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RESEARCH ARTICLE

Normal Skew Lattices

Jonathan Leech

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Recall that a band S is **normal** if each principal monoid eSe is a semilattice. Equivalently, a normal band is a band satisfying the identity, $wxyz = wyxz$. Recall also that a **skew lattice** is an algebra $(S, \vee, \wedge)$ such that $\vee$ and $\wedge$ are associative, idempotent binary operations on the set S which are connected by the absorption laws, $x \wedge (x \vee y) = x = (y \vee x) \wedge x$, and their duals. By a **normal skew lattice** we mean a skew lattice $(S, \vee, \wedge)$ such that each principal subalgebra $x \wedge S \wedge x$ is a sublattice of S. Skew lattices naturally arise as multiplicative bands of idempotents in rings. In particular, *every maximal normal band of idempotents in a ring forms a normal skew lattice which is the full set of idempotents in the subring it generates; and conversely, when the idempotents of a ring are closed under multiplication, they form a normal skew lattice.* (See [6] 2.2.) Upon examination of the semigroup ring of a normal band, one thus obtains that *every normal band can be embedded in a normal skew lattice.* Hence there is a sense in which normal skew lattices form completions of normal bands; as such, the theory of normal skew lattices may be seen to extend the theory of normal bands initiated in [9], [12], and [13].

This paper is divided into three sections, the first of which is a preliminary section giving the skew lattice analogues of some basic results of Yamada and Kimura. The remaining sections focus attention on the principal theme of this paper, the connection between normality and distributivity. The main result of the middle section, Theorem 2.8, gives a canonical factorization of a normal symmetric skew lattice into the fibered product of a lattice with a distributive skew lattice; this effectively reduces the study of normal symmetric skew lattices to the distributive case. The distributive case is considered in the final section which contains analogues of several fundamental results about distributive lattices. Finally, the needed background on bands, distributive lattices, and universal algebras may be found in [2], [3], and [8].

1. Basic Theory

1.1. Recall that a **rectangular skew lattice** is a skew lattice S for which $(S, \wedge)$ is a rectangular band and $(S, \vee)$ is its dual where $x \vee y = y \wedge x$. The first important theorem on skew lattices is the Clifford-McLean Theorem: *the maximal rectangular subalgebras of a skew lattice form a congruence partition for which the induced quotient algebra is the maximal lattice image.* The congruence classes are called **equivalence classes** and two members, x and y, of the same class are said to be **equivalent**, denoted $x \equiv y$. The **natural partial ordering** of a skew lattice is defined by $x \geq y$ iff $x \wedge y = y = y \wedge x$, or dually, $x \vee y = x = y \vee x$. Our goal is to show how a normal skew lattice is constructed from its equivalence classes. For normal bands there is the construction of

Yamada and Kimura utilizing a system of homomorphisms between rectangular bands which, in essence, mimics the natural partial ordering. To extend this construction to normal skew lattices, the homomorphisms must be projections.

1.2. A **projection** is a pair $(K, \mathbf{k})$ where $K : A \to B$ is an epimorphism of rectangular skew lattices and $\mathbf{k}$ is a set of monomorphisms $k : B \to A$ called **coprojections** such that: (i) there is a factorization $J \times B \cong A$ which composed with K yields the B-coordinate projection; (ii) under this factorization the coprojections correspond to the canonical injections, $B \to \{j\} \times B$. Clearly the coprojections decompose the inverse relation K^{-1}. Any factorization for which (i) and (ii) hold is said to be **compatible** with the projection. *Compatible factorizations are essentially unique:* given compatible factorization, $J \times B \cong A$ and $J' \times B \cong A$, there is an isomorphism $J \cong J'$ that transforms $J' \times B \cong A$ into $J \times B \cong A$. For each b in B a compatible factorization $K^{-1}b \times B \cong A$ is given by $(u, y) \to ky$ where k is the unique coprojection for which $kb = u$. *Projections form a category:* given projection $(L, \mathbf{l})$ with $L : B \to C$, one has $(L, \mathbf{l})(K, \mathbf{k}) = (LK, \mathbf{kl})$ where $\mathbf{kl} = \{kl \mid k \text{ in } \mathbf{k} \text{ and } l \text{ in } \mathbf{l}\}$. Finally, a projection $(K, \mathbf{k})$ will usually be denoted by K. Specializing [7] 1.3 to the normal case yields:

Proposition 1.3. *Let $A > B$ be two classes in a normal skew lattice and let $K = K(A, B) : A \to B$ be defined implicitly by $x \geq Kx$. Then all maps $k : B \to A$ of the form $ky = y \vee a \vee y$, for a fixed in A, are either equal or have disjoint images; together they form the coprojections for the projection $K : A \to B$.* (We refer to K and its k as the **natural projection and coprojections** for A and B.)

1.4. A **rectangular functor** on a lattice T is a functor K from $(T, \geq)$ to the category of rectangular skew lattices and projections such that $s \neq t$ implies $K(s)$ and $K(t)$ are disjoint. For $s \geq t$ in T, the projection from $K(s)$ to $K(t)$ is denoted by $K(s, t)$ and its coprojections are denoted by $k(s, t, i)$. Given T and K, let S denote the union of the $K(t)$. One can use the $K(s, t)$ to define a normal meet operation on S; one would like to use the $k(s, t, i)$ to define a join opertion on S, thus turning it into a normal skew lattice. This leads us to consider pairs of inverse relations of the form $K(n, s)^{-1}$ and $K(n, t)^{-1}$ where $n = s \vee t$ in T. We call two such relations **orthogonal** if for each a in $K(s)$, the **image** of a in $K(n)$ given by $K(n, s)^{-1}[a]$ lies in the image of a unique coprojection $k(n, t, j)$, and similarly for each b in $K(t)$ the image of b in $K(n)$ lies in the image of a unique coprojection $k(n, s, i)$. Clearly, *orthogonality holds in each join situation when the lattice is totally ordered;* moreover:

Proposition 1.5. *For a finite chain T, every rectangular functor K over T is isomorphically obtained from a T-indexed family of rectangular algebras $\{X(t) \mid t \text{ in } T\}$ by setting $K(t) = \prod\{X(s) \mid s \leq t\}$ and letting all projections and coprojections be the canonical coordinate projections and injections.*

Specializing [7] 3.10,12, we extend a well known theorem given in [13].

Theorem 1.6. *Let K be a rectangular functor defined on the lattice T such that in each join situation $n = s \vee t$ the inverse relations $K(n, s)^{-1}$ and $K(n, t)^{-1}$ are orthogonal. Then $S = \cup\{K(t) \mid t \text{ in } T\}$ becomes a normal skew lattice, with K providing the system of natural projections and coprojections, if for each pair a in $K(s)$ and b in $K(t)$, the meet and join are defined by*

$$a \wedge b = K(s, m)a \wedge K(t, m)b \text{ and } a \vee b = k(n, s, i)a \vee k(n, t, j)b,$$

where $m = s \wedge t$, $n = s \vee t$, *the image of* $k(n, s, i)$ *contains* $K(n,t)^{-1}[b]$ *and the image of* $k(n,t,j)$ *contains* $K(n,s)^{-1}[a]$. *Conversely, every normal skew lattice arises in this manner.*

1.7. Normal skew lattices form a variety of algebras which includes the subvariety of **right normal** skew lattices ($x \wedge y \wedge z = y \wedge x \wedge z$) and the subvariety of **left normal** skew lattices ($x \wedge y \wedge z = x \wedge z \wedge y$). By [5] 1.15, every normal skew lattice is the fibered product of its maximal right normal image with its maximal left normal image over its maximal lattice image. (See also [4], [13].)

