_y |y| + c(X)^2 \geq c(X)^2 + c(Y)^2$. □

Proof of Theorem 5. Suppose that $K(n, n)$ is δ-embeddable in a Euclidean space. Then there exist two dispersed sets X, Y each of size n, such that $\text{dis}(X, Y) < \delta$. By Lemma 1, we have $c(X)^2 + c(Y)^2 \leq \text{dis}(X, Y)^2 < \delta^2$. Hence we may assume $c(X)^2 < \delta^2/2$. Since $c(X) > c(n)$ by Theorem A, we have $c(n)^2 = (n-1)/(2n) < \delta^2/2$, i.e., $n < 1/(1-\delta^2)$. Therefore, if $n > 1/(1-\delta^2)$, then $K(n, n)$ is not δ-embeddable. □

3. Focal Set

Let X be a nonempty set in E^k. A point y of E^k is called a *focal point* of X if $|y-x| \leq 1$ for all points x of X. The set of all focal points of X is called the *focal set* of X and it is denoted by $F(X)$. Thus, by definition,

$$F(X) = \{y \in E^k : |y-x| \leq 1 \text{ for all } x \text{ of } X\} = \bigcap_{x \in X} B(x, 1),$$

where $B(x, 1)$ denotes the unit (closed) ball centered at $x \in E^k$. Note that $F(X) = \varnothing$ if the c-radius $c(X)$ is >1.

The operation F is interesting in its own right. For example, if X is a set of c-radius ≤ 1, then the set

$$\tfrac{1}{2}[F(X) + F(F(X))]$$

is a set of constant width 1 containing X, where + denotes the vector sum of two sets, see [5] for detail.

The following lemma will be clear.

Lemma 2. *There exists a dispersed set X of size m in E^k such that* dpn $F(X) \geq n$ *if and only if* sph $K(m, n) \leq k$.

Example. For two points x, y $(x \neq y)$ in E^2, the set $F(\{x, y\})$ is the intersection of the two unit disks with centers x, y. It is easy to see that if $|x-y| > 1$, then dpn $F(\{x, y\}) \leq 2$. Hence sph $K(2, 3) > 2$.

By the *dimension* of a set X in a Euclidean space, we mean the dimension of the flat (=affine subspace) spanned by X. In the rest of this section we assume the following:

> X is a c-spherical set in E^k of dimension $j < k$ with c-center at the origin O and c-radius $c(X) < 1$.

The subspace spanned by X is called the *tangent space* of X, and is denoted by $T(X)$. Clearly, $T(X)$ contains the origin O and dim $T(X) = j$. The orthogonal complement of $T(X)$ in E^k is called the *normal space* of X, which is denoted by $N(X)$. Then the intersection of the normal space $N(X)$ and the focal set $F(X)$ is a $(k-j)$-dimensional disk of radius $s(X) = (1 - c(X)^2)^{1/2}$, because $|x| = c(X)$ for all x of X. This disk is called the *normal disk* and its boundary is called the *normal sphere* of X. For example, if $X = \{(-\tfrac{1}{2}, 0, 0), (\tfrac{1}{2}, 0, 0)\}$ in E^3, then the normal sphere of X is $\{(0, s, t) : s^2 + t^2 = \tfrac{3}{4}\}$. Note that the normal sphere of X is the intersection of all unit hyperspheres $S(x, 1)$ with center $x \in X$.

Proof of Theorem 1. Let X be the vertex set of an $(m-1)$-dimensional regular simplex of side-length $1+\varepsilon$ in E^{m-1+k}. Then the c-radius of X is $c(X) = (1+\varepsilon)c(m)$, and the normal sphere of X is a k-dimensional sphere with radius $s(X) = (1-c(X)^2)^{1/2}$. By the definition of $N(k, m)$, a k-dimensional sphere of radius $s(m)$ contains a dispersed set of size $N(k, m)$. Hence, if $\varepsilon > 0$ is sufficiently small, the normal sphere of X also contains a dispersed set of size $N(k, m)$. Hence dpn $F(X) \geq N(k, m)$. Then by Lemma 2, we have sph $K(m, N(k, m)) \leq m+k-1$. □

Let

$$p: E^k \to T(X),$$
$$q: E^k \to N(X),$$

be the orthogonal projections. Then for any point z of E^k,

$$z = p(z) + q(z).$$

Any noncollinear three points x, y, z determine a circle. By the circular arc xyz, we mean the arc xyz of the circle determined by x, y, z. The boundary of $F(X)$ is denoted by $\partial F(X)$.

