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RESEARCH ARTICLE

Individual Stability of C_0-semigroups With Uniformly Bounded Local Resolvent

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It is well-known that a C_0-semigroup $\mathbf{T} = \{T(t)\}_{t\geq 0}$ on a Hilbert space is uniformly exponentially stable, i.e. $\|T(t)\| \leq Me^{-\omega t}$ for some $\omega > 0$ and all $t \geq 0$, if and only if the resolvent $R(z, A) := (z - A)^{-1}$ of its generator A exists and is uniformly bounded in the right half plane $\{\operatorname{Re} z > 0\}$ [2]. Many different proofs of this result exist; see [6] for further references.

For semigroups acting on a Banach space, this result is false (see for instance [3], [6, Example A-IV.1.2(b)]), and only some weaker statements are true. Define, for $n = 0, 1, 2, \ldots$, the growth bounds $\omega_n(\mathbf{T})$ as the infimum of all $\omega \in \mathbb{R}$ such that

$$\|T(t)x\| \leq Me^{\omega t}\|x\|_{D(A^n)}$$

for all $x \in D(A^n)$, $t \geq 0$, and some $M \geq 1$. Here, $\|x\|_{D(A^n)} := \|x\| + \|A^n x\|$ is the graph norm in $D(A^n)$. It was shown by Slemrod [11] that $\omega_2(\mathbf{T}) < 0$ for every semigroup with uniformly bounded resolvent in $\{\operatorname{Re} z > 0\}$. In some special cases it was known that even $\omega_1(\mathbf{T}) < 0$ holds, viz. if the underlying Banach space is B-convex [14] or if $\mathbf{T}$ is a positive semigroup on a Banach lattice [6, Theorem C-IV.1.3]. It was an open question whether this holds for arbitrary semigroups with uniformly bounded resolvent in $\{\operatorname{Re} z > 0\}$. In [5] an affirmative solution is claimed, but the proof depends on a lemma that is wrong; cf. [8] for a counterexample. Recently, the problem was settled by Weis and Wrobel [12]. In a nutshell, their argument is the following: first, using complex interpolation theory, it is shown that the map $\alpha \mapsto \omega_\alpha(\mathbf{T})$ is convex, hence continuous, as a map $\mathbb{R}_+ \to \mathbb{R}_+$. Here, the growth bounds $\omega_\alpha(\mathbf{T})$ are defined in terms of the fractional powers of $-A$. Then this is combined with the fact [8] that $\omega_{1+\epsilon}(\mathbf{T}) < 0$ for all $\epsilon > 0$.

In this note, we prove an individual stability result for elements in $D(A)$ that has the Weis-Wrobel result as an immediate corollary. The proof uses a complex inversion formula for the Laplace transform of $\mathbf{T}$. The basic idea is to deform the path of integration to a suitable piecewise-linear path and estimate the pieces separately. Thus, our proof is elementary and uses first principles only.

Lemma 1. *For all $r > 0$ and $t > 0$,*

$$\left| \int_r^\infty \frac{e^{i\lambda t}}{\lambda}\, d\lambda \right| \leq \frac{3\pi}{2rt}.$$

Proof. Integrate $z \mapsto z^{-1}e^{itz}$ along the closed contour consisting of the semicircle Γ_r of radius r in the upper half plane, the interval $[r, R]$, the semicircle Γ_R, and the interval $[-R, -r]$. By letting $R \to \infty$, we find that

$$\left| \int_r^\infty \frac{\sin \lambda t}{\lambda}\, d\lambda \right| \leq \left| \frac{1}{2i} \int_{\Gamma_r} \frac{e^{izt}}{z}\, dz \right| \leq \frac{1}{2} \int_0^\pi e^{-rt\sin\theta}\, d\theta \leq \frac{\pi}{2rt},$$

in the last estimate using the obvious facts that $\sin(\pi-\theta)=\sin\theta$ and $\sin\theta \geq \frac{2\theta}{\pi}$ for all $0 \leq \theta \leq \frac{\pi}{2}$. We also have

$$\left|\int_r^\infty \frac{\cos\lambda t}{\lambda}\,d\lambda\right| = \left|\int_{rt}^\infty \frac{\sin(\tau+\frac{\pi}{2})}{\tau}\,d\tau\right|$$
$$\leq \frac{\pi}{2(rt+\frac{\pi}{2})} + \int_{rt}^\infty \left|\sin(\tau+\frac{\pi}{2})\left(\frac{1}{\tau}-\frac{1}{\tau+\frac{\pi}{2}}\right)\right|\,d\tau \leq \frac{\pi}{rt}.$$

