Semigroup Forum Vol. 52 (1996) 225–239

RESEARCH ARTICLE

The Spectral Mapping Theorem for Evolution Semigroups on Spaces of Vector–valued Functions*

Frank Räbiger and Roland Schnaubelt

Communicated by Rainer Nagel

Introduction

The solutions of a non–autonomous linear Cauchy problem on a Banach space X are given, under appropriate conditions, by a family $(U(t,s))_{t\geq s}$ contained in the space $\mathcal{L}(X)$ of bounded linear operators on X, for which the following properties hold:

(E1) the mapping $(t,s) \mapsto U(t,s)$ from $D := \{(t,s) \in \mathbb{R}^2 \mid t \geq s\}$ into $\mathcal{L}(X)$ is strongly continuous,

(E2) $U(s,s) = Id_X$, $U(t,r)U(r,s) = U(t,s)$ for all $t \geq r \geq s$,

(E3) there are constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{\omega(t-s)}$ for all $(t,s) \in D$

(see e.g. [3], [5], [17], [26]). In the following a family $(U(t,s))_{(t,s)\in D}$ in $\mathcal{L}(X)$ satisfying (E1)–(E3) is called an *evolution family.* It has been noticed by several authors (see [7], [8], [9], [13], [18], [20], [21], [22], [23] and the references therein) that asymptotic properties of the evolution family $(U(t,s))_{(t,s)\in D}$ are strongly related to the asymptotic behaviour of an associated *evolution semigroup* $(T_E(t))_{t\geq 0}$ of operators on a Banach space $E(X)$ of X–valued functions (see Section 1). For a large class of these function spaces this evolution semigroup is strongly continuous and hence has a generator G_E. It has been shown by R. Rau [20, Prop. 1.7] and Y. Latushkin and S. Montgomery–Smith [7, Thm. 3.1], [8, Thm. 4] that on the function spaces $C_0(\mathbb{R},X)$ and $L^p(\mathbb{R},X)$, $1 \leq p < \infty$, these semigroups always satisfy the spectral mapping theorem

$$\text{(SMT)} \qquad \sigma(T_E(t)) \setminus \{0\} = \exp(t\sigma(G_E)), \qquad t \geq 0.$$

As a consequence the asymptotic behaviour of the evolution family is determined by the spectrum $\sigma(G_E)$ of the generator of the corresponding evolution semigroup.

In this paper we show that for a large class of X–valued function spaces $E(X)$ including $C_0(\mathbb{R},X)$ and $L^p(\mathbb{R},X)$, $1 \leq p < \infty$, each evolution semigroup satisfies (SMT). Moreover, the spectra of the semigroup operators and the generator are independent of the space. As a consequence, we obtain a characterization of

* This paper is part of a research project supported by the Deutsche Forschungsgemeinschaft DFG.

hyperbolic evolution families through the hyperbolicity of the corresponding evolution semigroups on spaces $E(X)$. This improves previous results of R. Rau [22, Thm. 6], Y. Latushkin, S. Montgomery–Smith, T.W. Randolph [7, Thm. 3.4], [9, Thm. 3.3] and the authors [18, Thm. 1.5].

Throughout the following $(U(t,s))_{(t,s)\in D}$ always denotes an evolution family on the Banach space X. We use the terminology from [12] and recall some notions from spectral theory. Let $(A, D(A))$ be an operator on the Banach space Y. The *resolvent set* $\rho(A)$ is the set of all $\lambda \in \mathbb{C}$ such that $R(\lambda, A) := (\lambda - A)^{-1}$ exists in $\mathcal{L}(Y)$. Its complement $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is called the *spectrum* of A. A complex number $\lambda \in \mathbb{C}$ belongs to the *residual spectrum* $R\sigma(A)$ if $(\lambda - A)D(A)$ is not dense in Y and belongs to the *approximate point spectrum* $A\sigma(A)$ if there is a sequence (x_n) in $D(A)$ such that $\|x_n\| \geq 1$ for all $n \in \mathbb{N}$ and $\lim_n(\lambda - A)x_n = 0$.

1. Evolution Semigroups

For the investigation of an evolution family $(U(t,s))_{(t,s)\in D}$ it is useful to associate a semigroup of operators on a space of (equivalence classes of) Banach space valued functions (see e.g. [4], [6], [7], [11], [13], [15], [16], [18], [20], [23] and the references therein). Here we consider this construction in a more general framework. Denote by $\mathcal{B}$ the Borel algebra and by λ the Lebesgue measure on $\mathbb{R}$. A vector space E of real-valued Borel-measurable functions on $\mathbb{R}$ (modulo λ-nullfunctions) is called a *Banach function space* (over $(\mathbb{R}, \mathcal{B}, \lambda)$) if

(BFS1) E is an ideal in the space $M(\mathbb{R}, \mathcal{B}, \lambda)$ of measurable functions modulo λ-nullfunctions, i.e., if $\varphi \in E$, $\psi \in M(\mathbb{R}, \mathcal{B}, \lambda)$ and $|\varphi(\cdot)| \leq |\psi(\cdot)|$ λ-a.e., then $\psi \in E$,

(BFS2) the characteristic functions χ_A belong to E for all $A \in \mathcal{B}$ of finite measure,

(BFS3) each $\varphi \in E$ is locally integrable, i.e. $\int_A |\varphi|\, d\lambda < \infty$ for all $A \in \mathcal{B}$ of finite measure,

(BFS4) E is a Banach lattice with respect to a norm $\|\cdot\|_E$, i.e. $(E, \|\cdot\|_E)$ is complete and $\|\psi\|_E \leq \|\varphi\|_E$ for all $\varphi, \psi \in E$ such that $|\psi(\cdot)| \leq |\varphi(\cdot)|$ λ-a.e..

Let E be a Banach function space and X a Banach space. We set

$$E(X) := \{f\colon \mathbb{R} \to X \mid f \text{ is strongly measurable and } \|f(\cdot)\|_X \in E\}$$

(modulo λ-nullfunctions). If it causes no confusion we identify elements of E and $E(X)$ with functions on $\mathbb{R}$. Then $E(X)$ (with the obvious linear operations) is a linear space, and for the norm

$$\|f\|_{E(X)} := \|\, \|f(\cdot)\|_X \,\|_E\,, \qquad f \in E(X),$$

$E(X)$ is a Banach space. If we further assume that E is *translation invariant*, i.e., $\varphi(\cdot - t) \in E$ for $\varphi \in E$ and $t \in \mathbb{R}$, then

$$\text{(E)} \qquad T_E(t)f(\cdot) := U(\cdot, \cdot - t)f(\cdot - t)\,, \qquad t \geq 0,\ f \in E(X),$$

defines a semigroup $(T_E(t))_{t\geq 0}$ of bounded linear operators on $E(X)$. (Notice that the translation is a positive operator on the Banach lattice E and is therefore bounded (see [24, II.5.3]).)

In the same way (E) yields a semigroup $(T_\infty(t))_{t\geq 0}$ of bounded linear operators on the space $C_0(\mathbb{R}, X)$ of continuous functions vanishing at $\pm\infty$ (endowed with the sup–norm $\|\cdot\|_\infty$).

Definition 1.1. The semigroup $(T_E(t))_{t\geq 0}$, resp. $(T_\infty(t))_{t\geq 0}$ is called the *evolution semigroup* on $E(X)$, resp. $C_0(\mathbb{R}, X)$ associated with $(U(t,s))_{(t,s)\in D}$.