2. Symmetric Algebras

2.1. The simplifications encountered in 1.5 may be extended to a wider class of algebras. A skew lattice is **symmetric** iff commutativity is unambiguous: $x \vee y = y \vee x$ iff $x \wedge y = y \wedge x$. Since normality insures that $x \vee y = y \vee x$ implies $x \wedge y = y \wedge x$, it follows that a normal skew lattice is symmetric whenever $x \wedge y = y \wedge x$ implies $x \vee y = y \vee x$. This implication is equivalent to the identity, $x \vee y \vee (x \wedge y \wedge x) = (x \wedge y \wedge x) \vee y \vee x$; thus *symmetric normal skew lattices form a subvariety of the variety of normal skew lattices.* Normal skew lattices in rings are symmetric as are skew lattices for which the underlying lattice is totally ordered.

2.2. We generalize the construction of Proposition 1.5. Let T be a lattice, let P be a prime filter of T and let X be a rectangular algebra. Then $T[X,P]$ is the symmetric normal skew lattice defined as the union of two subalgebras, $P \times X$ and $T - P$, where for (p,x) in $P \times X$ and t in $T - P$ the mixed joins and meets are given by $(p,x) \vee t = (p \vee t, x) = t \vee (p,x)$ and $(p,x) \wedge t = p \wedge t = t \wedge (p,x)$. Any skew lattice isomorphic with $T[X,P]$ is said to be P-**primary** over T with **fiber** X. Prime filters arise as inverse images $f^{-1}(1)$ for lattice epimorphisms $f : T \to \mathbf{2}$, where $\mathbf{2}$ denotes the lattice 1^0. Thus $T[X,P]$ may be viewed as the fibered product $T \times_{\mathbf{2}} X^0$ obtained by pulling the surjection $X^0 \to \mathbf{2}$ back along $f : T \to \mathbf{2}$. More generally, let $F(T)$ be the family of all prime filters of T, including T, and let $\{X(P) \mid P \text{ in } F(T)\}$ be a corresponding family of rectangular algebras; then the fibered product over T, $\prod_T T[X(P), P]$, is symmetric and normal. Its functor K is given by setting $K(t) = \prod\{X(P) \mid t \text{ in } P\}$ and using the canonical coordinate projections and injections. Any skew lattice isomorphic to such a fibered product is said to be **decomposable** and the fibered product is said to be its **primary decomposition**. An immediate consequence of the theory to be developed in the remainder of this paper is the following result.

Theorem 2.3. *A symmetric normal skew lattice with a finite maximal distributive lattice image is decomposable. In particular, a symmetric normal skew lattice with a finitely generated maximal lattice image is decomposable.*

2.4. The above discussion indicates that upon assuming symmetry there arises a fundamental connection between normality and distributivity. Clearly this has no analogue for bands. The precise connection is spelled out in Theorems 2.5 and 2.8. But first recall that a skew lattice is **distributive** if both the identity $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$ and its dual hold; it is called **meet bidistributive** if the stated identity can be strengthened to: $x \wedge (y \vee z) \wedge w = (x \wedge y \wedge w) \vee (x \wedge z \wedge w)$. Recall also that in a **right handed** skew lattice ($x \wedge y \wedge x = y \wedge x$ and $x \vee y \vee x = x \vee y$) distributivity reduces to $(y \vee z) \wedge x = (y \wedge x) \vee (z \wedge x)$ and its dual, $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Distributivity and normality mix as follows:

Theorem 2.5. *A skew lattice S is meet bidistributive if and only if it is normal and its maximal lattice image T is distributive, in which case S is also distributive. Both $x \wedge (y \vee z) = (x \wedge z) \vee (x \wedge z)$ and $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$ hold in S if and only if S is both meet bidistributive and symmetric.*