Lemma 3. *Let z be a point on the boundary $\partial F(X)$ such that $p(z) \neq 0$, $q(z) \neq 0$. Let z^+, z^- be the two points where the line $Op(z)$ meets the normal sphere of X. Then the circular arc $M(z) = z^+ z z^-$ lies entirely on $\partial F(X)$.*

The arc $M(z)$ is called the *meridian* of $\partial F(X)$ passing through the point z. Note that $z^- = -z^+$.

Proof. Since z is a boundary point of $F(X)$, z lies on some sphere $S(x_i, 1)$. And since z^+, z^- are two points of the normal sphere of X, z^+ and z^- lie on $S(x_j, 1)$ for all $j = 1, \dots, m$. Here we note the following fact: if a circle and a sphere have more than two points in common, then the circle lies entirely on the sphere. Therefore, if $z \in S(x_j, 1)$, then $M(z) \subset S(x_j, 1)$, while if $z \notin S(x_j, 1)$, then, since $z \in B(x_j, 1)$, all points of $M(z)$ other than the two endpoints are interior points of $B(x_j, 1)$. Therefore $M(z) \subset F(X)$ and $M(z) \subset S(x_i, 1)$. Thus $M(z) \subset \partial F(X)$. □

4. Dispersed Points in a Focal Set

In this section we prove some lemmas that are useful in the computation of the dispersed point number of a focal set. Throughout this section, we assume that

> $X = \{x_1, \dots, x_m\} \subset E^k$ $(m \leq k)$ is a dispersed set with c-center O(origin) and c-radius $c(X) \leq \sqrt{\tfrac{1}{2}}$.

Then by Theorem B, X is affinely independent and c-spherical; the tangent space $T(X)$ of X is an $(m-1)$-dimensional subspace of E^k, and the normal space $N(X)$ (the orthogonal complement of $T(X)$ in E^k) is $(k-m+1)$-dimensional. The projections $E^k \to T(X)$, $E^k \to N(X)$ are denoted by p and q, respectively. Further, we denote by

$$f\colon E^k - \{O\} \to \partial F(X)$$

the central projection from the origin, i.e., for any $z \neq O$, $f(z)$ is the point where the ray $\overrightarrow{Oz}$ meets the boundary $\partial F(X)$. (Note that $F(X)$ is a convex body containing O inside.)

Lemma 4. *For any dispersed pair x, y in $F(X)$,*

$$y \neq O \quad \text{and} \quad |x - f(y)| \geq |x - y|.$$

Proof. Since $|x| < 1$ (because $|x - x_i| \leq 1$ for $i = 1, \ldots, m$, and O is contained in the convex hull of X) and $|y| < 1$, it follows that $x \neq O$, $y \neq O$, and that the angle at O is the largest angle in the triangle Oxy. Then the angle $\angle Oyx$ is acute, and hence it follows that $|x - f(y)| \geq |x - y|$. □

Let us denote by $\Delta(X)$ the simplex spanned by X.

Lemma 5. *Let z be a point of $F(X)$. If $|z| \geq c(X)$, then $p(z) \in \Delta(X)$.*

Proof. We prove the contraposition. Suppose that $z' = p(z)$ lies outside $\Delta(X)$. Then, for some $0 < t < 1$, tz' lies on a face of $\Delta(X)$, say on the face opposite to the point x_1. Then tz' is expressed as a convex combination of $x_2, \ldots, x_m$:

$$tz' = a_2 x_2 + \cdots + a_m x_m.$$

Since $1 < |x_1 - x_i|^2 = 2c(X)^2 - 2\langle x_1, x_i\rangle$ for $i \geq 2$, we have $2\langle x_1, x_i\rangle < 2c(X)^2 - 1$, and

$$\begin{aligned} 2\langle x_1, tz'\rangle &= 2\langle x_1, a_2 x_2 + \cdots + a_m x_m\rangle \\ &< (a_2 + \cdots + a_m)(2c(X)^2 - 1) = 2c(X)^2 - 1 < 0. \end{aligned}$$

So $2\langle x_1, z'\rangle < 2c(X)^2 - 1$. Since $\langle x_1, z\rangle = \langle x_1, z'\rangle$,

$$1 \geq |x_1 - z|^2 = c(X)^2 + |z|^2 - 2\langle x_1, z\rangle > c(X)^2 + |z|^2 - 2c(X)^2 + 1.$$

Therefore $|z| < c(X)$. □

Lemma 6. $c(F(X) \cap T(X)) < c(X)$.