It can be easily verified that the evolution semigroup $(T_\infty(t))_{t\geq 0}$ is strongly continuous. By $(G_\infty, D(G_\infty))$ we denote its generator. The evolution semigroup $(T_E(t))_{t\geq 0}$ is not always strongly continuous (see e.g. [12, A-I.3.4]). We obtain strong continuity for a large class of Banach function spaces which we introduce in the following. A Banach function space E is called *admissible*, if

(A1) E has *order continuous norm*, i.e., if $(\varphi_\alpha)_{\alpha\in A}$ is a decreasing net of positive functions in E with $\inf_{\alpha\in A}\varphi_\alpha = 0$, then $\lim_\alpha \|\varphi_\alpha\| = 0$,

(A2) E is translation–invariant and the group $(S(t))_{t\in\mathbb{R}}$ of translations on E, given by $S(t)\varphi = \varphi(\cdot - t)$, $\varphi \in E$, $t \in \mathbb{R}$, is strongly continuous and uniformly bounded.

Besides the spaces $L^p(\mathbb{R})$, $1 \leq p < \infty$, many other function spaces occuring in interpolation theory, e.g. the Lorentz spaces $L_{p,q}(\mathbb{R})$, $1 < p < \infty$, $1 \leq q < \infty$, (see [2, Thm. 3 and p. 284], [27, 1.18.6, 1.19.3]) and, more general, the class of rearrangement invariant function spaces over $(\mathbb{R}, \mathcal{B}, \lambda)$ (see [10, 2.a]) are admissible. Our next proposition is a special case of [19, Prop. 2.3].

Proposition 1.2. *Let E be an admissible Banach function space. Then every evolution semigroup on $E(X)$ is strongly continuous.* ∎

In case $(T_E(t))_{t\geq 0}$ is strongly continuous we denote by $(G_E, D(G_E))$ its generator.

The aim of this paper is to show that for an admissible Banach function space E every evolution semigroup $(T_E(t))_{t\geq 0}$ satisfies the spectral mapping theorem

$$\sigma(T_E(t)) \setminus \{0\} = \exp(t\sigma(G_E)), \qquad t \geq 0, \tag{1.1}$$

and the spectra of the semigroup operators and the generator coincide with the spectra of the corresponding operators on $C_0(\mathbb{R}, X)$. In particular, the spectra are independent of the space E. For the spaces $L^p(\mathbb{R})$, $1 \leq p < \infty$, this has been shown first in [7] using completely different methods.

It is known that for any strongly continuous semigroups certain 'spectral inclusions' hold. To be more precise, let $(T(t))_{t\geq 0}$ be a strongly continuous semigroup on the Banach space Y with generator A. Then

$$\exp(t\sigma(A)) \subseteq \sigma(T(t)), \qquad t \geq 0, \tag{1.2}$$

and the residual spectra of the semigroup and the generator satisfy

$$\exp(tR\sigma(A)) = R\sigma(T(t)) \setminus \{0\}, \qquad t \geq 0 \tag{1.3}$$

(see [12, A-III.6.2-6.3]). Notice that $\sigma(T(t)) = R\sigma(T(t)) \cup A\sigma(T(t))$ (see [12, A-III.2.1]). Hence, by (1.2) and (1.3) the spectral mapping theorem (1.1) holds for $(T(t))_{t\geq 0}$ if (and only if)

$$A\sigma(T(t)) \setminus \{0\} \subseteq \exp(t\sigma(A)), \qquad t \geq 0. \tag{1.4}$$

If we consider an evolution semigroup $(T(t))_{t\geq 0}$ induced by an evolution family $(U(t,s))_{(t,s)\in D}$, then (1.4) and hence the spectral mapping theorem is a consequence of the following weaker assertion:

$$\text{if } 1 \in A\sigma(T(t_0)) \text{ for some } t_0 > 0, \text{ then } 0 \in \sigma(A). \tag{1.5}$$

We show how (1.4) follows from (1.5). Let $0 \neq \mu \in A\sigma(T(t_0))$ for some $t_0 > 0$. Choose $\alpha \in \mathbb{C}$ such that $\mu = e^{\alpha t_0}$. Then $1 \in A\sigma(e^{-\alpha t_0}T(t_0))$. Moreover, $(e^{-\alpha t}T(t))_{t\geq 0}$ is the evolution semigroup associated with the evolution family $(e^{-\alpha(t-s)}U(t,s))_{(t,s)\in D}$. Now (1.5) applied to $(e^{-\alpha t}T(t))_{t\geq 0}$ yields $0 \in \sigma(A) - \alpha$ and hence $\alpha \in \sigma(A)$. Thus $\mu = e^{\alpha t_0} \in \exp(t_0\sigma(A))$.

In Section 2 we give a direct proof of (1.5) for evolution semigroups on $C_0(\mathbb{R}, X)$ (Theorem 2.3). In the subsequent sections we show $\sigma(G_\infty) \subseteq \sigma(G_E)$ (Proposition 3.8) and $\sigma(T_E(t)) \subseteq \sigma(T_\infty(t))$ (Proposition 4.3) for every admissible Banach function space E. Together with the spectral inclusion (1.2) this yields the spectral mapping theorem (1.1).

2. The Spectral Mapping Theorem on $C_0(\mathbb{R}, X)$

Let $(U(t,s))_{(t,s)\in D}$ be an evolution family on the Banach space X. We consider the evolution semigroup $(T_\infty(t))_{t\geq 0}$ with generator $(G_\infty, D(G_\infty))$ on the space $C_0(\mathbb{R}, X)$. By $C^1(\mathbb{R})$ we denote the *space of continuously differentiable complex-valued functions* and set $U(t,s) := 0$ for $t < s$. We start with the following lemma.

Lemma 2.1. *Let $(U(t,s))_{(t,s)\in D}$ be an evolution family on the Banach space X. Let $x \in X$, $r \in \mathbb{R}$, I a compact interval in $\mathbb{R}$, and $\alpha \in C^1(\mathbb{R})$ such that* supp $\alpha :=$ $\{s \in \mathbb{R} \mid \alpha(s) \neq 0\} \subseteq I$ *and $r \leq s$ for each $s \in I$. Then $g := \alpha(\cdot)U(\cdot, r)x \in D(G_\infty)$ and $G_\infty g = -\alpha'(\cdot)U(\cdot, r)x$.*

Proof. Clearly, $g \in C_0(\mathbb{R}, X)$. Let $s \in \mathbb{R}$ and $t > 0$. If $s - t \leq r$, then

$$T_\infty(t)g\,(s) = U(s, s-t)g(s-t) = 0 = \alpha(s-t)\,U(s,r)x\,.$$

For $s - t > r$ we have

$$T_\infty(t)\,g(s) = U(s, s-t)\,\alpha(s-t)\,U(s-t, r)x = \alpha(s-t)\,U(s,r)x.$$

Thus

$$\frac{1}{t}\left(T_\infty(t)g - g\right)(s) = \frac{1}{t}\left(\alpha(s-t) - \alpha(s)\right)U(s,r)x\,.$$

Hence, $\lim_{t\downarrow 0}\frac{1}{t}\left(T_\infty(t)g - g\right)$ exists in $C_0(\mathbb{R}, X)$ and $G_\infty g = \lim_{t\downarrow 0}\frac{1}{t}\left(T_\infty(t)g - g\right) = -\alpha'(\cdot)U(\cdot, r)x$. ∎

Now we show that each evolution semigroup on $C_0(\mathbb{R}, X)$ satisfies (1.5).