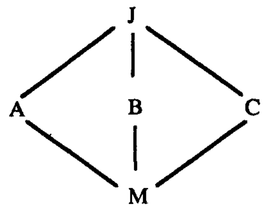
Proof. By [5] 1.15, we may also assume S is right handed. First let S be meet bidistributive. Then T is distributive. Let x, y, z in S be such that $x \geq y, z$ and $y \equiv z$. Then $z = z \wedge (y \vee z) \wedge x = (z \wedge y \wedge x) \vee (z \wedge x \wedge x) = y \vee z = y$. Thus $x \wedge S \wedge x$ is a sublattice for each x is S so that S is normal. Conversely, given S is normal and T is distributive, one has $x \wedge (y \vee z) \wedge w \equiv (x \wedge y \wedge w) \vee (x \wedge z \wedge w)$ with both sides in $(x \wedge w) \wedge S \wedge (x \wedge w)$. By normality, equivalence becomes equality and meet bidistributivity follows. Also $x \vee (y \wedge z) \equiv (x \vee y) \wedge (x \vee z)$ in S with $x \vee z \geq (x \vee y) \wedge (x \vee z)$. Since $x \vee (y \wedge z) \vee x \vee z = x \vee (y \wedge z) \vee z = x \vee z$, $x \vee z \geq x \vee (y \wedge z)$, again equivalence becomes quality, and S is distributive. Suppose next that S is also symmetric. Then y meet commutes with both $x \wedge (y \vee z)$ and $(x \wedge y) \vee (x \wedge z)$ with $x \wedge y$ being their common meet. By established identities, the commuting join of y with either $x \wedge (y \vee z)$ or $(x \wedge y) \vee (x \wedge z)$ is $(y \vee x) \wedge (y \vee z)$ and thus $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ follows. If S is not symmetric, then there exist x and y with $x \wedge y = y \wedge x$, but $x \vee y \neq y \vee x$. Thus $x \wedge (y \vee x) \neq x$, but $(x \wedge y) \vee (x \wedge x) = x$.

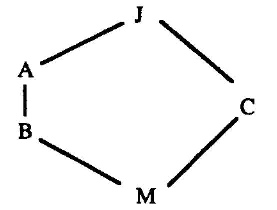
2.6. A skew lattice is **binormal** if both of its operations are normal. In this case its natural projections are isomorphisms and the skew lattice factors as the product of a lattice with a rectangular algebra. (This was first observed by Schein in [10].) Thus *the following are equivalent conditions on a skew lattice: it is both meet and join bidistributive; it factors as the product of a distributive lattice and a rectangular algebra.* (See also [1] and [11].)

2.7. Let S be symmetric and normal, having maximal lattice image T and rectangular functor of projections K. If $K(A, B)$ is an isomorphism for classes $A > B$, then for any intermediate class, C, both $K(A, C)$ and $K(C, B)$ are also isomorphisms. Elements, x and y, are said to be **reflections** in S if there exist $A \geq B$, with x and y lying in intermediate classes, such that (i) $K(A, B)$ is an isomorphism; and (ii) x and y have the same image both in A and in B. (This includes the possibility of x or y lying in A or B or even being equal.) It is easily seen that reflection is an equivalence relation. One can show that whenever $K(A, B)$ is an isomorphism, then for all classes C both $K(A \vee C, B \vee C)$ and $K(A \wedge C, B \wedge C)$ are isomorphisms. It follows that *reflection is a congruence.* Indeed, it is the maximal congruence inducing isomorphisms between corresponding equivalence classes in S and the induced quotient algebra. The quotient algebra induced by reflection is called the **reduced algebra** and is denoted by S^r. Clearly $S^{rr} = S^r$. We say that S is **reduced** if $S^r = S$.

2.8. The Reduction Theorem. *Let S be a symmetric normal skew lattice, let T be its maximal lattice image, let S^r be its reduced algebra, and let D be the maximal lattice image of S^r. Then the canonical epimorphisms, $S \to T$ and $S \to S^r$, induce an isomorphism of S with the fibered product over D of T with S^r : $S \cong T \times_D S^r$; moreover, the reduced algebra S^r is distributive.*

Proof. We need only show that when S is reduced, then it is distributive. We do so by showing that neither of the following types of subalgebras can arise in S, where A, B, C, J, and M are distinct equivalence classes in S.