From these two estimates, the lemma follows. ∎

Theorem 2. *Let* $\mathbf{T}$ *be a* C_0*-semigroup on a Banach space* X*, with generator* A*. If, for some* $x_0 \in X$*, the map* $z \mapsto R(z,A)x_0$ *admits a bounded analytic continuation to the half plane* $\{\mathrm{Re}\, z > 0\}$*, then for each* $\lambda \in \varrho(A)$ *there exists a constant* M *such that*

$$\|T(t)R(\lambda,A)x_0\| \leq M(1+t) \qquad \forall t \geq 0.$$

Proof. By the resolvent identity, it is enough to prove the theorem for one $\lambda \in \varrho(A)$. Fix $\omega \geq 0$ large enough such that the semigroup $\mathbf{T}_\omega$ defined by $T_\omega(t) := e^{-\omega t}T(t)$ has negative type $\omega_0(\mathbf{T}_\omega) < 0$. We shall prove that $\|T(t)R(\omega,A)x_0\| \leq M(1+t)$ for some M and all $t \geq 0$.

Put $A_\omega := A-\omega$ and let $F_0(z)$ denote the bounded analytic continuation of $R(z,A_\omega)x_0$ to $\{\mathrm{Re}\, z > -\omega\}$. Choose a constant K such that $\sup\{\|F_0(z)\| : \mathrm{Re}\, z > -\omega\} \leq K\|x_0\|$. Fix $t > 0$. By [9] (see also [6], p.116) and the resolvent identity, for all $\xi > \omega_0(\mathbf{T}_\omega)$ we have

$$T_\omega(t)A_\omega^{-1}x_0 = \frac{1}{2\pi i}\int_{\xi+i\mathbb{R}} e^{zt}R(z,A_\omega)A_\omega^{-1}x_0\,dz$$
$$= \frac{1}{2\pi i}\int_{\xi+i\mathbb{R}} e^{zt}z^{-1}(A_\omega^{-1}x_0 + F_0(z))\,dz.$$

By Cauchy's theorem, we may shift the path of integration to $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4 \cup \Gamma_5$, where

$$\Gamma_1 = \{z = i\eta :\ \eta \leq -r\};$$
$$\Gamma_2 = \{z = \xi + i\eta :\ -\delta \leq \xi \leq 0,\ \eta = -r\};$$
$$\Gamma_3 = \{z = \xi + i\eta :\ \xi = -\delta,\ -r \leq \eta \leq r\};$$
$$\Gamma_4 = \{z = \xi + i\eta :\ -\delta \leq \xi \leq 0,\ \eta = r\};$$
$$\Gamma_5 = \{z = i\eta :\ \eta \geq r\}.$$

Here, $-\omega < -\delta < 0$ is arbitrary and $r > 0$ is to be chosen later. We are going to estimate the integrals over Γ_i, $i = 1, \ldots, 5$, separately.

We start with the integral over Γ_1. Since $\omega_0(\mathbf{T}_\omega) < 0$, there is a constant N such that

$$\left(\int_0^\infty |\langle x^*, T_\omega(t)x\rangle|^2\,dt\right)^{\frac{1}{2}} \leq N\|x\|\,\|x^*\| \qquad \forall x \in X,\ x^* \in X^*.$$

By the Plancherel theorem,

$$\left(\frac{1}{2\pi}\int_{-\infty}^\infty |\langle x^*, R(i\eta,A_\omega)x\rangle|^2\,d\eta\right)^{\frac{1}{2}} \leq N\|x\|\,\|x^*\| \qquad \forall x \in X,\ x^* \in X^*.$$