Proposition 2.2. *If $1 \in A\sigma(T_\infty(t_0))$ for some $t_0 > 0$, then $0 \in \sigma(G_\infty)$.*

Proof. Let $(U(t,s))_{(t,s)\in D}$ be the evolution family on X corresponding to the evolution semigroup $(T_\infty(t))_{t\geq 0}$. For each $n \in \mathbb{N}$ there exists $f_n \in C_0(\mathbb{R}, X)$ such that $\|f_n\| = 1$ and $\|f_n - T_\infty(kt_0)f_n\| < \frac{1}{2n}$ for all $0 \leq k \leq 2n$, $k \in \mathbb{N}$. Then

$$\frac{1}{2} < \sup_{s\in\mathbb{R}} \|U(s, s-kt_0)f_n(s-kt_0)\| < \frac{3}{2} \tag{2.1}$$

for $0 \leq k \leq 2n$. For each $n \in \mathbb{N}$ choose $s_n \in \mathbb{R}$ such that

$$\|U(s_n, s_n - nt_0)x_n\| \geq \frac{1}{2} \qquad \text{where } x_n := f_n(s_n - nt_0).$$

Let $I_n := [s_n - nt_0, s_n + nt_0]$ and $\tilde{g}_n := \chi_{I_n} U(\cdot, s_n - nt_0)x_n$. Thus $\|\tilde{g}_n(s_n)\| \geq \frac{1}{2}$. On the other hand, each $s \in I_n$ can be written as $s = s_n + (k + \sigma - n)t_0$ where $0 \leq k \leq 2n$ and $\sigma \in [0,1[$. Then, by (E3) and (2.1),

$$\begin{aligned}
&\|\tilde{g}_n(s)\| \\
&\quad = \|U(s_n + (k+\sigma-n)t_0, s_n + (k-n)t_0)\, U(s_n + (k-n)t_0, s_n - nt_0)\, x_n\| \\
&\quad \leq Me^{|\omega|t_0} \|U(s_n + (k-n)t_0, s_n - nt_0)x_n\| \\
&\quad = Me^{|\omega|t_0} \|U(s_n + (k-n)t_0, s_n + (k-n)t_0 - kt_0)\, f_n(s_n + (k-n)t_0 - kt_0)\| \\
&\quad \leq \frac{3}{2} Me^{|\omega|t_0}.
\end{aligned}$$

Thus there is a positive constant c such that

$$\frac{1}{2} \leq \sup_{s\in\mathbb{R}} \|\tilde{g}_n(s)\| \leq c \qquad \text{for all } n \in \mathbb{N}.$$

Now choose $\alpha_n \in C^1(\mathbb{R})$ such that $\alpha_n(s_n) = 1$, $0 \leq \alpha(s) \leq 1$, $\operatorname{supp} \alpha_n \subseteq I_n$ and $\|\alpha_n'\|_\infty \leq \frac{2}{nt_0}$. Define $g_n := \alpha_n \tilde{g}_n = \alpha_n(\cdot)U(\cdot, s_n - nt_0)x_n \in C_0(\mathbb{R}, X)$. Then $\frac{1}{2} \leq \|g_n\|_\infty \leq c$ for all $n \in \mathbb{N}$. Moreover, Lemma 2.1 implies $g_n \in D(G_\infty)$ and $G_\infty g_n = -\alpha_n' \tilde{g}_n$. In particular, we obtain $\|G_\infty g_n\|_\infty \leq \frac{2c}{nt_0}$, and hence 0 is an approximate eigenvalue of G_∞. ∎

As explained in Section 1 this proposition implies the spectral mapping theorem for evolution semigroups on $C_0(\mathbb{R}, X)$.

Theorem 2.3. *Let $(T_\infty(t))_{t\geq 0}$ be an evolution semigroup on $C_0(\mathbb{R}, X)$ with generator $(G_\infty, D(G_\infty))$. Then $\sigma(T_\infty(t)) \setminus \{0\} = \exp(t\sigma(G_\infty))$ for all $t \geq 0$.* ∎

3. The Spectrum of the Generator

In the following E is an admissible Banach function space and X a Banach space. Then $E_\infty(X) := E(X) \cap C_0(\mathbb{R}, X)$ endowed with the norm $\|f\| := \max(\|f\|_{E(X)}, \|f\|_\infty)$ is a Banach space. For Banach spaces Y and Z we write $Z \hookrightarrow Y$ if $Z \subseteq Y$ and the identity map from Z into Y is continuous. If $(A, D(A))$ is a linear operator on Y and $Z \subseteq Y$, we denote by $(A_|, D(A_|))$ the part of A in Z, i.e.

$$D(A_|) := \{x \in D(A) \cap Z \mid Ax \in Z\} \quad \text{and } A_| x := Ax.$$

Obviously, $E_\infty(X) \hookrightarrow E(X)$ and $E_\infty(X) \hookrightarrow C_0(\mathbb{R}, X)$. Every evolution family $(U(t,s))_{(t,s)\in D}$ on X induces a strongly continuous semigroup $(T_{E,\infty}(t))_{t\geq 0}$ on $E_\infty(X)$ which is the restriction of $(T_E(t))_{t\geq 0}$ and $(T_\infty(t))_{t\geq 0}$, respectively. Then by [14, Lemma 2.6] the generator $(G_{E,\infty}, D(G_{E,\infty}))$ of $(T_{E,\infty}(t))_{t\geq 0}$ is the part of $(G_E, D(G_E))$ and $(G_\infty, D(G_\infty))$ in $E_\infty(X)$, respectively. In this section we show

$$\sigma(G_\infty) \subseteq \sigma(G_E) = \sigma(G_{E,\infty}) \tag{3.1}$$

for every admissible Banach function space E.

First we discuss some properties of admissible Banach function spaces E and the induced spaces $E(X)$ which will be used later on. The order continuity of the norm of E implies that the dual space E' is again a Banach function space (over $(\mathbb{R}, \mathcal{B}, \lambda)$) where the duality is given by

$$< \psi, \varphi > := \int_{\mathbb{R}} \varphi\psi \, d\lambda \qquad \text{for } \varphi \in E \text{ and } \psi \in E' \tag{3.2}$$

(see [10, p. 29]). Moreover, the translation invariance of E and (3.2) yield that E' is also translation invariant and the adjoint of the translation $S(t)$ is given by $S(t)'\psi = \psi(\cdot + t)$ for all $\psi \in E'$ and $t \in \mathbb{R}$. In particular, the group $(S(t)')_{t\in\mathbb{R}}$ of translations on E' is uniformly bounded. Our next lemma shows that certain exponential functions belong to E'. For $r \in \mathbb{R}$ let $\varepsilon_r : \mathbb{R} \to \mathbb{R}$ be defined by $\varepsilon_r(s) := e^{rs}$, $s \in \mathbb{R}$.

Lemma 3.1. *Let E be an admissible Banach function space and $r \in \mathbb{R}_+ \setminus \{0\}$. Then $\chi_{]-\infty,0]}\, \varepsilon_r \in E'$.*

Proof. Since $\chi_{]-1,0]} \in E'$ and the group of translations on E' is uniformly bounded, we obtain $\chi_{]-n,-n+1]} \in E'$ for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} \|\chi_{]-n,-n+1]}\| < \infty$. Then $\varphi := \sum_{n\in\mathbb{N}} e^{r(-n+1)} \chi_{]-n,-n+1]} \in E'$ and $0 \leq \chi_{]-\infty,0]}\, \varepsilon_r \leq \varphi$. Since E' is a Banach function space, the assertion follows. ∎

The following result is shown in [19, Lemma 2.2]. By $C_c(\mathbb{R}, X)$ we denote the space of X-valued continuous functions having compact support.

Lemma 3.2. *Let E be an admissible Banach function space. Then $C_c(\mathbb{R}, X)$ is dense in $E(X)$.* ∎

Consider now the evolution semigroup $(T_E(t))_{t\geq 0}$ on $E(X)$. The following smoothing property of the resolvent $R(\lambda, G_E) := (\lambda - G_E)^{-1}$, $\lambda \in \rho(G_E)$, of the generator G_E plays a central role (see also [18, Lemma 2.1], [15, Cor. 4.7]). Recall that for an operator $(A, D(A))$ the graph norm on $D(A)$ is defined as $\|x\|_A := \|x\| + \|Ax\|$, $x \in D(A)$.

