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SURVEY

Recent Developments in the Theory of Skew Lattices

Jonathan Leech

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0. From Bands and Lattices to Skew Lattices

The study of skew lattices has a pair of origins. Skew lattices were originally considered in various attempts to generalize the concept of a lattice in noncommutative, and even nonassociative, directions. More recently, skew lattices have been studied as structurally enriched bands possessing a dual multiplication, much as a semilattice might be a reduct of a lattice; indeed this often happens with bands occurring as multiplicative subsemigroups of rings. While this paper surveys recent developments, since each approach complements the other we begin by recalling some of the issues and results of the "generalized lattice" approach.

0.1. A lattice is often defined to be an algebra $(S; \vee, \wedge)$ on a set S having associative, commutative binary operations $\vee$ and $\wedge$ which are related by the absorption identities, $x \wedge (x \vee y) = x$ and $x \vee (x \wedge y) = x$. Both operations are necessarily idempotent, for $x \vee x = x \vee [x \wedge (x \vee y)] = x$, and similarly, $x \wedge x = x$. Equivalently, a lattice may be defined to be a partially ordered set $(S; \geq)$ having suprema and infima existing for finite nonempty subsets. For given a lattice algebra, upon defining $x \geq y$ to hold if $x \vee y = x$, and dually $x \wedge y = y$, one obtains a lattice ordered set. Conversely, given a lattice ordered set, then upon setting $x \vee y = \sup\{x, y\}$ and $x \wedge y = \inf\{x, y\}$, one obtains a lattice algebra. Both processes are clearly reciprocal to each other.

0.2. It was inevitable that attempts would be made to generalize the concept of a lattice. For instance, one could take the algebraic definition above, drop commutativity, but keep associativity and the stated absorption identities, and so define a type of skew lattice. What can be said about such an algebra? To begin, absorption still guarantees both $\vee$ and $\wedge$ to be idempotent. Most surprising, perhaps, is the observation that any band can be transformed into this type of skew lattice upon setting either $x \vee y = x$ and $x \wedge y = xy$, or else setting $x \vee y = xy$ and $x \wedge y = x$. In either case the two operations are at best minimally related. This seems to indicate that without commutativity, the stated absorption identities are not enough. Indeed, we may need to consider also their left-right dualizations: $(y \vee x) \wedge x = x$ and $(y \wedge x) \vee x = x$.

0.3. Thus it may prove helpful to begin with a system $(S; \vee, \wedge)$ where we only assume both $(S; \vee)$ and $(S; \wedge)$ to be bands. In this case each of the four possible absorption identities expresses an implication between two of the four cases of algebraic absorption as follows:

(1) $x \wedge (x \vee y) = x$ expresses the implication, $r \vee s = s \Rightarrow r \wedge s = r$;

(2) $(x \vee y) \wedge y = y$ expresses the implication, $r \vee s = r \Rightarrow r \wedge s = s$;

(3) $x \vee (x \wedge y) = x$ expresses the implication, $r \wedge s = s \Rightarrow r \vee s = r$;

(4) $(x \wedge y) \vee y = y$ expresses the implication, $r \wedge s = r \Rightarrow r \vee s = s$.

The identities (1) and (3) given in our definition of a lattice are thus equivalent to the partial dualities, $r \vee s = s \Rightarrow r \wedge s = r$ and $r \wedge s = s \Rightarrow r \vee s = r$, in which each of the four possible types of algebraic absorption occurs exactly once, with each related to exactly one other type. It is these partial dualities which provide some connection between $\vee$ and $\wedge$. We will provisionally call any system $(S; \vee, \wedge)$ satisfying (1) and (3) a "skew lattice of type (1,3)."

0.4. Suppose that instead we pick identities (1) and (4), and so talk about skew lattices of type (1,4) for which the duality, $r \vee s = s \Leftrightarrow r \wedge s = r$, holds. Examples of skew lattices of this type are obtained as follows. Given a quasi-lattice, that is a quasi-ordered set $(S; \subset)$ for which each pair of equivalence classes A and B has both a join class J and a meet class M, one can define a skew lattice operations of type (1,4) on S so that $x \subset y$ if and only if $x \vee y = y$, and dually, $x \wedge y = x$. To begin, choose in each equivalence class A a representative A^*. For $x \subset y$, set $x \vee y = y$ and $x \wedge y = x$; otherwise let $x \vee y$ and $x \wedge y$ be the respective representatives of their join and meet classes. While every quasi-lattice supports a skew lattice structure of type (1,4), not all skew lattices of this type arise this way as may be seen in the following simple example:

On $S = \{0, 1, 2\}$ let $\vee$ be determined by $0 \vee n = n = n \vee 0$ and $n \vee m = n$ for $m, n \neq 0$, and let $\wedge$ be determined by $1 \wedge 2 = 0 = 2 \wedge 1$. It is easily checked that $(S; \vee, \wedge)$ is a type (1,4) skew lattice for which the quasi-order $x \subset y$ given by $x \vee y = y$, and dually $x \wedge y = x$, does not form a quasi-lattice structure on S. Notice also that in this example the $\mathcal{D}$-relations for the two operations disagree. Indeed for $\vee$ the $\mathcal{D}$-class partition of S is $\{0 \mid 1, 2\}$, while for $\wedge$ the $\mathcal{D}$-class partition is given by $\{0 \mid 1 \mid 2\}$. *Thus for both type* (1,3) *and type* (1,4) *skew lattices, no analogue of the Clifford-McLean Theorem for bands exists.*

0.5. The kind of considerations encountered in the above paragraphs typify much of what occurred in early studies of skew lattices. The first person to study noncommutative versions of lattice theory was the physicist, Pascual Jordan, who published numerous articles on the subject beginning in 1949 and continuing through the early 1960s. A summary of his research may be found in [9] which includes references to nearly all of his publications on the subject. Besides Jordan, others engaged in limited publishing at this time, including S. Matsushita [16]. These individuals were motivated by the appearance of nonstandard logic in the study of quantum mechanics and a related interest in the structure of ideals in noncommutative rings. While their contributions were largely exploratory in nature and rather fragmented, we may identify several themes. (i) There was an early interest in inducing skew lattice operations on quasi-ordered sets, as well as defining quasi-orderings on skew lattices. (ii) This resulted in an interest in flat skew lattices, where in semigroup parlance "flat" meant that the Green's relation $\mathcal{D}$ reduced to either $\mathcal{L}$ or $\mathcal{R}$ for both operations.