Proof. First note that $x_i \notin F(X)$, $i = 1, \ldots, m$, for X is dispersed. Let $z \in F(X) \cap T(X)$. Then $z = p(z)$. If $z \in \Delta(X)$, then since $z \neq x_i$, $i = 1, \ldots, m$, $|z| < c(X)$. If $z \notin \Delta(X)$, then, by Lemma 5, $|z| < c(X)$. □

For a point z of $F(X)$, let us define

$$z^\circ = \begin{cases} f(p(z)) & \text{if} \quad p(z) \neq O, \\ O & \text{if} \quad p(z) = O, \end{cases} \qquad z^+ = \begin{cases} f(q(z)) & \text{if} \quad q(z) \neq O, \\ O & \text{if} \quad q(z) = O. \end{cases}$$

If $p(z) \neq O$ and $q(z) \neq O$, then the circular arc $z^+ z^\circ(-z^+)$ is the meridian $M(f(z))$ passing through the point $f(z)$ (cf. Lemma 3). In this case, z° bisects the meridian $M(f(z))$, and z lies on the angular region $\angle z^\circ O z^+$. Note also that if $z \neq O$, then $z^+ = f(z)^+$ and $z^\circ = f(z)^\circ$.

Lemma 7. *Let x, y be a dispersed pair on $\partial F(X)$ such that $\langle x, q(y)\rangle \geq O$. If $|y| < c(X)$, then $|x - y^\circ| \geq |x - y|$, while if $|y| \geq c(X)$, then $|x - y^+| \geq |x - y|$.*

Proof. If $p(y) = O$, then $|y| = |y^+| = s(X) > c(X)$, and $|x - y^+| = |x - y|$. If $q(y) = O$, then $|y| = |y^\circ| < c(X)$, and $|x - y^\circ| = |x - y|$. Assume $p(y) \neq O$ and $q(y) \neq O$. Let $M(y)$ be the meridian of $\partial F(X)$ passing through y, and let P be the plane determined by $M(y)$ (i.e., P is determined by O, $p(y)$, $q(y)$). Let w be the center of the circle C determined by $M(y)$. Then since the circle C passes through y^+ and $y^- := -y^+$, the center w must lie on the line Oy°. Since $|y^\circ| < c(X) < s(X) = |y^+|$, the origin lies between w and y°, and hence there is a unique point z on the subarc $y^+ y^\circ$ of $M(y)$ such that $|z| = c(X)$. The point y lies on the arc $y^+ z$ or on the arc zy° of $M(y)$ accordingly as $|y| \geq c(X)$ or $|y| < c(X)$. Let x' be the projection of the point x on the plane P determined by C.

Suppose first $|y| < c(X)$. Since $\langle x, q(y)\rangle \geq 0$, the points x' and y^+ lie on the same side of the line Oy° in the plane P. In this case one of the angles $\angle x'wz$, $\angle x'wy^\circ$ is greater than $\angle x'wy$. And hence one of $|x' - z|$, $|x' - y^\circ|$ is greater than $|x' - y|$, from which we can deduce that one of $|x - z|$, $|x - y^\circ|$ is greater than $|x - y|$. Thus if we prove $|x - z| \leq 1$, then we have $|x - y^\circ| \geq |x - y|$. Since $|z| = c(X)$, $p(z)$ is contained in $\Delta(X)$ by Lemma 5. Hence $p(z)$ is expressed as a convex combination of $x_1, \ldots, x_m$:

$$p(z) = a_1 x_1 + \cdots + a_m x_m \qquad (a_1 + \cdots + a_m = 1,\ a_i \geq 0).$$

For each x_i,

$$1 \geq |x - x_i|^2 = |x|^2 + |x_i|^2 - 2\langle x, x_i\rangle.$$

Multiplying both sides by a_i and summing on $i = 1, \ldots, m$, we have

$$1 \geq |x|^2 + c(X)^2 - 2\langle x, p(z)\rangle = |x|^2 + |z|^2 - 2\langle x, p(z)\rangle.$$

But since $\langle x, z\rangle = \langle x, p(z)\rangle + \langle x, q(z)\rangle$, we have

$$|x - z|^2 = |x|^2 + |z|^2 - 2\langle x, p(z)\rangle - 2\langle x, q(z)\rangle \leq 1 - 2\langle x, q(z)\rangle \leq 1.$$

Thus $|x - z| \leq 1$.