Therefore, by the Lemma and Hölder's inequality, for all $x^* \in X^*$ we have

$$\begin{aligned}\left|\int_{\Gamma_1} e^{zt}z^{-1}\langle x^*, A_\omega^{-1}x_0 + F_0(z)\rangle\, dz\right| &= \left|\int_{\Gamma_1} e^{zt}z^{-1}\langle x^*, A_\omega^{-1}x_0 + R(z,A_\omega)x_0\rangle\, dz\right| \\ &\le \frac{3\pi}{2rt}\|A_\omega^{-1}x_0\|\,\|x^*\| + N\sqrt{2\pi}\,\|x_0\|\,\|x^*\|\cdot\left(\int_r^\infty \frac{1}{\eta^2}\,d\eta\right)^{\frac12} \\ &= \frac{3\pi}{2rt}\|A_\omega^{-1}x_0\|\,\|x^*\| + \frac{N\sqrt{2\pi}}{r^{\frac12}}\|x_0\|\,\|x^*\|.\end{aligned}$$

The same estimate holds for the integral over Γ_5. Also, we have

$$\left\|\int_{\Gamma_2} e^{zt}z^{-1}(A_\omega^{-1}x_0 + F_0(z))\, dz\right\| \le \delta\, r^{-1}(\|A_\omega^{-1}\| + K)\|x_0\|,$$

and the same estimate holds for the integral over Γ_4. Finally, for the integral over Γ_3 we have

$$\begin{aligned}\left\|\int_{\Gamma_3} e^{zt}z^{-1}(A_\omega^{-1}x_0 + F_0(z))\, dz\right\| &\le e^{-\delta t}(\|A_\omega^{-1}\| + K)\|x_0\|\cdot\int_{-r}^{r}\frac{1}{|-\delta + i\eta|}\,d\eta \\ &\le 2e^{-\delta t}\ln(1+\frac{r}{\delta})(\|A_\omega^{-1}\| + K)\|x_0\|.\end{aligned}$$

Putting everything together, we find that for all $x^* \in X^*$,

$$|\langle x^*, T_\omega(t)A_\omega^{-1}x_0\rangle| \le C\left(\frac{1}{rt} + \frac{1}{r^{\frac12}} + \frac{\delta}{r} + e^{-\delta t}\ln(1+\frac{r}{\delta})\right)\|x_0\|\,\|x^*\|,$$

where C is a constant depending on N, K, and $\|A_\omega^{-1}\|$. Letting $\delta \to \omega$ and taking the supremum over all functionals x^* of norm ≤ 1, we find that

$$\|T_\omega(t)A_\omega^{-1}x_0\| \le C\left(\frac{1}{rt} + \frac{1}{r^{\frac12}} + \frac{\omega}{r} + e^{-\omega t}\ln(1+\frac{r}{\omega})\right)\|x_0\|.$$

So far, $r > 0$ was arbitrary. For fixed $t > 0$ we now take $r = e^{2\omega t}$. It follows that

$$\|T_\omega(t)A_\omega^{-1}x_0\| \le C'\big(t^{-1}e^{-2\omega t} + e^{-\omega t} + e^{-2\omega t} + e^{-\omega t}(1+2\omega t)\big)\|x_0\|,$$

where C' is a constant depending only on N, K, $\|A_\omega^{-1}\|$, and ω. Since $T_\omega(t)A_\omega^{-1}x_0$ is bounded for $0 \le t \le 1$, it follows that $\|T_\omega(t)A_\omega^{-1}x_0\| \le M(1+t)e^{-\omega t}$ for some M and all $t \ge 0$. By scaling back to $\mathbf{T}$, we obtain the desired result. ∎

In the theorem no assumption whatsoever is made about the location of the spectrum, nor on the growth of the semigroup. One should compare this to the following result of Arendt and Batty [1]: If $\mathbf{T}$ is a C_0-semigroup with generator A, and $x_0 \in X$ is such that $z \mapsto R(z,A)x_0$ admits an analytic continuation to a neighbourhood of $\{\mathrm{Re}\, z \ge 0\}$ and $t \mapsto T(t)x_0$ is bounded, then $\lim_{t\to\infty} T(t)A^{-1}x_0 = 0$. The proof is based on a Tauberian result for Laplace transforms.