Suppose that M is trivial, say $M = \{m\}$, so that in both diagrams each element of C commutes with every element in either A or B. In the left diagram, each element of C commutes with all elements in J so that C is trivial; likewise, A, B, and finally J are trivial classes. In the right diagram, each element in J has unique factorization $a \vee c = b \vee c$, with (a,c) in $A \times C$, (b,c) in $B \times C$, and $a > b$. As a consequence, $K(A,B)$ is an isomorphism. Even when M is not trivial, the general case may be reduced to the special case upon intersecting all classes with the principal subalgebra $m \vee S \vee m$ for any m in M. Thus, in general, $K(J,M)$ is an isomorphism in the left diagram, while $K(A,B)$ is an isomorphism in the right diagram. But S is assumed reduced. Hence neither type of subalgebra can arise. Thus the lattice T is distributive, and by Theorem 2.5, so is S.

3. Distributive Symmetric Algebras

3.1. Our goal is to obtain noncommutative analogues of the following basic facts about distributive lattices: every distributive lattice can be embedded into a power of the lattice **2**; when the lattice is finite, then each element is a unique irredundant join of join-irreducible elements. We begin by examining subdirect decomposition. In the case of normal bands, the subdirectly irreducible algebras consist of isomorphic copies of the following bands: $\mathbf{R}_2$, the right normal band on $\{1,2\}$ with multiplication given by the rule $xy = y$, $\mathbf{L}_2$, its left normal dual; **2**, the semilattice 1^0; **3**, the band $\mathbf{R}_2^0$; and $\mathbf{3}^*$, its left normal dual. (This was shown by Schein in [9].) All five bands become subdirectly irreducible skew lattices (which are necessarily distributive, normal, and symmetric) upon letting the meet be the given multiplication, and letting the join dualize the meet: $x \vee y = y \wedge x$, if neither are zero, and $x \vee 0 = x = 0 \vee x$, otherwise. The following theorem extends some basic results given for the case of right normal bands by Schein in [9] and Wagner in [12]; in its statement **5** denotes the fibered product $\mathbf{3} \times_{\mathbf{2}} \mathbf{3}^*$.

Theorem 3.2. *The only subdirectly irreducible distributive, symmetric normal skew lattices are copies of* **2**, $\mathbf{R}_2$, $\mathbf{L}_2$, **3**, *or* $\mathbf{3}^*$. *Every distributive, symmetric normal skew lattice is thus a subdirect product of these algebras and hence can be embedded in a power of* **5**.

Proof. Let S be subdirectly irreducible. We show that it is a copy of one of the five algebras. Since S is subdirectly irreducible, it must have distinct elements a and b which are congruent under all nonidentity congruences. If a and b are not equivalent, then equivalence is the identity congruence. Thus S is a lattice which must be a copy of **2**. So suppose $a \equiv b$ in class A. Define the congruence $\sim$ (rel a) by $x \sim y$(rel a) iff $a \wedge x = a \wedge y$, and let $\sim^*$ (rel a) be its left-right dual. Then either $\sim$ (rel a) or $\sim^*$ (rel a) separates a from b and is thus the identity. This forces the class A to be maximal. If this classs is all of S, then S must be a copy of either $\mathbf{R}_2$ or $\mathbf{L}_2$. So suppose that $a > c$. If $b \not> c$, then either $\sim$ (rel c) or $\sim^*$ (rel c) are nonidentity congruences separating a and b. Thus for all c in S, $a > c$ iff $b > c$. Together with symmetry, this implies that the class A is join irreducible in the underlying lattice. If S' denotes $S - A$ and T' denotes the sublattice $a \wedge S' \wedge a = b \wedge S' \wedge b$, then define the congruence $\sim$ (rel T') by $x \sim y(\text{rel } T')$ iff $t \vee x \vee t = t \vee y \vee t$ for some t in T'. This congruence clearly separates a and b, and thus is the identity. Since S' is a single congruence class it must reduce to the zero of S and thus $S = A^0$. Applying subdirect irreducibility again, S must be a copy of either $\mathbf{3}$ or $\mathbf{3}^*$.