An example of this would be the quasi-lattice construction given in 1.4 above. (iii) Attention was given to various noncommutative versions of distribution which, like absorption, breaks up when commutativity is dropped. For example, even when all four absorption identities hold, the two identities,

$$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z) \quad \text{and} \quad x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$$

and their two left-right duals are mutually independent in that no three together imply the fourth. (iv) Along with skew lattices there was an interest in bands. In particular, what are now termed (left, right) normal bands and (left, right) regular bands were recognized. Apparently missing, however, was an awareness of results on bands published by semigroup researchers. In particular, a lack of familiarity with Kimura's work on regular bands [10] appearing in the latter 1950s may have contributed to a continued emphasis on formulations assuming or implying flatness. (v) Skew lattices were usually defined so that (partial) dualities occurred between the $\mathcal{R}$-quasi-orders on both $(S;\vee)$ and $(S;\wedge)$ and between their $\mathcal{L}$-quasi-orders. This was expressed by such identities as $x \wedge (y \vee x) = x$, or $(x \wedge y) \vee x = x$. Indeed these were the absorption laws in Jordan's initial definition of a skew lattice. Upon replacing $\vee$ by the operation $\vee^*$ defined by $x \vee^* y = y \vee x$, one passes from a skew lattice of either approach to the other. Thus Jordan's definition corresponds to our type (1,3).

0.6. Following Jordan, M. D. Gerhardts published a number of articles on skew lattices during the latter half of the 1960s through 1970. While in the tradition of Jordan, Gerhardts was more focused. In particular he considered the problem of determining when a skew lattice factors as the direct product of a nest (a type of retangular skew lattice) and a lattice. In [7] he showed that type (1,3) skew lattices which could be thus factored were characterized by the identities: $(x\vee y\vee z)\wedge(y\vee x) = x \vee y$ and $(x \wedge y \wedge z) \vee (y \wedge x) = x \wedge y$. This followed an earlier paper [6] in which skew lattices of both types (1,3) and $(1^*, 3^*)$ and satisfying certain distributive identities were shown to decompose as the direct product of a nest and a distributive skew lattice. Again, publications about bands by semigroup theorists are never referenced in Gerhardts' papers, even though both volumes of *The Algebraic Theory of Semigroups* by Clifford and Preston had appeared by 1967.

0.7. Further developments occurred in a 1972 paper of B. M. Schein [22] which studied a certain class of skew lattices described in terms of various quasi-orders which induced their operations. In this paper Schein became the first to observe that *a skew lattice S satisfying all four absorption identities* (1)-(4) *factors as the direct product of a lattice and a rectangular skew lattice if and only if both $(S,\vee)$ and $(S,\wedge)$ are normal bands.* (Schein actually used the $*$-absorption identities, but his result immediately translates into the scheme of this paper. A brief discussion of some of the ideas in Schein's paper is given in the introductory remarks of [12].)

0.8. The 1980s witnessed the appearance of articles by Cornish [5] and Schweigert [23, 24]. Of particular interest in this survey is [5] in which Cornish considered what are here termed "left handed skew Boolean algebras." It was while writing his dissertation under Cornish, that Bignall [1], following ideas of Keimal and Werner

[10], observed a connection between certain types of skew Boolean algebras and discriminator varieties. The work of Bignall and Cornish, as well as Werner and Keimal, are but part of a growing body of literature about generalized Boolean structures. (A good source in this area, though by now somewhat dated, is the fourth chapter of [4].) More recently in [2] Bignall used these algebras in a study of multiple valued logic. An updated version of Bignall's results forms a major part of [3].

0.9. The author's first paper on the subject [12] appeared in 1989. The approach taken in a series of papers appearing over the last five years is to consider only skew lattices satisfying all four absorption identities, so that both of the following dualities hold: $r \vee s = s \Leftrightarrow r \wedge s = r$ and $r \vee s = r \Leftrightarrow r \wedge s = s$. This was motivated in part by a study of multiplicative bands of idempotents in rings, where often a polynomial counter-multiplication can be found turning the band into a skew lattice of this type. This was also motivated by the extent to which concepts and results from semigroup theory about bands prove useful. For full duality implies that the $\mathcal{L}$-classes of either $(S; \vee)$ or $(S; \wedge)$ are precisely the $\mathcal{R}$-classes of the other. Thus the relation $\mathcal{D}$ is unambiguously defined for this type of skew lattice, in contrast to those encountered above. Hence $\mathcal{D}$ is a congruence, and a skew lattice version of the Clifford-McLean Theorem for bands must hold. Another consequence of full duality is that both of the bands $(S; \vee)$ and $(S; \wedge)$ must be regular ($xyxzx = xyzx$). Thus, in contrast to the situation for the (1,3) case, a band can be embedded in a skew lattice of this type if and only if it is regular. Thus also, results obtained by N. Kimura about regular bands in his 1957 Tulane dissertation written under A. H. Clifford and published in [10] are of major relevance in the study of skew lattices. But a prior motivation in the author's mind was the belief that the study of bands might ultimately lead one to consider noncommutative variants of lattices. What appears in the following pages is the outgrowth of that belief.

0.10. This paper is a survey of recent developments in the study of skew lattices. What follows is divided into five sections, the first of which presents some basic definitions and theorems, and shows how results about bands yield fundamental theorems about this class of skew lattices. The second section considers conditions under which multiplicative bands of idempotents in rings acquire a dual multiplication and so become skew lattices, and also introduces common properties of this class of skew lattices. The next two sections consider skew lattice versions of normal bands. Section Three is about local lattices $(S; \vee, \wedge)$ for which the band $(S; \wedge)$ is normal. Local lattices are of interest because of their connection to skew Boolean algebras. Thanks to local lattices, there exists a fundamental link between the normal bands of semigroup theory and the study of Boolean structures. The subclass of skew Boolean algebras first considered by Bignall in [1], is considered in the fourth section and connections with discriminator varieties are indicated. In the final section we return to skew lattices proper and show how comparable $\mathcal{D}$-classes create coset partitions of each other, which in turn reveals a sensitivity to order between distinct $\mathcal{D}$-classes. Thus, while every skew lattice satisfying full duality lives on a quasi-lattice (the Clifford-McLean Theorem), in contrast with the (1,4) case not every quasi-lattice can support a skew lattice structure of this type. Coset partitions are then used to design an example of an infinite skew lattice on two generators (in contrast to the situation for either bands or lattices where the free algebra on two generators is finite).

Finally, it should be mentioned that skew lattices form part of a broader area of "associative, idempotent algebraic systems." Besides bands, lattices and skew lattices, various authors have also studied idempotent semirings ([17], [18]), restrictive bisemigroups ([21]), and idempotent right sided quantales (or "non-commutative frames"). An introductory treatment of the latter, along with references, is found in the fifth chapter of [20].