Proposition 3.3. *Let E be an admissible Banach function space and $\lambda \in \rho(G_E)$. Then $R(\lambda, G_E) : E(X) \to E_\infty(X)$ is continuous. Moreover, $(D(G_E), \|\cdot\|_{G_E}) \hookrightarrow E_\infty(X) \hookrightarrow C_0(\mathbb{R}, X)$ and $D(G_E)$ is dense in $E_\infty(X)$ and $C_0(\mathbb{R}, X)$.*

Proof. Let $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|U(t,s)\| \leq Me^{\omega(t-s)}$ for $(t,s) \in D$. The group $(S(t))_{t\in\mathbb{R}}$ of translations on E is uniformly bounded, i.e., $N := \sup_{t\in\mathbb{R}} \|S(t)\| <$

∞. Then $\|T_E(t)\| \leq Me^{\omega t}\|S(t)\| \leq MNe^{\omega t}$ and $\|T_\infty(t)\| \leq Me^{\omega t}$ for $t \geq 0$, i.e., ω dominates the growth bounds of $(T_E(t))_{t\geq 0}$ and $(T_\infty(t))_{t\geq 0}$.

Let $\mu > \omega$. Then $\mu \in \rho(G_E) \cap \rho(G_\infty)$ and $R(\mu, G_E) = \int_0^\infty e^{-\mu t} T_E(t)\, dt$ and $R(\mu, G_\infty) = \int_0^\infty e^{-\mu t} T_\infty(t)\, dt$, where the integrals are defined strongly. If $f \in E_\infty(X)$, then $R(\mu, G_E)f = R(\mu, G_\infty)f$, and hence $R(\mu, G_E)E_\infty(X) \subseteq E_\infty(X)$. Moreover, for $f \in E_\infty(X)$ and $s \in \mathbb{R}$ we have

$$\begin{aligned} \|R(\mu, G_E)f(s)\| &= \left\| \int_0^\infty e^{-\mu t}\, U(s, s-t) f(s-t)\, dt \right\| \\ &\leq M \int_0^\infty e^{(\omega-\mu)t}\, \|f(s-t)\|\, dt \\ &= M \int_{-\infty}^s e^{(\mu-\omega)(t-s)}\, \|f(t)\|\, dt \\ &= M \int_{-\infty}^s S(-s)'(\chi_{]-\infty,0]}\, \varepsilon_{\mu-\omega})(t)\, \|f(t)\|\, dt \\ &\leq MN\, \|\chi_{]-\infty,0]}\, \varepsilon_{\mu-\omega}\|_{E'}\, \|f\|_{E(X)} \\ &\leq \tilde{M}\, \|f\|_{E(X)}, \end{aligned}$$

where the last inequalities follow from Lemma 3.1. Hence, $\|R(\mu, G_E)f\|_\infty \leq \tilde{M}\, \|f\|_{E(X)}$ for all $f \in E_\infty(X)$. Since $E_\infty(X)$ is dense in $E(X)$, the resolvent $R(\mu, G_E)$ maps $E(X)$ continuously into $C_0(\mathbb{R}, X) \cap E(X) = E_\infty(X)$.
If $\lambda \in \rho(G_E)$ is arbitrary, then the resolvent equation yields

$$R(\lambda, G_E) \left[R(\mu, G_E) - \frac{1}{\mu - \lambda} \right] = \frac{1}{\lambda - \mu}\, R(\mu, G_E)$$

where $R(\mu, G_E) - \frac{1}{\mu-\lambda}$ is invertible. Hence,

$$R(\lambda, G_E) = \frac{1}{\lambda - \mu}\, R(\mu, G_E) \left[R(\mu, G_E) - \frac{1}{\mu - \lambda} \right]^{-1} : E(X) \to E_\infty(X)$$

is continuous.

Since $D(G_E) = R(\lambda, G_E)E(X) \subseteq E_\infty(X)$ and $(G_{E,\infty}, D(G_{E,\infty}))$ is the part of $(G_E, D(G_E))$ in $E_\infty(X)$, the space $D(G_E)$ must be dense in $E_\infty(X)$ and $C_0(\mathbb{R}, X)$. Finally, the identity $Id\colon (D(G_E), \|\cdot\|_{G_E}) \to E_\infty(X)$ is the composition of the continuous operators $\lambda - G_E\colon (D(G_E), \|\cdot\|_{G_E}) \to E(X)$ and $R(\lambda, G_E)\colon E(X) \to E_\infty(X)$, $\lambda \in \rho(G_E)$. Hence, $(D(G_E), \|\cdot\|_{G_E}) \hookrightarrow E_\infty(X)$, whereas $E_\infty(X) \hookrightarrow C_0(\mathbb{R}, X)$ is obvious. ∎

The following lemma contains a spectral inclusion for certain parts of an operator (see also [1, Prop. 1.1], [17, Lemma 4.5.2]).

Lemma 3.4. *Let Y and Z be Banach spaces such that $Z \subseteq Y$. Moreover, let $(A, D(A))$ be an operator on Y and denote with $(A_|, D(A_|))$ its part in Z. If $D(A) \subseteq Z$ and $(A_|, D(A_|))$ is closed, then $\sigma(A_|) \subseteq \sigma(A)$.*

Proof. Let $\lambda \in \rho(A)$. Obviously, $\lambda - A_|$ is injective. On the other hand, if $z \in Z$ there exists $x \in D(A) \subseteq Z$ such that $(\lambda - A)x = z$. Therefore, $x \in D(A_|)$ and $(\lambda - A_|)x = z$. Thus $\lambda - A_|$ is a bijection. Since $(A_|, D(A_|))$ is closed, this implies $\lambda \in \rho(G_E)$. ∎

Now Lemma 3.4 and Proposition 3.3 yield the equality stated in (3.1).

Proposition 3.5. *Let E be an admissible Banach function space. Then* $\sigma(G_E) = \sigma(G_{E,\infty})$.

Proof. We mentioned already that $E_\infty(X) \subseteq E(X)$ and $(G_{E,\infty}, D(G_{E,\infty}))$ is the part of $(G_E, D(G_E))$ in $E_\infty(X)$. Thus Lemma 3.4 and Proposition 3.3 imply $\sigma(G_{E,\infty}) \subseteq \sigma(G_E)$.

Conversely, let $F := D(G_E)$ be endowed with the graph norm and let $(T_E^1(t))_{t\geq 0}$ be the restriction of $(T_E(t))_{t\geq 0}$ to F. Then $(G_F, D(G_F)) = (G_E, D(G_E^2))$ is the generator of $(T_E^1(t))_{t\geq 0}$. In addition, $(T_E^1(t))_{t\geq 0}$ and $(T_E(t))_{t\geq 0}$ are similar in the following sense: if $\lambda \in \rho(G_E)$, then $\lambda - G_E: F \to E(X)$ is an isomorphism and $T_E^1(t) = R(\lambda, G_E) T_E(t)(\lambda - G_E)$ for all $t \geq 0$ (cf. [12, A-I.3.5]). Therefore $(G_F, D(G_F))$ and $(G_E, D(G_E))$ are similar, and hence $\sigma(G_F) = \sigma(G_E)$. By Proposition 3.3 we have $F \hookrightarrow E_\infty(X)$. Furthermore, $(T_E^1(t))_{t\geq 0}$ is the restriction of $(T_{E,\infty}(t))_{t\geq 0}$ to F, and hence [14, Lemma 2.6] implies that $(G_F, D(G_F))$ is the part of $(G_{E,\infty}, D(G_{E,\infty}))$ in F. Thus for $\lambda \in \rho(G_E) \cap \rho(G_{E,\infty})$ we obtain

$$(\lambda - G_E)D(G_{E,\infty}) = (\lambda - G_{E,\infty})D(G_{E,\infty}) = E_\infty(X) \subseteq E(X) = (\lambda - G_E)D(G_E),$$

and hence $D(G_{E,\infty}) \subseteq F$. Now Lemma 3.4 yields $\sigma(G_F) \subseteq \sigma(G_{E,\infty})$ and the proof is finished. ∎

It is an immediate consequence of Proposition 3.5 that the residual spectrum $R\sigma(G_\infty)$ is contained in $\sigma(G_E)$.