Corollary 3.3. *Distributive, symmetric normal skew lattices form the subvariety of skew lattices generated by* $\mathbf{5}$. *Symmetric normal skew lattices form the subvariety of skew lattices generated from the variety of lattices and* $\mathbf{5}$.

3.4. When the underlying lattice is also finite, the skew lattice is decomposable. To see this, let S be such an algebra,let T be its maximal lattice image, and let $\pi = \pi(T)$ be the set of all join-irreducible elements of T, including the minimal element 0. The class of prime filters of T is given by $F(T) = \{p \vee T \mid p \text{ in } \pi\}$. Recall that the **center** of S, denoted by $Z(S)$, coincides with the union of all singleton classes of S. Since S is normal, its center correspponds to a (possibly empty) ideal in T; in particular, $Z(S)$ is empty precisely when the minimal class of S is nontrivial; however, all classes minimal in the complement of $Z(S)$ correspond to join-irreducible elements in π.

Lemma 3.5. *Let X be a minimal equivalence class in the complement of $Z(S)$, let x be fixed in X, and let P be the prime filter in T induced by the image of X in T. Set $S' = (S - S \vee x \vee S) \cup x \vee S \vee x$ and let $T[X,P]$ be the P-primary algebra induced by X and P. Then S' is a subalgebra which is also mapped onto T by the canonical epimorphism from S. Moreover, there is an isomorphism θ decomposing S into a fibered product, $\theta : S \cong S' \times_T T[X,P]$, which is given by $\theta(y) = (x \vee y \vee x, y \wedge x \wedge y)$ for all y in $S \vee x \vee S$, and $\theta(y) = y$ otherwise. Finally, upon comparison in T the center of S' is properly larger than the center of S.*

Proof. S' is a subalgebra since x commutes with elements in the complement of $S \vee x \vee S$. θ is at least an isomorphism off of $S \vee x \vee S$ and by Theorem 1.6, the complementary restriction is at least a bijection of $S \vee x \vee S$ with the subalgebra of the fibered product lying over the same filter. We leave it to the reader to show that the latter is also an isomorphism between subalgebras. Suppose that u lies in $S \vee x \vee S$, while w lies in the complement. Then $\theta(u \vee w) = \theta(u) \vee \theta(w)$ is equivalent to $x \vee u \vee w \vee x = x \vee u \vee x \vee w$ and $(u \vee w) \wedge x \wedge (u \vee w) = u \wedge x \wedge u$. Since x commutes with w, the first identity holds. Because $x \wedge w$ lies in $Z(S)$, $x \wedge u \geq x \wedge w$ and $x \wedge (u \vee w) = x \wedge u$. Similarly, $(u \vee w) \wedge x = u \wedge x$ and the second identity also holds. Finally, $\theta(u \wedge w) = \theta(u) \wedge \theta(w)$ is equivalent to $u \wedge w = (x \vee u \vee x) \wedge w$, which is clear.

3.6. The Primary Decomposition Theorem. *Let S be a distributive, symmetric normal skew lattice with finite maximal lattice image T and let $F(T)$ be the set of prime filters of T, including T. Then each P in $F(T)$ corresponds to a rectangular algebra $X(P)$, which is unique to within isomorphism, such that the $X(P)$ induce an isomorphism of S with the fibered product $\prod_T T[X(P), P]$. S is reduced if and only if $X(P)$ is nontrivial for each proper prime filter P.*

Proof. Repeated applications of the previous lemma enable one to pass through the prime filters of T and successfully strip primary factors off of S to obtain the decomposition. To see uniqueness, let X be a join irreducible class of S corresponding to the prime filter P. If $P = T$, then X is the minimal class of S and $X(P) = X$. Otherwise, there is a maximal class lying beneath X, call it Y. Upon applying the lemma to the subalgebra $X \cup Y$, X must factor as $X(P) \times Y$, where to within isomorphism $X(P)$ is given as $y \vee X \vee y$ for any y in Y. The final assertion about S being reduced now follows. If $X(P)$ is trivial for P not T, then $K(X,Y)$ is an isomorphism; but if no $X(P)$ is trivial, except possibly for $P = T$, then no $K(A,B)$ with $A > B$ can be an isomorphism.