Similarly, if $|y| \geq c(X)$, then we have $|x - y^+| \geq |x - y|$. □

Lemma 8. *Let x, y be a dispersed pair in $F(X)$. Then $|x^+ - y^+| > 1$ or $|x^\circ - y^\circ| > 1$ holds. If $|y| \geq c(X)$, then $|x^+ - y^+|$ is always greater than* 1.

Proof. We may assume that x, y lie on the boundary $\partial F(X)$ (otherwise, by Lemma 4, we can replace x, y by $f(x), f(y)$). If $\langle x, q(y)\rangle < 0$, then $\langle x^+, y^+\rangle < 0$, and hence $|x^+ - y^+|^2 > |x^+|^2 + |y^+|^2 = 2s(X)^2 > 1$. Suppose now $\langle x, q(y)\rangle \geq 0$. Then, by the above lemma, $|x - y^+| > 1$ or $|x - y^\circ| > 1$ accordingly as $|y| \geq c(X)$ or $|y| < c(X)$. Since $\langle y^\circ, q(x)\rangle = 0$ and $\langle y^+, q(x)\rangle \geq 0$, applying Lemma 7 again, and noting that $|x^\circ - y^+|, |x^+ - y^\circ| \leq 1$, we have $|x^\circ - y^\circ| > 1$ or $|x^+ - y^+| > 1$. If $|y| \geq c(X)$, then $|x^+ - y^+|$ is always greater than 1. □

5. Proof of Theorems 2–4

Proof of Theorem 2. Suppose that sph $K(n, n) \leq n$ for some $n \geq 3$. Then there exist two dispersed sets X, Y, each of size n in E^n such that dis$(X, Y) \leq 1$. We may assume $c(X) \leq c(Y)$ and the c-center of X is at the origin O. Then, by Lemma 1, $c(X) \leq \sqrt{\frac{1}{2}}$, and hence the tangent space $T(X)$ of X is $(n-1)$-dimensional by Theorem B. Hence the normal space of X is one-dimensional, and the normal sphere of X consists of only two points. Since $c(X) \leq c(Y)$, there is at least one point y in Y such that $|y| \geq c(X)$. Then, since Y is a dispersed set in $F(X)$, applying Lemma 8, we have $|y^+ - w^+| > 1$ for all w of $Y - \{y\}$. Hence, all points w^+ must coincide with each other. Then, again by Lemma 8, we must have $|v^\circ - w^\circ| > 1$ for $v, w \in Y - \{y\}$. This implies that X and $\{w^\circ\colon w \in Y, w \neq y\}$ together induce a unit neighborhood graph isomorphic to $K(n, n-1)$, and hence sph $K(n, n-1) \leq n-1$. Thus sph $K(n-1, n-1) \leq n-1$. Repeating the same argument, we finally reach sph $K(2, 3) \leq 2$, which is a contradiction (see the example after Lemma 2). □

Proof of Theorem 3. Suppose that sph $K(m, M) \leq m - 1 + k$. Then there exist two dispersed sets X, Y of size m and M in E^{m-1+k} such that $F(X) \supset Y$. If $c(Y) < c(X)$, then $c(Y) < \sqrt{\frac{1}{2}}$ by Lemma 1, and Y is affinely independent and c-spherical by Theorem B. In this case, Y spans E^{m-1+k}, and since $X \subset F(Y)$, we must have $c(X) < c(F(Y)) < c(Y)$ by Lemma 6, which is a contradiction. Hence $c(X) \leq c(Y)$. Therefore X is affinely independent and c-spherical. We may assume that the c-center of X is at the origin O. Let

$$W = \{w \in Y\colon |w| < c(X)\}.$$

Then $|W| \leq m + k$. However, $|W| = m + k$ is impossible, because if $|W| = m + k$, then W spans E^{m-1+k}, and since $X \subset F(W)$, we must have $c(X) < c(W) < c(X)$, a contradiction. Thus $|W| \leq m - 1 + k$. Therefore, there are at least

$$N(k, m) - n + 1$$

points y_i, $i = 1, \ldots, N(k, m) - n + 1$, in Y such that $|y_i| \geq c(X)$. Let

$$U = Y - \{y_i\colon i = 1, \ldots, N(k, m) - n + 1\}.$$

Let G° be the graph with vertex set U and edge set

$$\{uv: u, v \in U \quad \text{and} \quad |u^\circ - v^\circ| \leq 1\}.$$

Then since $|U| \geq m-1+k \geq \max\{n, R(m-1, n)\}$ and since $n = R(2, n)$, the graph G° contains either an independent set of size $\max\{2, m-1\}$ or a clique of size n. We show that G° must contain a clique of size n. If $m \leq 3$, then since

$$\text{sph } K(1,2)=1, \qquad \text{sph } K(2,2)=2, \quad \text{and} \quad \text{sph } K(3,2)>2,$$