1. Some Definitions and Basic Theorems

1.1. Throughout the remainder of this paper, by a *skew lattice* is meant an algebra $(S; \vee, \wedge)$ consisting of a pair of associative binary operations $\wedge$ and $\vee$ satisfying the absorption laws:

$$x \wedge (x \vee y) = x = (y \vee x) \wedge x,$$
$$x \vee (x \wedge y) = x = (y \wedge x) \vee x.$$

As was seen in the introduction, absorption causes both operations to be idempotent, and in addition, asserts that the bands $S_\vee = (S; \vee)$ and $S_\wedge = (S; \wedge)$ have dualized multiplications in that $x \wedge y = x$ iff $x \vee y = y$, and likewise $x \wedge y = y$ iff $x \vee y = x$. The $\mathcal{L}$-quasiorder on either band is thus dualized by the $\mathcal{R}$-quasiorder on the other. This allows us to define the *natural partial order* by letting $x \geq y$ in S denote either the condition $x \wedge y = y = y \wedge x$, or its dual, $x \vee y = x = y \vee x$.

1.2. All lattices are skew lattices. At the other extreme, a *rectangular skew lattice* is a skew lattice S for which both $S_\wedge$ and $S_\vee$ are rectangular bands whose multiplications dualize each other , $x \wedge y = y \vee x$. Any rectangular band S can be turned into a rectangular skew lattice by setting $x \wedge y = xy$ and $x \vee y = yx$. Conversely, all such skew lattices are obtained in this manner.

1.3. Thanks to duality, the Green's equivalence relations are defined for a skew lattice as follows: $\mathcal{R} = \mathcal{R}_\wedge = \mathcal{L}_\vee$; $\mathcal{L} = \mathcal{L}_\wedge = \mathcal{R}_\vee$; and $\mathcal{D} = \mathcal{D}_\wedge = \mathcal{D}_\vee$. Here $\mathcal{R}_\wedge, \mathcal{L}_\wedge$ and $\mathcal{D}_\wedge$ denote the various Green's relations for the band $S_\wedge$, while $\mathcal{R}_\vee, \mathcal{L}_\vee$ and $\mathcal{D}_\vee$ denote the corresponding relations for $S_\vee$. The Clifford-McLean Theorem on the structure of bands ([8] IV.3.1) clearly extends to skew lattices:

Theorem 1.4. *Given any skew lattice S, the relation $\mathcal{D}$ is a congruence. The $\mathcal{D}$-classes of S are its maximal rectangular subalgebras and the quotient algebra $S/\mathcal{D}$ forms the maximal lattice image of S.* ∎

1.5. A skew lattice S is *right-handed* if $\mathcal{D} = \mathcal{R}$, or equivalently, if the identity $x \wedge y \wedge x = y \wedge x$ and its dual $x \vee y \vee x = x \vee y$ hold on S. Dually, S is *left-handed* if $\mathcal{D} = \mathcal{L}$ and the dual identities hold. There is a sense in which the study of arbitrary skew lattices may be reduced to the right-handed case. To begin, any left-handed skew lattice can be obtained as the left-right dualization of a right-handed skew lattice. What is more, toward the end of this section it will be shown that for any skew lattice S, both $S_\wedge$ and $S_\vee$ are *regular bands*, that is, both operations $\wedge$ and $\vee$ satisfy the identity $xyxzx = xyzx$. One may thus apply results of Kimura [11] to skew lattices and obtain:

Theorem 1.6. *The relations $\mathcal{L}$ and $\mathcal{R}$ are both congruences on any skew lattice S. Moreover,*

1) *$S/\mathcal{L}$ is the maximal right-handed image of S.*

2) *$S/\mathcal{R}$ is the maximal left-handed image of S.*

3) *The induced epimorphisms $S \to S/\mathcal{L}$ and $S \to S/\mathcal{R}$ together yield an isomorphism of S with the fibered product $X/\mathcal{L} \times_{S/\mathcal{D}} S/\mathcal{R}$.*

■

Two more elementary results about skew lattices are as follows:

Theorem 1.7. *Given a skew lattice S and $x \in S$, the following are equivlaent:*

1) *The $\mathcal{D}$-class of x is a singleton class: $\mathcal{D}_x = \{x\}$.*

2) *$x \wedge y = y \wedge x$, for all $y \in S$.*

3) *$x \vee y = y \vee x$, for all $y \in S$.*

Thus the center of S is the sublattice formed as the union of all its singleton $\mathcal{D}$-classes. In particular, S is a lattice if either $\vee$ or $\wedge$ is commutative. ■

Theorem 1.8. *In a skew lattice, the union of all finite $\mathcal{D}$-classes forms a subalgebra.* ■

With the exception of Theorem 1.8, which was stated in [15] 3.6, all of the definitions and results appearing thus far were given in the first section of [12]. We now introduce a special class of bands about which we will prove several simple results from which, in turn, follow the assertion of regularity in 1.5 and the previous two theorems. We thus provide verifications that are independent of the orginal arguments.

1.9. A band satisfies the *class covering condition* (CCC) if for any pair of comparable $\mathcal{D}$-classes $X > Y$ in the band, the restriction of the natural partial ordering is surjective: given $y \in Y$, there exists an $x \in X$ such that $x \geq y$.

Lemma 1.10. *Given a skew lattice S, both $S_\vee$ and $S_\wedge$ satisfy the class covering condition.*

Proof. Given $\mathcal{D}$-classes $X > Y$ in S with $x \in X$ and $y \in Y: x \geq x \wedge y \wedge x \in Y$ and $y \leq y \vee x \vee y \in X$. ■

(It should not be lost on the reader that whereas bands satisfying the CCC fail to form a subvariety of bands, skew lattices do form a variety.)

Theorem 1.11. *In a band S satisfying the class covering condition, the following hold:*

1) *S is regular ($xyxzx = xyzx$), and thus $\mathcal{L}$ and $\mathcal{R}$ are congruences.*

2) *An element* x *lies in the center of* S *if and only if* $\mathcal{D}_x = \{x\}$.

3) *Given* $x, y \in S$, *if both* $\mathcal{D}_x$ *and* $\mathcal{D}_y$ *are finite*, then so is $\mathcal{D}_{xy}$.

4) In general, $\mathcal{D}_{xy} = \mathcal{D}_x\mathcal{D}_y$, where the latter is $\{uv \mid u \in \mathcal{D}_x, v \in \mathcal{D}_y\}$.