Corollary 3.6. *Let E be an admissible Banach function space. Then* $R\sigma(G_\infty) \subseteq \sigma(G_E)$.

Proof. Let $\lambda \in \rho(G_E) = \rho(G_{E,\infty})$. Then

$$C_c(\mathbb{R}, X) \subseteq E_\infty(X) = (\lambda - G_{E,\infty})D(G_{E,\infty}) \subseteq (\lambda - G_\infty)D(G_\infty) \subseteq C_0(\mathbb{R}, X).$$

Thus $(\lambda - G_\infty)D(G_\infty)$ is dense in $C_0(\mathbb{R}, X)$ and hence $\lambda \notin R\sigma(G_\infty)$. ∎

For the proof of (3.1) it remains to show $A\sigma(G_\infty) \subseteq \sigma(G_E)$. The following lemma is the essential step to that inclusion.

Lemma 3.7. *Let E be an admissible Banach function space. If $0 \in A\sigma(G_\infty)$, then $0 \in \sigma(G_E)$.*

Proof. Let $0 \in A\sigma(G_\infty)$. The spectral inclusion $\exp(tA\sigma(G_\infty)) \subseteq A\sigma(T_\infty(t))$ for the approximate point spectrum (see [12, A-III.6.2]) yields $1 \in A\sigma(T_\infty(1))$. As in the proof of Proposition 2.2 we find intervals $I_n := [s_n - n, s_n + n] \subseteq \mathbb{R}$, elements $x_n \in X$ and a positive constant c such that $\tilde{g}_n := \chi_{I_n} U(\cdot, s_n - n)x_n: \mathbb{R} \to X$ satisfies

$$\|\tilde{g}_n(s_n)\| \geq \frac{1}{2} \quad \text{and} \quad \sup_{s\in\mathbb{R}} \|\tilde{g}_n(s)\| \leq c \quad \text{for all } n \in \mathbb{N}. \tag{3.3}$$

(Here we set $U(t, s) := 0$ for $t < s$.) Let $J_n := [s_n - n + 1, s_n + n - 1] \subseteq I_n$ and $\tilde{h}_n := \chi_{J_n} \tilde{g}_n$. Then

$$\|\tilde{h}_n(s_n)\| \geq \frac{1}{2} \quad \text{and} \quad \sup_{s\in\mathbb{R}} \|\tilde{h}_n(s)\| \leq c \quad \text{for all } n \in \mathbb{N}. \tag{3.4}$$

Clearly, $\tilde{g}_n, \tilde{h}_n \in E(X)$. The group $(S(t))_{t\in\mathbb{R}}$ of translations on E satisfies $N := \sup_{t\in\mathbb{R}} \|S(t)\| < \infty$. Then for each $n \in \mathbb{N}$ we have

$$\begin{aligned} \|\tilde{g}_n - \tilde{h}_n\|_{E(X)} &\leq \|\chi_{[s_n-n, s_n-n+1]}\, \tilde{g}_n\|_{E(X)} + \|\chi_{[s_n+n-1, s_n+n]}\, \tilde{g}_n\|_{E(X)} \\ &\leq c\Big(\|\chi_{[s_n-n,s_n-n+1]}\|_E + \|\chi_{[s_n+n-1,s_n+n]}\|_E \Big) \\ &\leq 2cN\, \|\chi_{[0,1]}\|_E . \end{aligned}$$

Thus there is a constant $\tilde{M}$ such that

$$\|\tilde{g}_n - \tilde{h}_n\|_{E(X)} \leq \tilde{M} \quad \text{for all } n \in \mathbb{N}. \tag{3.5}$$

Let $c_n := \|\tilde{h}_n\|_{E(X)}$, $n \in \mathbb{N}$. We have to distinguish two cases.

First case: $\displaystyle\liminf_{n\to\infty} \frac{c_n}{n} = 0$.

For each $n \in \mathbb{N}$ choose $\alpha_n \in C^1(\mathbb{R}, [0,1])$ such that $\alpha_n(s_n) = 1$, $\operatorname{supp} \alpha_n \subseteq I_n$ and $\|\alpha_n'\|_\infty \leq \frac{2}{n}$. Let $g_n := \alpha_n \tilde{g}_n = \alpha_n(\cdot) U(\cdot, s_n - n) x_n \colon \mathbb{R} \to X$. By Lemma 2.1 we have $g_n \in D(G_\infty)$ and $G_\infty g_n = -\alpha_n' \tilde{g}_n$. Obviously, $g_n, G_\infty g_n \in E_\infty(X)$. Since $(G_{E,\infty}, D(G_{E,\infty}))$ is the part of $(G_\infty, D(G_\infty))$ in $E_\infty(X)$, we obtain $g_n \in D(G_{E,\infty})$ and $G_{E,\infty} g_n = -\alpha_n' \tilde{g}_n$. Now (3.3) implies $\|g_n\|_{E_\infty(X)} \geq \|g_n\|_\infty \geq \|g_n(s_n)\|_X \geq \frac{1}{2}$. Moreover, (3.5) yields

$$\begin{aligned} \|G_{E,\infty}\, g_n\|_{E_\infty(X)} &\leq \|\alpha_n' \tilde{g}_n\|_\infty + \|\alpha_n' \tilde{g}_n\|_{E(X)} \\ &\leq \frac{2c}{n} + \|\alpha_n'(\tilde{g}_n - \tilde{h}_n)\|_{E(X)} + \|\alpha_n' \tilde{h}_n\|_{E(X)} \\ &\leq \frac{2c}{n} + \frac{2\tilde{M}}{n} + \frac{2c_n}{n}. \end{aligned}$$

Hence $\liminf_{n\to\infty} \|G_{E,\infty}\, g_n\|_{E(X)} = 0$. Thus $0 \in A\sigma(G_{E,\infty}) \subseteq \sigma(G_E)$ by Proposition 3.5.

Second case: $\displaystyle\liminf_{n\to\infty} \frac{c_n}{n} > 0$.

Then $\lim_{n\to\infty} c_n = \infty$. We may assume $c_n > 0$ for all $n \in \mathbb{N}$. Choose $\beta_n \in C^1(\mathbb{R})$ such that $0 \leq \beta_n \leq \frac{1}{c_n}$, $\beta_{n|J_n} \equiv \frac{1}{c_n}$, $\operatorname{supp} \beta_n \subseteq I_n$ and $\|\beta_n'\|_\infty \leq \frac{2}{c_n}$. Let $h_n := \beta_n \tilde{g}_n$, $n \in \mathbb{N}$. As in the first case (for the functions g_n) we obtain $h_n \in D(G_{E,\infty}) \subseteq D(G_E)$ and $G_{E,\infty} h_n = -\beta_n' \tilde{g}_n = G_E h_n$. Now $\|h_n\|_{E(X)} \geq \frac{1}{c_n} \|\tilde{h}_n\|_{E(X)} = 1$. In addition,

$$\begin{aligned} \|G_E h_n\|_{E(X)} &\leq \frac{2}{c_n} \left(\|\chi_{[s_n-n,s_n-n+1]}\, \tilde{g}_n\|_{E(X)} + \|\chi_{[s_n+n-1,s_n+n]}\, \tilde{g}_n\|_{E(X)} \right) \\ &\leq \frac{4c}{c_n} N\, \|\chi_{[0,1]}\|_E . \end{aligned}$$

Hence $\lim_{n\to\infty} \|G_E h_n\|_{E(X)} = 0$.