As an immediate corollary of the theorem, we recover the result of Weis and Wrobel [12]:

Corollary 3. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X. If the resolvent of the generator of $\mathbf{T}$ exists and is uniformly bounded in the right half plane, then $\omega_1(\mathbf{T}) < 0$.*

Proof. The uniform boundedness of the resolvent implies the existence of a $\delta > 0$ such that the resolvent exists and is uniformly bounded in $\{\operatorname{Re} z > -\delta\}$. In particular, for each $x \in X$, the map $z \mapsto R(z, A)x$ is uniformly bounded in $\{\operatorname{Re} z > -\delta\}$. By Theorem 2, the exponential type of $t \mapsto T(t)A^{-1}x$ is at most $-\delta$. Therefore, by the uniform boundedness theorem, $\omega_1(\mathbf{T}) \leq -\delta$. ∎

Actually, in [12] it is proved that $\omega_{\alpha+1}(\mathbf{T}) \leq s_\alpha(A)$ for all $\alpha \geq 0$, where $\omega_\alpha(\mathbf{T})$ denotes the growth bound of elements in $D((A_\omega)^\alpha)$ (which is independent of $\omega > \omega_0(\mathbf{T})$) and $s_\alpha(A)$ denotes the abscissa of polynomial growth of order α of the resolvent. By [NSW, Lemma 3.3], for all $\alpha \geq 0$ and $x \in X$ we have

$$T(t)(-A_\omega)^{-\alpha-1}x = \frac{1}{2\pi i}\int_{\operatorname{Re} z=-c} e^{zt}(-z)^{-\alpha}R(z, A_\omega)A_\omega^{-1}x\, dz,$$

where $0 < c < \omega - \omega_0(\mathbf{T})$ is arbitrary. Using this identity, it is easy to modify the proof of Theorem 2 to obtain the corresponding individual stability result for all $\alpha \geq 0$. By the uniform boundedness theorem, the inequality $\omega_{\alpha+1}(\mathbf{T}) \leq s_\alpha(A)$ then follows from this. In Hilbert space, the stronger inequalities $\omega_\alpha(\mathbf{T}) \leq s_\alpha(A)$ hold; see Weiss [13] (for integers α) and Weis and Wrobel [12].

We now turn to an application which says that, roughly speaking, if the improper convergence on the imaginary axis of the Laplace transform of the orbit of x_0 is uniform with respect to $i\lambda \in i\mathbb{R}$, we can estimate the growth of the orbit of $R(\lambda, A)x_0$.

Theorem 4. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X, with generator A. If, for some $x_0 \in X$,*

$$\sup_{\lambda\in\mathbb{R}}\sup_{s>0}\left\|\int_0^s e^{-i\lambda t}T(t)x_0\, dt\right\| < \infty,$$

then for each $\lambda \in \varrho(A)$ there is a constant M such that

$$\|T(t)R(\lambda, A)x_0\| \leq M(1+t) \qquad \forall t \geq 0.$$

Proof. We shall prove that $R(z, A)x_0$ admits a bounded analytic continuation to $\{\operatorname{Re} z > 0\}$. The proof is modelled after [7], Thm. 1.3.

Choose a constant K such that

$$\sup_{\lambda\in\mathbb{R}}\sup_{s>0}\left\|\int_0^s e^{-i\lambda t}T(t)x_0\, dt\right\| \leq K\|x_0\|.$$

Consider the X-valued entire functions $F_s(z) = \int_0^s e^{-zt}T(t)x_0\, dt$. By assumption, each F_s is bounded on the imaginary axis, say with bound K. Also, a simple estimate shows that each F_s is bounded on vertical lines. Choose constants N and $\omega \geq 0$ such that $\|T(t)\| \leq Ne^{\omega t}$ for all $t \geq 0$, and let $\xi = \omega + 1$. Then,

$$\|F_s(\xi + i\eta)\| \leq \int_0^s e^{-\xi t}\|T(t)x_0\|\, dt \leq \int_0^s Ne^{-t}\|x_0\|\, dt \leq N\|x_0\|.$$

Therefore, by the Phragmen-Lindelöf theorem, each F_s is uniformly bounded in the strip $S_\xi := \{0 \le \operatorname{Re} z \le \xi\}$, with bound $\max\{K, N\}\|x_0\|$. Moreover, for $\operatorname{Re} z > \omega$ we have

$$\lim_{s\to\infty} F_s(z) = R(z, A)x_0.$$

By Vitali's theorem [4], Thm. 3.14.1, the limit $\lim_{s\to\infty} F_s(z)$ exists for all $z \in S_\xi$, the convergence being uniformly on compacta. The limit function F is analytic in the interior of S_ξ and coincides with $R(z, A)x_0$ for $\omega < \operatorname{Re} z \le \xi$. Moreover, F is uniformly bounded in S_ξ, with bound $\max\{K, N\}\|x_0\|$. This proves that $R(z, A)x_0$ admits a bounded analytic continuation to the interior of S_ξ. By the Hille-Yosida Theorem, $R(z, A)x_0$ is also uniformly bounded in $\{\operatorname{Re} z \ge \xi\}$. Therefore, the analytic continuation F is uniformly bounded in $\{\operatorname{Re} z > 0\}$. ∎

We recall from [10] (see also [6, Theorem A-IV.1.4]) that $\omega_1(\mathbf{T})$ coincides with the abscissa of simple convergence of the Laplace transform of $\mathbf{T}$. The following result is a uniform version of this. The proof, which is based on the same observation as Corollary 3, is left to the reader.