Proof. (1) Given $x, y, z \in S$, by the CCC, $u, v \in \mathcal{D}_x$ exist such that $u \geq xy$ and $v \geq zx$. Since $uxv = uv$ in the rectangular class $\mathcal{D}_x$, we obtain: $xyzx = (xy)uv(zx) = (xy)uxv(zx) = xyxzx$. (2) Clearly $\mathcal{D}_x = \{x\}$ if x commutes with all $y \in S$. So suppose that $\mathcal{D}_x = \{x\}$ and that $y \in S$ is given. By the CCC, one must have $x \geq xy$, and so $xy = xyx$. Likewise $yx = xyx$, and so $xy = yx$. (4) Let $w \in \mathcal{D}_{xy}$. By the CCC again, $u \in \mathcal{D}_x$ and $v \in \mathcal{D}_y$ exist such that $u \geq w$ and $v \geq w$. Since uv, $w \in \mathcal{D}_{xy}$, a rectangular class, we have $w = uvwuv = uv$ and (4) holds. ■

We conclude this section by stating a result which is not an application of results about bands, but instead reveals a sensitivity to order which is atypical of bands. Its justification is provided in the final section of the paper.

Theorem 1.12. *Given a prime number* p, *then the union of all* $\mathcal{D}$*-classes in a skew lattice* S *having* p*-power order (including all singleton classes) forms a subalgebra.* ■

2. Skew Lattices in Rings

2.1. Given a ring A, set $x \wedge y = xy$ and $x \vee y = x + y - xy$ (the so-called "circle" operation). Both $\wedge$ and $\vee$ are associative. Also, for any $s \in A$, $x \wedge x = x$ holds if and only if $x \vee x = x$. Upon letting $\mathrm{E}(A)$ denote the set of idempotent elements in A, we obtain the following easily checked theorem.

Theorem 2.2. *If* $S \subseteq \mathrm{E}(A)$ *is closed under both* $\vee$ *and* $\wedge$, *then* $(S; \vee, \wedge)$ *is a skew lattice satisfying the following properties:*

1) $x \vee y = y \vee x$ *if and only if* $x \wedge y = y \wedge x$.

2) $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$.

3) $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$.

■

2.3. A skew lattice satisfying (1) is called *symmetric*, as all instances of commutativity are symmetric with respect to the two operations. Thus, given a nonempty, element-wise $\wedge$-commuting [or $\vee$-commuting] subset X, the subalgebra $\langle X \rangle$ that is generated from X is a sublattice. Symmetric skew lattices form a subvariety jointly characterized by the identity,

4) $x \wedge y \wedge (x \vee y \vee x) = (x \vee y \vee x) \wedge y \wedge x$,

and its dual. A pleasing property of symmetric skew lattices is stated in the next theorem whose proof is given in [12]. But first, by a *lattice cross section* is meant a sublattice of a skew lattice S intersecting each $\mathcal{D}$-class of S at a single element.

Theorem 2.4. *Let S be a symmetric skew lattice for which the maximal lattice image S/D is at most countably infinite. Then S has a lattice cross section S_0.* ∎

2.5. A skew lattice satisfying both (2) and (3) in 2.2 above is called *middle distributive.* Such skew lattices likewise form a subvariety. Are the identities (2) and (3) equivalent for skew lattices, as they are for lattices? At present, the answer is unknown to the author; but for local lattices (to be considered in the next section) the answer is affirmative.

How does one find skew lattices in $\mathrm{E}(A)$? Answer: *By looking at bands in* $\mathrm{E}(A)$, especially bands that are maximal with respect to some constraint. This is illustrated by the following results from [12] and [13].

Theorem 2.6. *Every multiplicative band in* $\mathrm{E}(A)$ *that is maximal with respect to being right regular* ($\mathcal{D} = \mathcal{R}$) *is also closed under* $\vee$ *and so forms a right handed skew lattice. In general, every right regular band in* $\mathrm{E}(A)$ *generates a right handed skew lattice in the derived algebra* $(A; \vee, \wedge)$. ∎

We may begin by taking a specified lattice cross section and then proceed as follows:

Theorem 2.7. *Given that* $S_0 \subseteq \mathrm{E}(A)$ *is a lattice under* $\wedge$ *and* $\vee$*, a right handed skew lattice* $S \subseteq \mathrm{E}(A)$ *exists that is uniquely maximal with respect to having* S_0 *as a lattice cross section. Indeed,* $S_\wedge$ *is uniquely maximal subject to the joint condition of being right regular and having* S_0 *as a semilattice cross section.* ∎

2.8. A maximal regular band need not be closed under $\vee$ as defined. Indeed examples are easily found of multiplicative rectangular bands that are not closed under $\vee$. Such examples, however, are closed under the following cubic variant ∇ of $\vee$ defined by

$$x \nabla y = x + y + yx - xyx - yxy,$$

since in the rectangular case the polynomial expression reduces to yx, so that one has a rectangular skew lattice. Let us replace the condition that the band be regular with the condition that it be normal. (Recall that a band is *normal* if it is middle commutative $xyzw = xzyw$. This is equivalent to asserting that all subsets $x\downarrow = \{u \mid u \leq x\}$ are element-wise commutative.) For normal bands we have the following results. (The adjective "Boolean" appearing in their statements will be defined in the next section.)

Theorem 2.9. *Every maximal normal multiplicative band* S *in* $\mathrm{E}(A)$ *is also closed under* ∇ *and so forms a Boolean skew lattice* $(S; \nabla, \wedge, 0)$. ∎

Corollary 2.10. *If the set of all idempotents* $\mathrm{E}(A)$ *in a ring* A *is closed under multiplication, then* $\mathrm{E}(A)$ *is a normal band and thus* $(\mathrm{E}(A); \nabla, \wedge, 0)$ *is a Boolean skew lattice. (For the case where* A *has a multiplicative identity, the condition that* $\mathrm{E}(A)$ *be multiplicatively closed is well-known to imply that* $\mathrm{E}(A)$ *forms a Boolean lattice under the operations* $\wedge$ *and* $\vee$*.)* ∎

3. Local Lattices and Distributivity

3.1. A *local lattice* is a skew lattice S for which each subalgebra $x\downarrow$, defined as either $\{x \wedge s \wedge x \mid s \in S\}$ or $\{u \in S \mid u \leq x\}$, is a lattice under the given operations. Since this is equivalent to asserting that $S_\wedge$ is a normal band, local lattices form a subvariety of skew lattices. If each principal sublattice $x\downarrow$ is also distributive, then S is called a **distributive** local lattice. For this class of skew lattices, we have the following useful technical result ([14], Theorem 2.5).