In both cases $0 \in A\sigma(G_E)$ and the proof is finished. ∎

From the previous results we can deduce (3.1) easily using the same argument as in the proof that (1.5) implies (1.4).

Proposition 3.8. *Let E be an admissible Banach function space. Then $\sigma(G_\infty) \subseteq \sigma(G_E) = \sigma(G_{E,\infty})$.* ∎

4. The Spectral Mapping Theorem on Banach Function Spaces

Throughout this section let E be an admissible Banach function space. Our aim is to show the spectral mapping theorem for evolution semigroups on $E(X)$ and that the spectra of the semigroup operators and the generator are independent of the space. In the following we exploit the strong symmetry properties of the spectrum of an evolution semigroup and its generator. This has been observed e.g. on $C_0(\mathbb{R}, X)$ by Rau [22, Prop. 2]. Precisely, if $\Gamma := \{\lambda \in \mathbb{C} \mid |\lambda| = 1\}$ is the unit circle, then

$$(4.1) \qquad \sigma(T_\infty(t)) = \Gamma\sigma(T_\infty(t)) \quad \text{for } t > 0 \text{ and } \quad \sigma(G_\infty) = \sigma(G_\infty) + i\mathbb{R}\,.$$

The analogous result with the same proof holds for evolution semigroups on $E(X)$.

Lemma 4.1. *Let E be an admissible Banach space. Then*

(a) $\sigma(T_E(t)) = \Gamma\sigma(T_E(t))$ *for* $t > 0$.

(b) $\sigma(G_E) = \sigma(G_E) + i\mathbb{R}$.

Let $C_b(\mathbb{R}, \mathcal{L}_s(X))$ be the space of all bounded, strongly continuous, operator–valued functions V on $\mathbb{R}$, i.e., $V: \mathbb{R} \to \mathcal{L}(X)$ is bounded and $V(\cdot)x: \mathbb{R} \to X$ is continuous for every $x \in X$. Each $V \in C_b(\mathbb{R}, \mathcal{L}_s(X))$ induces a bounded, linear 'multiplication' operator $\mathcal{V}$ on $C_0(\mathbb{R}, X)$ defined by $\mathcal{V}f := V(\cdot)f(\cdot)$ with $\|\mathcal{V}\| = \sup_{s\in\mathbb{R}} \|V(s)\|$. For $f \in E(X)$ we have $V(\cdot)f(\cdot) \in E(X)$ and

$$\|V(\cdot)f(\cdot)\|_{E(X)} = \|\, \|V(\cdot)f(\cdot)\|_X \,\|_E \le \sup_{s\in\mathbb{R}} \|V(s)\| \; \|f\|_{E(X)}\,.$$

Therefore $\mathcal{V}f := V(\cdot)f(\cdot)$ defines a bounded linear operator on $E(X)$. It can be easily seen that its norm equals $\sup_{s\in\mathbb{R}} \|V(s)\|$. Thus $C_b(\mathbb{R}, \mathcal{L}_s(X))$ endowed with the sup–norm can be identified with a closed linear subspace of $\mathcal{L}(C_0(\mathbb{R}, X))$ and $\mathcal{L}(E(X))$, respectively.

Our next lemma connects the spectra of $T_\infty(t)$ and $T_E(t)$. With $\mathbb{D} := \{\lambda \in \mathbb{C} \mid |\lambda| \le 1\}$ we denote the unit disc.

Lemma 4.2. *Let E be an admissible Banach function space and $t_0 > 0$.*

(a) *If* $\Gamma \subseteq \rho(T_\infty(t_0))$, *then* $\Gamma \subseteq \rho(T_E(t_0))$.

(b) *If* $\mathbb{D} \subseteq \rho(T_\infty(t_0))$, *then* $\mathbb{D} \subseteq \rho(T_E(t_0))$.

Proof. Let $(U(t,s))_{(t,s)\in D}$ be the evolution family that induces $(T_E(t))_{t\ge0}$ and $(T_\infty(t))_{t\ge0}$, respectively.

(a) Assume $\Gamma \subseteq \rho(T_\infty(t_0))$. Then $\Gamma \subseteq \rho(T_\infty(1))$ (see [12, pp. 71–72]). Let $\mathcal{P} \in \mathcal{L}(E(X))$ be the spectral projection of $T_\infty(1)$ with respect to $\{\lambda \in \sigma(T_\infty(1)) \mid |\lambda| < 1\}$. By a result of Rau [21, Lemma 14], [22, Lemma 5] the projection $\mathcal{P}$ admits a representation as a strongly continuous, projection–valued multiplication operator, i.e., $\mathcal{P} = P(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$. Moreover, $P(\cdot)$ satisfies the following properties (see [18, Thm. 1.5], [22, Thm. 6]):

$$(\text{H1}) \qquad P(t)U(t,s) = U(t,s)P(s) \quad \text{for all } (t,s) \in D,$$

there exist constants $N \geq 1$ and $\alpha > 0$ such that

$$\|U(t,s)x\| \leq Ne^{-\alpha(t-s)}\|x\| \tag{H2}$$

for all $(t,s) \in D$ and $x \in P(s)X$ and

$$\|U(t,s)x\| \geq N^{-1}\, e^{\alpha(t-s)}\|x\| \tag{H3}$$

for all $(t,s) \in D$ and $x \in Q(s)X := (Id - P(s))X$. Finally, the operator

$$U_|(t,s) : Q(s)X \to Q(t)X \mid x \mapsto U(t,s)x \tag{H4}$$

is a bijection for all $(t,s) \in D$.

Clearly, $\mathcal{P} = P(\cdot)$ defines a bounded linear projection on $E(X)$. We set $E(X)^s := \mathcal{P}E(X)$ and $E(X)^u := (Id - \mathcal{P})E(X)$. By (H1) the projection $\mathcal{P}$ and $(T_E(t))_{t\geq 0}$ commute. Hence, $T_E(t)E(X)^s \subseteq E(X)^s$ and $T_E(t)E(X)^u \subseteq E(X)^u$ for $t \geq 0$. Denote by $(T_E^s(t))_{t\geq 0}$ and $(T_E^u(t))_{t\geq 0}$ the restrictions of $(T_E(t))_{t\geq 0}$ to $E(X)^s$ and $E(X)^u$, respectively. Let $C := \sup_{t\in\mathbb{R}} \|S(t)\|$ where $(S(t))_{t\in\mathbb{R}}$ is the group of translations on E. If $f \in E(X)^s$, then (H2) implies

$$\begin{aligned}
\|T_E^s(t)f\|_{E(X)} &= \|U(\cdot,\cdot - t)P(\cdot - t)f(\cdot - t)\|_{E(X)} \\
&\leq Ne^{-\alpha t}\|f(\cdot - t)\|_{E(X)} \\
&\leq CNe^{-\alpha t}\|f\|_{E(X)}.
\end{aligned}$$

Thus the spectral radius of $T_E^s(t_0)$ is less than 1, and hence $\Gamma \subseteq \rho(T_E^s(t_0))$.

If $f \in E(X)^u$, then (H3) yields

$$\begin{aligned}
\|T_E^u(t)f\|_{E(X)} &= \|U(\cdot,\cdot - t)(Id - P(\cdot - t))f(\cdot - t)\|_{E(X)} \\
&\geq N^{-1}\, e^{\alpha t}\|f(\cdot - t)\|_{E(X)} \\
&\geq C^{-1}N^{-1}\, e^{\alpha t}\|f\|_{E(X)}.
\end{aligned} \tag{4.2}$$

Therefore $T_E^u(t)$ has closed range and is injective for each $t \geq 0$.