Corollary 5. *Let $\mathbf{T}$ be a C_0-semigroup on a Banach space X. If*

$$\sup_{\lambda\in\mathbb{R}} \sup_{s>0} \left\| \int_0^s e^{-i\lambda t} T(t)x\, dt \right\| < \infty \qquad \forall x \in X,$$

then $\omega_1(\mathbf{T}) < 0$.

It is interesting to compare this result with [7], Cor. 2.3. There, it is shown that $\omega_0(\mathbf{T}) < 0$ if and only if

$$\sup_{s>0} \left\| \int_0^s T(t)g(t)\, dt \right\| < \infty \qquad \forall g \in AP(\mathbb{R}_+, X),$$

where $AP(\mathbb{R}_+, X)$ denotes the space of X-valued almost periodic functions on $\mathbb{R}_+$. Thus, Corollary 5 can be interpreted as saying what happens if instead of considering all of $AP(\mathbb{R}_+, X)$, one only considers the dense subspace spanned by functions $t \mapsto e^{-i\lambda t} \otimes x$.

As is well-known, for positive C_0-semigroups on Banach lattices the spectral bound $s(A) := \sup\{\operatorname{Re} z :\ z \in \sigma(A)\}$ and the growth bound $\omega_1(\mathbf{T})$ coincide [6, Theorem C-IV.1.3]. The following theorem generalizes this to individual elements. It says that, for positive x_0, the growth bound of $t \mapsto T(t)R(\lambda, A)x_0$ can be estimated by the local spectral bound of A at x_0.

Theorem 6. *Let $\mathbf{T}$ be a positive C_0-semigroup on a Banach lattice X, with generator A. If, for some $0 \le x_0 \in X$, the map $z \mapsto R(z, A)x_0$ has an analytic continuation to $\{\operatorname{Re} z > 0\}$, then for each $\lambda \in \varrho(A)$ the map $t \mapsto T(t)R(\lambda, A)x_0$ has exponential type less than or equal to 0.*

Proof. Let $\omega_s(A)$ denote the abscissa of simple convergence of the Laplace transform of $t \mapsto T(t)x_0$. By the vector-valued Pringsheim-Landau theorem [3], Thm. 2.1, $\omega_s(A)$ is a singular point for the analytic function

$$z \mapsto \lim_{s\to\infty} \int_0^s e^{-zt} T(t)x_0\, dt.$$

Therefore, the fact that $R(z,A)x_0$ has an analytic extension $F(z)$ to $\{\operatorname{Re} z > 0\}$ implies that $\omega_s(A) \leq 0$. Then it is evident that for all z_0, $z_1 \in \mathbb{C}$ with $\operatorname{Re} z_1 \geq \operatorname{Re} z_0 > 0$,

$$|F(z_1)| = \left| \lim_{s\to\infty} \int_0^s e^{-z_1 t} T(t)x_0 \, dt \right|$$
$$\leq \lim_{s\to\infty} \int_0^s e^{-\operatorname{Re} z_1 t} T(t)x_0 \, dt \leq \lim_{s\to\infty} \int_0^s e^{-\operatorname{Re} z_0 t} T(t)x_0 \, dt = F(\operatorname{Re} z_0).$$

This implies that $\|F(z)\| \leq \|F(\operatorname{Re} z)\|$, so that $F(z)$ is uniformly bounded in each half plane $\{\operatorname{Re} z > \epsilon\}$. It then follows from Theorem 2 that $t \mapsto T(t)R(\lambda, A)x_0$ has exponential type $\leq \epsilon$ for each $\epsilon > 0$. ∎

The proof shows that actually it is enough to have an analytic continuation of $R(z,A)x_0$ to a neighbourhood of $(0,\infty)$.

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