3.7. For right handed algebras, the results of this section can be recast in terms analogous to rings of sets. Let A and B be nonempty sets and let $\mathcal{P}(A,B)$ denote the set of all partial functions from A to B. $\mathcal{P}(A,B)$ becomes a skew lattice which is easily seen to be distributive, symmetric, and right normal upon setting $f \vee g = f \cup (g \mid G - F)$ and $f \wedge g = g \mid (F \cap G)$, where F and G denote the functional domains of f and g, respectively. By a **ring of partial functions** is meant any subalgebra of $\mathcal{P}(A,B)$ for some A and B. Since the power algebra $\mathbf{3}^A$ is an isomorphic copy of $\mathcal{P}(A,\{1,2\})$, we can state the following variation of Theorem 3.2: *every distributive, symmetric right normal skew lattice is isomorphic with a ring of partial functions.* To recast Theorem 3.6, we first refine the ring concept to allow partial functions with variable codomains, $\mathbf{B} = \{B(a) \mid a \text{ in } A\}$; that is, partial functions that are restrictions of functions in the Cartesian product $\prod \mathbf{B}$. This class of partial functions is denoted by $\mathcal{P}(A,\mathbf{B})$. If $B = \cup\mathbf{B}$, then this class forms a subalgebra of $\mathcal{P}(A,B)$ and is thus a ring of partial functions. If $\mathcal{R}$ is a ring of subsets of A, then the **full ring of partial functions** over $\mathcal{R}$, $\mathcal{P}(R,\mathbf{B})$, is the ring of all partial functions which have their domain in $\mathcal{R}$. This leads to the following variation of the Primary Decomposition Theorem: *every distributive, symmetric right normal skew lattice with a finite maximal lattice image is isomorphic with a full ring of partial functions over a ring of subsets of a finite set.* For if T, $F(T)$, P, and $X(P)$ are as in 3.6, then just set $A = F(T)$, $\mathbf{B} = \{X(P) \mid P \text{ in } F(T)\}$ and $\mathcal{R} = \{F(x) \mid x \text{ in } T\}$ where $F(x) = \{P \mid x \text{ in } P\}$.

3.8. We conclude with yet another analogy with distributive lattices. Recall that a symmetric skew lattice S with zero 0 is said to be **quasi-Boolean** if for each x in S the subalgebra $x \wedge S \wedge x$ forms a Boolean lattice; in this case S has a **difference operation** defined by setting $x - y$ equal to the complement of $x \wedge y \wedge x$ in $x \wedge S \wedge x$. Thus one may define a **skew quasi-Boolean algebra** to be an algebra $(B, \vee, \wedge, -, 0)$, where $-$ is a binary operation and 0 is a distinguished constant, such that: $(B, \vee, \wedge, 0)$ is a distributive, symmetric, and normal skew lattice with a zero element, 0; $(x-y) \vee (x \wedge y \wedge x) = x$ and $(x-y) \wedge (x \wedge y \wedge x) = 0$. The $\mathcal{P}(A,B)$ above, along with maximal normal bands in rings, from examples of such algebras. If a maximal class exists, the algebra is called a **skew Boolean algebra**. Clearly **5** becomes a skew Boolean algebra and by [6] 1.14, every skew Boolean algebra can be embedded in a power of the algebra **5**. Thus, a *skew lattice can be embedded in a skew Boolean algebra if and only if it is distributive, symmeric, and normal.*

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Department of Mathematics
Westmont College
955 La Paz Road
Santa Barbara, CA 93108
USA

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