On the other hand, $(Id - \mathcal{P})C_c(\mathbb{R},X)$ is dense in $E(X)^u$ and for $f \in (Id - \mathcal{P})C_c(\mathbb{R},X) \subseteq C_c(\mathbb{R},X)$ we have $T_\infty^u(t)f = T_E^u(t)f$. ¿From (H4) it follows that $f \in (Id - \mathcal{P})C_0(\mathbb{R},X)$ has compact support if and only if $T_\infty(t)^u f$ has compact support. Hence, by the invertibility of $T_\infty^u(t)$,

$$T_E^u(t)(Id - \mathcal{P})C_c(\mathbb{R},X) = T_\infty^u(t)(Id - \mathcal{P})C_c(\mathbb{R},X) = (Id - \mathcal{P})C_c(\mathbb{R},X).$$

In particular, $T_E^u(t)$ has dense range for every $t \geq 0$. Hence, $T_E^u(t)$ is invertible and (4.2) yields $\|T_E^u(t)^{-1}\| \leq CNe^{-\alpha t}$ for $t \geq 0$. Thus the spectral radius of $T_E^u(t_0)^{-1}$ is less than 1, and hence $\Gamma \subseteq \mathbb{D} \subseteq \rho(T_E^u(t_0))$.

(b) Let $\mathbb{D} \subseteq \rho(T_\infty(t_0))$. Then $\mathbb{D} \subseteq \rho(T_\infty(1))$ (see [12, pp. 71–72]) and the spectral projection $\mathcal{P}$ mentioned above is 0. Therefore $E(X) = E(X)^u$ and $T_E(t) = T_E^u(t)$ for $t \geq 0$. In part (a), however, we have shown $\mathbb{D} \subseteq \rho(T_E^u(t_0)) = \rho(T_E(t_0))$. ■

From (4.1) and Lemma 4.2 we obtain the following spectral inclusion.

Proposition 4.3. *Let E be an admissible Banach function space. Then $\sigma(T_E(t)) \subseteq \sigma(T_\infty(t))$ for all $t \geq 0$.*

Proof. Let $(U(t,s))_{(t,s)\in D}$ be the evolution family corresponding to $(T_\infty(t))_{t\geq 0}$ and $(T_E(t))_{t\geq 0}$. Fix $t_0 > 0$.

(a) Let $0 \neq \lambda \in \rho(T_\infty(t_0))$. Choose $r \in \mathbb{R}$ and $\lambda_1 \in \Gamma$ such that $\lambda = \lambda_1 e^{rt_0}$. Then $\lambda_1 \in \rho(e^{-rt_0}T_\infty(t_0))$. Moreover, $(e^{-rt}T_\infty(t))_{t\geq 0}$ is the evolution semigroup on $C_0(\mathbb{R}, X)$ induced by the evolution family $(e^{-r(t-s)}U(t,s))_{(t,s)\in D}$. By (4.1) we obtain $\Gamma \subseteq \rho(e^{-rt_0}T_\infty(t_0))$ and Lemma 4.2 implies $\Gamma \subseteq \rho(e^{-rt_0}T_E(t_0))$. Thus $\lambda \in \rho(T_E(t_0))$.

(b) Let $0 \in \rho(T_\infty(t_0))$. Choose $r \geq 0$ such that $\mathbb{D} \subseteq \rho(e^{rt_0}T_\infty(t_0))$. Then Lemma 4.2 implies $\mathbb{D} \subseteq \rho(e^{rt_0}T_E(t_0))$, and hence $0 \in \rho(T_E(t_0))$. ∎

The spectral mapping theorem is now an easy consequence of the previous results.

Theorem 4.4. *Let E be an admissible Banach function space. Then*

$$\sigma(T_E(t)) \setminus \{0\} = \exp(t\sigma(G_E)) = \exp(t\sigma(G_\infty)) = \sigma(T_\infty(t)) \setminus \{0\}$$

for all $t \geq 0$.

Proof. The spectral inclusion (1.2), Proposition 4.3, Theorem 2.3 and Proposition 3.8 lead to the inclusions

$$\exp(t\sigma(G_E)) \subseteq \sigma(T_E(t)) \setminus \{0\} \subseteq \sigma(T_\infty(t)) \setminus \{0\} = \exp(t\sigma(G_\infty)) \subseteq \exp(t\sigma(G_E))$$

proving the theorem. ∎

Recall that an evolution family $(U(t,s))_{(t,s)\in D}$ is called *hyperbolic* if there is a projection–valued function $P(\cdot) \in C_b(\mathbb{R}, \mathcal{L}_s(X))$ and constants $N \geq 1$ and $\alpha > 0$ such that (H1)–(H4) holds. We refer to [2], [3], [25] for the relevance of this concept in the theory of differential equations. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach Y is called *hyperbolic* if $\sigma(T(t)) \cap \Gamma = \emptyset$ for one (and hence all) $t > 0$.

The previous theorem allows a characterization of hyperbolic evolution families through the hyperbolicity of the induced evolution semigroup $(T_E(t))_{t\geq 0}$ on $E(X)$ (see [20], [7], [8], [9], [18] for special cases).

Corollary 4.5. *Let $(U(t,s))_{(t,s)\in D}$ be an evolution family on the Banach space X and let E be an admissible Banach function space. Then the following assertions are equivalent:*

(1) *$(U(t,s))_{(t,s)\in D}$ is hyperbolic.*

(2) *$(T_\infty(t))_{t\geq 0}$ is hyperbolic.*

(3) $\rho(G_\infty) \cap i\mathbb{R} \neq \emptyset$.

(4) *$(T_E(t))_{t\geq 0}$ is hyperbolic.*

(5) $\rho(G_E) \cap i\mathbb{R} \neq \emptyset$.

Proof. The equivalence of (1) and (2) is essentially due to Rau [22, Thm. 6] (see also [18, Thm. 1.5]). The remaining equivalences follow from Theorem 4.4, (4.1) and Lemma 4.1. ∎

From $\exp(t\sigma(G_E)) = \exp(t\sigma(G_\infty))$ for $t \geq 0$, the identity (4.1) and Lemma 4.1 we obtain the following improvement of Proposition 3.8.

Corollary 4.6. *Let E be an admissible Banach function space. Then $\sigma(G_E) = \sigma(G_{E,\infty}) = \sigma(G_\infty)$.* ∎

In addition, we obtain equality of the spectra for the operators belonging to the evolution semigroup. In the proof we make use of the following lemma (see [19, Lemma 3.1]).

Lemma 4.7. *Let E be an admissible Banach function space. Then*

$$\|f(s)\| = \lim_{\varepsilon\downarrow 0} \frac{1}{\|\chi_{[s,s+\varepsilon]}\|_E} \ \|\chi_{[s,s+\varepsilon]} f\|_{E(X)}$$

for every $f \in C_0(\mathbb{R}, X)$ and every $s \in \mathbb{R}$. ∎

Proposition 4.8. *Let E be an admissible Banach function space. Then $\sigma(T_E(t)) = \sigma(T_\infty(t))$ for all $t \geq 0$.*

Proof. Theorem 4.4 yields $\sigma(T_E(t))\setminus\{0\} = \sigma(T_\infty(t))\setminus\{0\} \subseteq \sigma(T_\infty(t))$. It remains to show that $0 \in \rho(T_E(t_0))$ for some $t_0 > 0$ implies $0 \in \rho(T_\infty(t_0))$.