$F(X) \cap T(X)$ cannot contain two dispersed points, and hence G° is a complete graph. Suppose that for $m>3$, G° contains an independent set of size $m-1$. Then the set $\{u^\circ: u \in U\}$ contains $m-1$ dispersed points, and hence $T(X)$ contains a unit neighborhood graph isomorphic to $K(m, m-1)$. But since

$$\text{sph } K(m-1, m) > m-1$$

by Theorem 2, this is impossible. Thus, in either case, G° must contain a clique of size n. Let $\{u_i: i=1, \ldots, n\}$ be a clique of G°. Then, by Lemma 8, the set

$$\{u_i^+: i=1, \ldots, n\} \cup \{y_i^+: i=1, \ldots, N(k, m)-n+1\}$$

is a dispersed set of size $N(k, m)+1$ on the normal sphere of X. However, since the normal sphere of X is a k-dimensional sphere of radius $s(X)<s(m)$, it cannot contain more than $N(k, m)$ dispersed points. Thus we have a contradiction. Therefore, sph $K(m, M) \geq m+k$.

Now assume $m=5$. By Corollary 3, and Table 1, we have sph $K(3,5)=5$ (note that to derive Corollary 3, we used only the case $m \leq 3$ of Theorem 3). Hence we have dpn $(F(X) \cap T(X)) \leq 2$. Therefore we may replace $\mathrm{R}(m-1, n)$ by $\mathrm{R}(3, n)$ in the condition of the theorem □

Proof of Theorem 4. We proved in Theorem 3 of [3] that, for any fixed $m>0$, $N(k, m)$ is exponentially large in k, that is,

$$N(k, m) > (k-2)/(k-1)^{1/2} \exp\{(k-2)\beta^2/2\},$$

where $\beta = \sin^{-1}\{1/(m+1)\}$. Hence, for infinitely many k, $N(k+1, m) - N(k, m) > k+m$ holds (for otherwise, $N(k, m) < k^2/2 + O(k)$, which is not exponentially large). Then, by Corollary 2 and Theorem 1,

$$m+k \leq \text{sph } K(m, N(k, m)+m+k-2) \leq \text{sph } K(m, N(k+1, m)) \leq m+k$$

holds for infinitely many k. For each of such k, let $n=n(k)=N(k+1, m)$. Then $d(m, n)=m+k$ and sph $K(m, n)=d(m, n)$. □

References

1. P. C. Fishburn, On the sphericity and cubicity of graphs, *J. Combin. Theory Ser. B* **35** (1983), 309-318.
2. P. Frankl and H. Maehara, Embedding the *n*-cube in lower dimensions, *European J. Combin.* **7** (1986), 221-225.
3. P. Frankl and H. Maehara, On the contact dimension of graphs, *Discrete Comput. Geom.* **3** (1988), 89-96.
4. L. C. Freeman, Spheres, cubes, and boxes: graph dimensionality and network structure, *Social Network* **5** (1983), 139-156.
5. H. Maehara, Convex bodies forming pairs of constant width, *J. Geom.* **22** (1984), 101-107.
6. H. Maehara, Space graphs and sphericity, *Discrete Appl. Math.* **7** (1984), 55-64.
7. H. Maehara, On the sphericity for the join of many graphs, *Discrete Math.* **49** (1984), 311-313.
8. H. Maehara, Contact patterns of equal nonoverlapping spheres, *Graphs Combin.* **1** (1985), 271-282.
9. H. Maehara, Sphericity exceeds cubicity for almost all complete bipartite graphs, *J. Combin. Theory Ser. B* **40** (1986), 231-235.
10. H. Maehara, On the sphericity of the graphs of semi-regular polyhedra, *Discrete Math.* **58** (1986), 311-315.
11. H. Maehara, J. Reiterman, V. Rödl, and E. Šimňajova, Embedding trees in Euclidean spaces, *Graphs Combin.* **4** (1988), 43-47.
12. J. Reiterman, V. Rödl, and E. Šiňajova, Geometrical embedding of graphs, *Discrete Math.*, to appear.
13. K. Schutte and B. L. Van der Waerden, Auf welcher kugel haben 5, 6, 7, 8, order 9 punkte mit Mindestabstand Ein Platz?, *Math. Ann.* **123** (1951), 96-124.

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