Theorem 3.2. *For a local lattice S, the following conditions are equivalent:*

1) *For each $x \in S$, the sublattice $x\downarrow$ is distributive.*

2) *The maximal lattice image $S/\mathcal{D}$ is distributive.*

3) *In S, $x \vee (y \wedge z) \vee x = (x \vee y \vee x) \wedge (x \vee z \vee x)$.*

4) *In S, $x \wedge (y \vee z) \wedge x = (x \wedge y \wedge x) \vee (x \wedge z \wedge x)$.*

5) *In S, $x \wedge (y \vee z) \wedge w = (x \wedge y \wedge w) \vee (x \wedge z \wedge w)$.*

In particular, (5) characterizes distributive local lattices as a subvariety of skew lattices. Moreover, the skew lattice subvariety of local lattices that are both symmetric and distributive is characterized by the following identities:

6) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z); \quad (y \vee z) \wedge w = (y \wedge w) \vee (z \wedge w)$.

■

3.3. A skew lattice S is called *Boolean* if: (i) S is symmetric; (ii) S has a *zero*, that is, an element 0 such that for all $x \in S$, $0 \wedge x = 0 = x \wedge 0$; (iii) each principal subalgebra $x\downarrow$ forms a Boolean lattice. Given $x, y \in S$, the *difference* $x - y$ is the complement of $x \wedge y \wedge x$ in $x\downarrow$. Upon including the difference and the nullary operation 0 in the list of operations, we obtain an enriched algebraic structure $(S; \vee, \wedge, -, 0)$ called a *skew Boolean algebra*. Skew Boolean algebras form a variety of algebras characterized by the standard skew lattice identities along with the two identities of Theorem 3.2 (6), and the following identities:

7) $(x - y) \wedge y = 0 = y \wedge (x - y)$.

8) $(x - y) \vee (x \wedge y \wedge x) = x = (x \wedge y \wedge x) \vee (x - y)$.

Indeed, $0 \wedge y = 0 = y \wedge 0$ follows from (7). Together, (7) and (8) yield $x - y$ as the relative complement of $x \wedge y \wedge x$ with respect to x.

To obtain insight into the structure of skew Boolean algebras, we consider the following noncommutative analogue of the Boolean algebra **2**:

3.4. A Boolean skew lattice S is *primitive* if it is formed by adjoining a zero element 0 to a rectangular skew lattice X, $S = X^0$, in which case:

$$x - y = \begin{cases} 0, & \text{when } x\mathcal{D}y, \\ x, & \text{otherwise.} \end{cases}$$

The induced algebra will be referred to as a *primitive algebra.*

3.5. A skew Boolean algebra that is a direct product of primitive algebras is said to be *completely reducible.* It can be shown that a skew Boolean algebra S is completely reducible if and only if (i) S is *complete* so that under the natural partial ordering both $\sup A$ and $\inf A$ exist for every element-wise commuting, nonempty subset A of S and (ii) S is *atomic* so that every nonzero element in S is bounded below by some 0-minimal element. Indeed, if S is complete and atomic with indexed 0-minimal $\mathcal{D}$-classes $\{X_\lambda \mid \lambda \in \Lambda\}$, then an isomorphism of $\prod_\Lambda X^0_\lambda$ upon S is given by the map $\langle u_\lambda \mid \lambda \in \Lambda\rangle \to \sup_\Lambda u_\lambda$. In particular, this occurs if the maximal lattice image $S/\mathcal{D}$ is finite. In general, every skew Boolean algebra is isomorphic to a subdirect product of primitive algebras.

3.6. Examples

a. Every maximal normal band of idempotents in a ring A forms a skew Boolean algebra. In particular, if $(\mathrm{E}(A), \leq)$ satisfies the finite chain condition, any such skew Boolean algebra factors as a direct product of a finite number of primitive algebras. (This applies in any ring satisfying either the ACC or the DCC on left [right] ideals.)

b. Given sets A and B, let $\mathbf{P} = \mathbf{P}(A, B)$ denote the set of all partial functions from A to B. For $f, g \in \mathbf{P}$ having respective domains $F, G \subseteq A$ we set:

$$f \vee g = f \cup (g \mid G - F); \quad f \wedge g = g \mid F \cap G; \quad f - g = f \mid (F - G).$$

Then $(\mathbf{P}; \vee, \wedge, -, 0)$ is a completely reducible, right handed skew Boolean algebra which is isomorphic to the right handed power algebra $(B^0)^A$. A ring theoretic analogue of $\mathbf{P}$ is given as follows:

c. (Cornish [5]) Let A be a C**-ring**. Thus for each $x \in A$, a *central idempotent cover* exists, that is, a central idempotent $C(x)$ that is minimal with respect to the condition: $C(x)x = x$. For $x, y \in A$ set:

$$x \vee y = x + y - C(x)y; \quad x \wedge y = C(x)y; \quad x - y = x - xC(y).$$

Then $(A; \vee, \wedge, -, 0)$ is a right handed skew Boolean algebra. (C-rings abound. Semisimple Artinian rings and, more generally, biregular rings are C-rings.)

We conclude this section with some results taken from [14] indicating the connections between the types of skew lattices considered in this section.

Theorem 3.7. *The skew lattice subvariety of symmetric local lattices is the join of the subvariety of lattices with the subvariety of distributive, symmetric local lattices; the latter subvariety is generated from the class of all [primitive] Boolean skew lattices.* ■

The above "join" assertion is a consequence of:

Theorem 3.8. *Let S be a symmetric local lattice with T its maximal distributive image. Then the natural epimorphisms $S \to S/\mathcal{D}$ and $S \to T$ induce a fibered product factorization: $S \cong S/\mathcal{D} \times_{T/\mathcal{D}} T$.* ■

Every distributive, symmetric local lattice S can be embedded in a product of primitive algebras. For $S/\mathcal{D}$ finite, this can be done in a precise way.

3.9. Construction

Suppose we are given (i) a family $\{X_\lambda \mid \lambda \in \Lambda\}$ of rectangular skew lattices and (ii) a **ring** $\Re$ of subsets of the indexing set Λ closed under $\cup$ and $\cap$. Then $\{X^0_\lambda \mid \lambda \in \Lambda\}$ is an induced family of primitive algebras. Let $\prod_\Re X^0_\lambda$ denote the subset of $\prod_\Lambda X^0_\lambda$ consisting of all $(x_\lambda)_\Lambda$ having *support* in $\Re$, that is $\{\lambda \in \Lambda | x_\lambda \neq 0\} \in \Re$. $\prod_\Re X^0_\lambda$ forms a distributive, symmetric local lattice under the skew lattice operations inherited from $\prod_\Lambda X^0_\lambda$ and is called a *full ring of functions.*

Theorem 3.10. *Every distributive, symmetric local lattice S for which $S/\mathcal{D}$ is finite is isomorphic to a full ring of functions over a ring of subsets of a finite set.* ∎

While the details of the proof are given in [14], suffice it to say that our choice for Λ is familiar: the set of all $\vee$-irreducible elements of $S/\mathcal{D}$. For $\Re$ we pick the ring of all subsets of Λ having the form $\{\lambda \in \Lambda \mid \lambda \leq x\}$ for some $x \in S/\mathcal{D}$.