Since $T_E(t_0)$ is invertible, $(T_E(t))_{t\geq 0}$ can be extended to a strongly continuous group (cf. [17, 1.6.5]). In particular, there are constants $M_1 \geq 1$ and $\omega_1 \in \mathbb{R}$ such that $\|T_E(t)f\|_{E(X)} \geq M_1^{-1} e^{-\omega_1 t} \|f\|_{E(X)}$ for $f \in E(X)$ and $t \geq 0$. Since the group $(S(t))_{t\in\mathbb{R}}$ of translations on E is uniformly bounded, there is a constant $c > 0$ such that $c\,\|\chi_{[a,b]}\|_E \leq \|\chi_{[a-t,b-t]}\|_E$ for every compact interval $[a,b] \subseteq \mathbb{R}$ and all $t \in \mathbb{R}$.

If $t \geq 0$ and $f \in C_c(\mathbb{R}, X)$, then $T_\infty(t)f = T_E(t)f \in C_c(\mathbb{R}, X)$. Moreover, Lemma 4.7 and the above-mentioned inequality yield

$$\begin{aligned}
\|T_\infty(t)f(s)\| &= \|T_E(t)f(s)\| \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\|\chi_{[s,s+\varepsilon]}\|_E} \ \|\chi_{[s,s+\varepsilon]}\, T_E(t)f\|_{E(X)} \\
&= \lim_{\varepsilon\downarrow 0} \frac{1}{\|\chi_{[s,s+\varepsilon]}\|_E} \ \|T_E(t)\,(\chi_{[s-t,s-t+\varepsilon]}\, f)\|_{E(X)} \\
&\geq c\, M_1^{-1}\, e^{-\omega_1 t} \lim_{\varepsilon\downarrow 0} \frac{1}{\|\chi_{[s-t,s-t+\varepsilon]}\|_E} \ \|\chi_{[s-t,s-t+\varepsilon]}\, f\|_{E(X)} \\
&= c\, M_1^{-1}\, e^{-\omega_1 t}\, \|f(s-t)\|
\end{aligned}$$

for each $s \in \mathbb{R}$. Thus $\|T_\infty(t)f\|_\infty \geq c\,M_1^{-1}\, e^{-\omega_1 t}\, \|f\|_\infty$ for $f \in C_0(\mathbb{R}, X)$ and $t \geq 0$. In particular, $T_\infty(t)$ has closed range and is injective for all $t \geq 0$. On the other hand, if $\lambda \in \rho(G_E)$, then Proposition 3.3 implies $R(\lambda, G_E)E(X) \subseteq C_0(\mathbb{R}, X)$ and $R(\lambda, G_E)E(X)$ is dense in $C_0(\mathbb{R}, X)$. Since

$$R(\lambda, G_E)E(X) = R(\lambda, G_E)T_E(t)E(X) = T_\infty(t)R(\lambda, G_E)E(X) \subseteq T_\infty(t)C_0(\mathbb{R}, X)$$

it follows that $T_\infty(t)$ has dense image for all $t \geq 0$. Thus $T_\infty(t)$ is invertible for all $t \geq 0$. ∎

Remark 4.9. There is an example due to Arendt [1, 3.4] of an evolution semigroup on $E_p = L^p(\mathbb{R}, e^s ds)$, $1 \le p < \infty$, such that the spectra of the generator and the semigroup operators depend on p and hence on the space E_p. Therefore in Theorem 4.4 (and its consequences) the uniform boundedness of the translation group on the Banach function space E cannot be omitted.

References

[1] Arendt, W., *Gaussian estimates and interpolation of the spectrum in L^p*, Differ. Integral Equ. **7** (1994), 1153–1168.

[2] Calderon, A. P., *Spaces between L^1 and L^∞ and the theorem of Marcinkiewicz,* Studia Math. **26** (1966), 273–299.

[2] Coppel, W.A., "Dichotomies in Stability Theory," Springer–Verlag, Berlin, 1978.

[3] Daleckij, Ju. L., and M. G. Krein, "Stability of Solutions of Differential Equations in Banach Spaces," Transl. Math. Monogr., Vol. 4, Amer. Math. Soc., Providence R.I., 1974.

[4] Evans, D. E., *Time dependent perturbations and scattering of strongly continuous groups on Banach spaces,* Math. Ann. **221** (1976), 275–290.

[5] Fattorini, H.O., "The Cauchy Problem," Addison–Wesley, Reading, 1983.

[6] Howland, J. S., *Stationary scattering theory for time–dependent Hamiltonians,* Math. Ann. **207** (1974), 315–335.

[7] Latushkin, Y., and S. Montgomery-Smith, *Evolutionary semigroups and Lyapunov theorems in Banach spaces,* J. Funct. Anal. (to appear).

[8] Latushkin, Y., and S. Montgomery–Smith, *Lyapunov theorems for Banach spaces,* Bull. Am. Math. Soc. **31** (1994), 44–49.

[9] Latushkin, Y., and T.W. Randolph, *Dichotomy of differential equations on Banach spaces and an algebra of weighted translation operators,* preprint.

[10] Lindenstrauss, J., and L. Tzafiri, "Classical Banach Spaces II, Function Spaces," Springer–Verlag, Berlin, 1979.

[11] Lumer, G., *Equations de diffusion dans le domaine (x,t) non-cylindriques et semigroupes "éspace-temps",* in: "Séminaire de Théorie du Potentiel Paris," No. 9, Springer–Verlag, 1989, 161–179.

[12] Nagel, R. (Ed.), "One–Parameter Semigroups of Positive Operators," Springer–Verlag, Berlin, 1984.

[13] Nagel, R., *Semigroup methods for non–autonomous Cauchy problems,* Tübinger Berichte zur Funktionalanalysis **2**, Tübingen, 1993.

[14] Nagel, R., and E. Sinestrari, *Inhomogeneous Volterra integrodifferential equations for Hille–Yosida operators,* in: "Functional Analysis," K. D. Bierstedt, A. Pietsch, W. M. Ruess, D. Vogt (Eds.), Proc. Essen Conference, 1993, 51–72.

[15] Neidhardt, H., *On abstract linear evolution equations I,* Math. Nachr. **103** (1981), 283–293.

[16] Paquet, L., *Semigroupes géneralisés et équations d'évolution,* in: "Séminaire de Théorie du Potentiel Paris," No. 4, Springer–Verlag, 1979, 243–263.

[17] Pazy, A., "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer–Verlag, Berlin, 1983.

[18] Räbiger, F., and R. Schnaubelt, *A spectral characterization of exponentially dichotomic and hyperbolic evolution families,* Tübinger Berichte zur Funktionalanalysis **3**, Tübingen, 1994.

[19] Räbiger, F., Rhandi, A., and R. Schnaubelt, *Perturbation and an abstract characterization of evolution semigroups,* Tübinger Berichte zur Funktionalanalysis **3**, Tübingen, 1994.

[20] Rau, R., Hyperbolic evolution semigroups, Dissertation, Tübingen, 1992.

[21] Rau, R., *Hyperbolic evolution groups and dichotomic evolution families,* J. Dyn. Differ. Equations **6** (1994), 335–350.

[22] Rau, R., *Hyperbolic evolution semigroups on vector valued function spaces,* Semigroup Forum **48** (1994), 107–118.

[23] Rhandi, A., Lipschitz stetige Evolutionsfamilien und die exponentielle Dichotomie, Dissertation, Tübingen, 1994.

[24] Schaefer, H. H., "Banach Lattices and Positive Operators," Springer–Verlag, Berlin, 1974.

[25] Sacker, R. J., and G. R. Sell, *Dichotomies for linear evolutionary equations in Banach spaces,* IMA preprint No. 838 (1991).

[26] Tanabe, H., "Equations of Evolution," Pitman, London, 1979.

[27] Triebel, H., "Interpolation Theory, Function Spaces, Differential Operators," North–Holland, Amsterdam, New York, Oxford, 1978.

Universität Tübingen
Mathematisches Institut
Auf der Morgenstelle 10
D-72076 Tübingen
Germany

Received July 5, 1994
and in final form October 20, 1994