4. Intersections

A significant subclass of skew Boolean algebras possess another natural operation. While this operation merges with the meet in the commutative case, in the general case it is necessary that we take it into account.

4.1. In a skew lattice S, the infimum (relative to $\geq$) of nonempty subset A, if it exists, is called its *intersection*, denoted $\cap A$. If $\cap A$ exists for all [finite] nonempty subsets, S is said to *have [finite] intersections.* A prime source for skew lattices having intersections is the class of local lattices. Indeed, any local lattice with a zero $(S; \vee, \wedge, 0)$ has intersections if either (i) S is complete, or (ii) each principal sublattice $x \downarrow$ is finite. In general, since the induced natural partial orderings for $\wedge$ and $\cap$ agree, it follows that skew lattices having finite intersections $(S; \vee, \wedge, \cap)$ form a variety of algebras. Even more is true:

Theorem 4.2. ([3], Theorem 2.8.) *The variety of skew lattices having finite intersections is congruence distributive.*

Proof. Upon setting $m(x,y,z) = (x \cap y) \vee (y \cap z) \vee (x \cap z)$, a majority term satisfying $m(x,x,y) = m(x,y,x) = m(y,x,x) = x$ is obtained. But this suffices to guarantee that the congruence lattice of S is distributive. (See [4], Theorem 12.3.) ∎

4.3. A *skew Boolean $\cap$-algebra* is a skew Boolean algebra with finite intersections $(S; \vee, \wedge, \cap, -, 0)$. Such algebras clearly form a variety, Moreover, many examples of skew Boolean algebras have finite intersections. These include complete algebras, and in particular algebras constructed from maximal normal bands in rings satisfying the FCC on idempotents. Algebras constructed from C-rings also have finite intersections: $x \cap y$ turns out to be $(1 - C(x-y))x$.

4.4. The presence of finite intersections permits a useful alternative to the standard difference called the *BCK-difference* (after "BCK algebras") and defined by: $x/y =$

$x - (x \cap y)$. Given a primitive algebra $S = X^0$ the interseciton and BCK-difference are as follows:

$$x \cap y = \begin{cases} x, & \text{if } x = y, \\ 0 & \text{if } x \neq y, \end{cases} \qquad \text{and} \qquad x/y = \begin{cases} 0, & \text{if } x = y, \\ x, & \text{if } x \neq y. \end{cases}$$

Skew Boolean $\cap$-algebras were just defined to be algebras having four binary operations and a single nullary operations. Identities such as $x \cap y = x/(x/y)$ and $x - y = x/(x \wedge y \wedge x)$ enable us to replace both "$-$" and "$\cap$" by "$/$" and so reduce the number of binary operations from four to three. Thus this class of algebras may be described as algebras of the form $(S; \vee, \wedge, /, 0)$. (This is how these algebras were originally considered by Bignall in his dissertation, [1].) The simplifying role of the BCK-difference is illustrated by the remaining results in this section, all of which are taken from [3].

4.5. To begin, all skew lattice congruences on skew Boolean algebras are in fact congruences of skew Boolean algebras; but while such congruences preserve the standard difference, they need not preserve finite intersections or BCK-differences. Moreover, Cornish has shown in [5] that the lattice of such congruences need not satisfy any lattice identity. Congruences preserving the operation $\cap$ are termed $\cap$-*congruences.* Given a skew Boolean $\cap$-algebra, its lattice of $\cap$-congruences is at least distributive. In fact it has a familiar description. But first:

4.6. An *ideal* in a skew lattice S is a nonempty subset I such that (i) given $x \in I$, it follows that $u \wedge x \wedge v \in I$ for all $u, v \in S$, and (ii) I is closed under $\vee$. Under the map $S \to S/\mathcal{D}$, every ideal of S arises as the inverse image of some ideal in $S/\mathcal{D}$.

Theorem 4.7. *Let Θ be an $\cap$-congruence on a skew Boolean $\cap$-algebra S with I_Θ being the congruence class of 0. Then:*

1) *I_Θ is an ideal, and*

2) *For all $x, y \in S$, $x\Theta y$ if and only if $(x/y \vee y/x) \in I_\Theta$.*

Conversely, if I is an ideal, then Θ_I, defined as in (2), is a $\cap$-congruence on S. Moreover, the maps $\Theta \to I_\Theta$ and $I \to \Theta_I$ are mutual inverses. ∎

Corollary 4.8. *The congruence lattice of a skew Boolean $\cap$-algebra S is isomorphic to the lattice of ideals of its maximal lattice image $S/\mathcal{D}$ (in the variety of skew lattices). In particular, finitely generated congruences correspond to principal ideals in $S/\mathcal{D}$ and thus form a relatively complemented sublattice in the congruence lattice of S.* ∎

Corollary 4.9. *Primitive skew Boolean $\cap$-algebras are simple (the only congruences are the two trivial ones) and thus are subdirectly irreducible.* ∎

4.10. The *spectrum* of a nonzero skew Boolean $\cap$-algebra S is the set Σ of all maximal ideals of S. For each $M \in \Sigma$, S/Θ_M is a primitive algebra. From the case for $S/\mathcal{D}$ it follows that the intersection $\cap\Sigma$ of all maximal ideals is the minimal ideal $\{0\}$. Thus the induced projections $\mu_M : S \to S/\Theta_M$ together yield S to be a subdirect product of primitive algebras:

$$\mu : S \to \prod_{M \in \Sigma} S/\Theta_M.$$

In fact, μ represents S as a *Boolean product.* Thus a Boolean space topology on Σ exists such that for any $x, y \in S$: (i) the equalizer $\{M \in \Sigma \mid \mu_M(x) = \mu_M(y)\}$ is clopen in Σ, and (ii) for any clopen subset U of Σ, an element $z \in S$ exists such that for $M \in U$, $\mu_M(z) = \mu_M(x)$, but for $M \notin U$, $\mu_M(z) = \mu_M(y)$. Indeed [4] IV.8.14 implies:

Theorem 4.11. *The map μ is a Boolean product representation of S with respect to the Boolean space topology on Σ generated from the subbasis of clopen sets having either the form $E_x = \{M \in \Sigma \mid x \in M\}$ or its complement $F_x = \{M \in \Sigma \mid x \notin M\}$ for some $x \in S$. In particular, given $x, y \in S$, the equalizer $\{M \mid \mu_M(x) = \mu_M(y)\}$ is the clopen subset $E_{x/y \vee y/x}$ of Σ.* ■

Since it is easily verified that a Boolean product of primitive algebras has finite intersections, we have:

Corollary 4.12. *Among skew Boolean algebras, those having finite intersections are characterized by being isomorphic to Boolean products of primitive algebras.* ■

4.13. These results fit into a larger picture. The *discriminator* on a set A is the function $d : A^3 \to A$ given by:

$$d(x, y, z) = \begin{cases} x, & \text{if } x \neq y, \\ z, & \text{otherwise.} \end{cases}$$

An algebra A in a variety of algebras is a *discriminator algebra* if the discriminator d on its underlying set is a term in the algebra, that is, d is obtained from the basis operations by a sequence of compositions. A variety V is a *discriminator variety* if it is generated from a class of discriminator algebras whose discriminators are instances of a common term. Such varieties are always congruence distributive and congruence permutable; moreover, they have been described as being "...the most successful generalization of Boolean algebras to date, successful because we obtain Boolean product representations (which can be used to provide a deep insight into algebraic and logical properties) [4, p.164]." In contrast to the case of skew Boolean algebras, we have:

Theorem 4.14. *Skew Boolean $\cap$-algebras form a discriminator variety.*

Proof. The term $(x/y) \vee [z - (x/y \vee y/x)] \vee (x/y)$ represents d on any primitive algebra. ■

Conversely, algebras in a discriminator variety for which a nullary term 0 is defined possess an induced skew Boolean $\cap$-algebra structure. This may be seen by appropriately modifying the following remarks.

4.15. Let $\mathbf{PD_0}$ denote the variety generated by the class of all *pointed* discriminator algebras $(A; d, 0)$ where d is the discriminator on A and 0 is a nullary operation.

Theorem 4.16. $\mathbf{PD_0}$ *is term equivalent to the variety of right handed skew Boolean* $\cap$*-algebras. In particular, the operations* $\vee$, $\wedge$ *and* $/$ *can be defined as terms in* $\mathbf{PD_0}$ *as follows:*

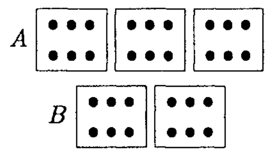
$$x \vee y = d(x, 0, y); \quad x \wedge y = d(x, x \vee y, y); \quad x/y = d(x, y, 0). \qquad \blacksquare$$

The use of the discriminator to define Boolean-like operations goes back to a 1974 study ([10]) of quasi-primal algebras by Keimel and Werner.

5. $\mathcal{D}$-Classes and Cardinality

5.1. A *primitive skew lattice* is a skew lattice P consisting of exactly two $\mathcal{D}$-classes, $A > B$. By a *coset* of B in A is meant a subset of A of the form $B \vee a \vee B$ for some $a \in A$; likewise a coset of A in B is any subset of B of the form $A \wedge b \wedge A$ for some $b \in B$. All cosets in either $\mathcal{D}$-class form rectangular subalgebras of their respective classes. The following fundamental result describes the extent to which the natural partial ordering $\geq$ determines the behavior of $\vee$ and $\wedge$.

Lemma 5.2. *Given a primitive skew lattice with* $\mathcal{D}$*-classes* $A > B$*, the cosets of* B *in* A *partition* A*, and likewise, the cosets of* A *in* B *partition* B*. Moreover, any coset* X *of* B *in* A *is isomorphic to any coset* Y *of* A *in* B*, under the map* $\varphi_{XY} : X \to Y$ *defined implicitly for* $x \in X$ *and* $y \in Y$ *by* $\varphi_{XY}(x) = y$ *iff* $x \geq y$*.*



Finally, for all $x \in X$ *and all* $y \in Y$*:*

$$x \vee y = x \vee \varphi_{XY}^{-1}(y) \quad \textit{and} \quad y \vee x = \varphi_{XY}^{-1}(y) \vee x \ \textit{in } A,$$
$$x \wedge y = x \wedge \varphi_{XY}(y) \quad \textit{and} \quad y \wedge x = \varphi_{XY}(y) \wedge x \ \textit{in } B.$$

$\blacksquare$

5.3. Isomorphisms between cosets of the form φ_{XY} will be called *coset bijections.* Lemma 5.2 suggests the following construction which, together with duality and fibered products (Theorem 1.6), provides a general description of primitive skew lattices.

Construction 5.4. Let A and B be disjoint, nonempty sets, with each partitioned, $A = \bigcup_I A_i$ and $B = \bigcup_J B_j$, so that all subsets in either partition have a common cardinality. For each pair of indices $i \in I$, $j \imath J$, let $\varphi_{ij} : A_i \to B_j$ be a fixed bijection. Define operations $\vee$ and $\wedge$ on $P = A \cup B$ as follows:

$$x \vee y = \begin{cases} x, & \text{if either } x \in A \text{ or } x, y \in B, \\ \varphi_{ij}^{-1}(x), & \text{for } x \in B_j \text{ and } y \in A_i. \end{cases}$$

$$x \wedge y = \begin{cases} y, & \text{if either } y \in B \text{ or } x, y \in A, \\ \varphi_{ij}(y), & \text{for } x \in B_j \text{ and } y \in A_i. \end{cases}$$

Theorem 5.5. *As constructed, P is a right handed, primitive skew lattice with $\mathcal{D}$-classes $A > B$, with coset partitions being the given partitions of A and B, and with coset bijections being the given family of bijections. Moreover, every right handed primitive skew lattice arises in this manner.* ∎

5.6. As a consequence of the above results, a primitive skew lattice on three generators is necessarily finite, with order at most nine in the right-handed case, and at most forty-five in the general case. An example of an infinite right handed, primitive skew lattice on four generators is constructed as follows. Let A and B both be copies of the set of integers with elements denoted by n_A and n_B respectively. Let $A = A_0 \cup A_1$ and $B = B_0 \cup B_1$ partition A and B into even and odd integers, with the even integers denoted by A_0 and B_0. Set $\varphi_{00}(2n_A) = 2n_B$, $\varphi_{01}(2n_A) = 2n+1_B$, $\varphi_{10}(2n+1_A) = 2n_B$ and $\varphi_{11}(2n+1_A) = 2n+3_B$. Let P be the infinite skew lattice constructed on $A \cup B$ using the φ_{ij}. Due to the shift on the odds given by φ_{11}, P is generated from 0_A, 0_B, 1_A and 1_B.

5.7. A trivial way of constructing a primitive skew lattice is to take two rectangular skew lattices A and B and then extend their separate operations as follows: $x \vee y = x = y \vee x$ and $x \wedge y = y = y \wedge x$, for $x \in A$, $y \in B$. If A and B are both finite with relatively prime orders, then since the order of any coset is a common divisor the orders of A and B, this construction is the only way to extend the operations and so form a primitive skew lattice with $\mathcal{D}$-classes $A > B$.

Cosets and their bijections reveal a sensitivity to order in $\mathcal{D}$-clàsses that does not occur with regular bands, even those satisfying the CCC. This is born out further in the following result, where we are given a pair of incomparable $\mathcal{D}$-classes A and B in an arbitrary skew lattice S, together with their join $\mathcal{D}$-class J. One thus has a pair of primitive skew lattices, $J \cup A$ and $J \cup B$, overlapping at J. (As stated, the result even makes sense for A and B comparable, so that $J = A$ or $J = B$, provided we understand a coset of any $\mathcal{D}$-class *in itself* to mean the entire $\mathcal{D}$-class.)

Theorem 5.8. *Let $\mathcal{D}$-classes A and B have join-class J in a skew lattice S. Let $J = \bigcup_I J_i^A = \bigcup_J J_j^B$ be respective partitions of J into A-cosets and B-cosets. Then all possible intersections $J_i^A \cap J_j^B$ have a common cardinality. In particular, if for any pair of indices J_i^A and J_j^B are both finite, then so is J, with*

$$|J| = \frac{\left|J_i^A\right| \left|J_j^B\right|}{\left|J_i^A \cap J_j^B\right|}.$$

Similar remarks hold for the cosets of A and B in the meet-class M. ∎

An immediate application is the following upgrading of Theorem 1.8 and consequent verification of Theorem 1.12.

Corollary 5.9. *If A and B are both finite, then so are J and M, with each of their orders dividing the product of the orders of A and B. In particular, if both A and B have p-power order for a common prime p, then J and M also have p-power order.* ∎

Does the partition of J by AB-coset intersections turn out to be precisely its partition by M-cosets, for $M = A \wedge B$? All that can be said in general is that M-cosets in J are just subsets of AB-coset intersections. We have, however, the following *geometric* interpretation of the symmetry condition.

Theorem 5.10. *If the skew lattice is symmetric, then the partition of J by AB-coset intersections is precisely the M-coset partition of J. Dually, the J-coset partition of M is the partition of M by AB-coset intersections in M. Conversely, when these coset relationships hold in $J = A \vee B$ and $M = A \wedge B$ for all pairs of $\mathcal{D}$-classes, A and B, then the skew lattice S is symmetric.* ■

This "coset geometry" proves useful in designing an example proving:

Theorem 5.11. *There exists an infinite skew lattice with just two generators.*

Proof. Consider the following symmetric right handed skew lattice S with four $\mathcal{D}$-classes: $A, B, J = A \vee B$ and $M = A \wedge B$. The operations restricted to each of the four classes is given trivially by $x \vee y = x$ and $x \wedge y = y$. The class members are given as follows. A and B are both copies of the set of integers

$$A = \{\ldots, -1_A, 0_A, 1_A, 2_A, \ldots\} \quad \text{and} \quad B = \{\ldots, -1_B, 0_B, 1_B, 2_B, \ldots\}$$

while $J = \{0_J, 1_J\}$ and $M = \{0_M, 1_M\}$. Operations between distinct classes are subject to the following constraints: (i) Both J and M are to be full cosets with respect to any $\mathcal{D}$-class. (ii) The natural partial ordering between classes is given by $0_J \geq 2n \geq 0_M$ and $1_j \geq 2n+1 \geq 1_M$ where $2n$ and $2n+1$ respectively denote even and odd members in either A or B. Because of these constraints we have: (iii) For $n \in \{0,1\}$, and any $x \in S$, $n_J \vee x = n_J$, but $x \vee n_J = (x \bmod 2)_J$. Likewise $n_M \wedge x = (x \bmod 2)_M$ and $x \wedge n_M = n_M$. (iv) Since $u \vee v \geq u \geq v \wedge u$ holds in any right handed skew lattice, we have in particular for x, y in distinct middle classes, A and B : $x \vee y = (x \bmod 2)_J$ and $x \wedge y = (y \bmod 2)_M$. It remains to describe the following cases for x in either A or B.

$$\begin{array}{llll} \text{For } x \in \{2n, 2n+1\}, & x \wedge 0_J = 2n & \text{and} & x \wedge 1_J = 2n+1, \\ \text{For } x \in \{2n-1, 2n\}, & 0_M \vee x = 2n & \text{and} & 1_M \vee x = 2n-1. \end{array}$$

Thus the J-cosets in both A and B are of the form $\{2n, 2n+1\}$ and the M-cosets are of the form $\{2n-1, 2n\}$. This allows a possible overlapping between J-cosets and M-cosets in both A and B that is crucial to the success of this example.

Claim: $S = \langle 0_A, 1_B \rangle$. Clearly 0_A and 1_B generate J and M. This leads to an expansion of intervals in A (and likewise in B) as follows: $[0,1] = 0_A \wedge J; [-1 \cdots 2] = M \vee [0,1]; [-2 \cdots 3] = [-1 \cdots 2] \wedge J$; and in general,

$$[-n \cdots n+1] = \begin{cases} M \vee [-n+1 \cdots n], & \text{for } n \text{ odd}, \\ [-n+1 \cdots n] \wedge J, & \text{for } n \text{ even}. \end{cases}$$

■

5.12. Notice that we have not attempted a detailed check of either the associativity of the two operations defined in 5.11, or their joint satisfaction of the absorption laws. By Theorem 5.5 these requirements are satisfied on all five possible maximal primitive subalgebras. That they hold all on S is a consequence of the system of cosets partitions and coset bijections. This coset geometry is the subject of [15] from which the results of this section is taken. The skew lattice S constructed above is, in fact, free on the generators 0_A and $1_{B'}$ as a right handed, symmetric skew lattice. By taking the fibered product of S with its left-handed dual S^{op} over their common maximal lattice image, one obtains (thanks to Theorem 1.6) a free symmetric skew lattice on two generators.

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Department of Mathematics
Westmont College
955 La Paz Road
Santa Barbara, CA 93108-1099
USA
email: leech@westmx.westmont.edu

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