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Probability that n Random Points Are in Convex Position*

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Abstract. We show that n random points chosen independently and uniformly from a parallelogram are in convex position with probability

$$\left(\frac{\binom{2n-2}{n-1}}{n!}\right)^2.$$

A finite set of points in the plane is called *convex* if its points are vertices of a convex polygon. In this paper we show the following result:

Theorem 1. *The set A of n random points chosen independently and uniformly from a parallelogram S is convex with probability*

$$\left(\frac{\binom{2n-2}{n-1}}{n!}\right)^2.$$

A large part of the studies in stochastic geometry deals with the convex hull C of a set of n points placed independently and uniformly in a fixed convex body K in $\mathbb{R}^d$.

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Typical questions are: How many vertices does C have? What is the volume of C? What is the surface area of C? See [WW] for a survey. In this paper we settle one very special case—the probability that C has n vertices in the case when K is a parallelogram. It is interesting that our approach is purely combinatorial, with no use of integration. We think that our method based on an approximation of the uniform distribution in a square by a large grid might have other applications. However, it is already unclear how to apply our method when K is a triangle or in three dimensions.

We now prove Theorem 1, and then we mention some applications of Theorem 1.

Proof of Theorem 1. Let $n > 2$ be a fixed integer. Since a proper affine transformation transfers the uniform distribution on S onto the uniform distribution on a square, we assume that S is a square. We approximate the square S by a grid whose size tends to infinity.

Let m be a positive integer (denoting the size of the grid). Partition the (axis-parallel) square S by $m - 1$ horizontal and by $m - 1$ vertical lines into m^2 squares $S_1, \dots, S_{m^2}$ of equal size. The centers of the squares $S_1, \dots, S_{m^2}$ form a square grid $m \times m$. Every point of A lies in each of the squares $S_1, \dots, S_{m^2}$ with the same probability $1/m^2$. Move every point of A to the center of the square S_i in which it lies, and denote the obtained multiset by $A(m)$. It is not difficult to see that

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \text{Prob}(A(m) \text{ is convex}).$$

Thus,

$$\text{Prob}(A \text{ is convex}) = \lim_{m \to \infty} \text{Prob}(R_m \text{ is convex}),$$

where, for every $m \geq 1$, R_m is a multiset of n points chosen randomly and independently from the square grid $G_m = \{(i, j): i, j = 1, 2, \dots, m\}$ (each point of G_m is always taken with the same probability $1/m^2$).

Let $\mathscr{M}(G_m)$ be the set of all multisets of size n with elements from G_m, and let $\mathscr{C}(G_m)$ be the set of all convex n-element subsets of G_m. It is easy to see that

$$\begin{aligned}\text{Prob}(A \text{ is convex}) &= \lim_{m \to \infty} \text{Prob}(R_m \text{ is convex}) \\ &= \lim_{m \to \infty} \frac{|\mathscr{C}(G_m)|}{|\mathscr{M}(G_m)|} = \lim_{m \to \infty} \frac{|\mathscr{C}(G_m)|}{\binom{m^2}{n}}.\end{aligned}$$

In what follows we estimate the size of $\mathscr{C}(G_m)$.

Every convex set $R \in \mathscr{C}(G_m)$ is uniquely defined by the smallest axis-parallel rectangle $Q(R)$ containing R and by the set $V(R)$ of the n integer vectors forming the boundary of the convex hull of R oriented in counterclockwise order.

Let $X(R)$ and $Y(R)$ be the multisets of the first and second coordinates of vectors in $V(R)$, respectively. Formally,

$$X(R) = \bigcup_{(x,y)\in V(R)} \{x\}, \qquad Y(R) = \bigcup_{(x,y)\in V(R)} \{y\}.$$

Let $\mathscr{C}'(G_m)$ be the set of all convex sets $R \in \mathscr{C}(G_m)$ such that $0 \notin X(R) \cup Y(R)$ and that the directions of the n^2 vectors (x, y) formed by all the n^2 pairs $x \in X(R)$, $y \in Y(R)$ are distinct. Thus, in particular, the multisets $X(R)$ and $Y(R)$ are sets for any $R \in \mathscr{C}'(G_m)$. It is not difficult to see that

$$\lim_{m\to\infty} \frac{|\mathscr{C}'(G_m)|}{|\mathscr{C}(G_m)|} = 1.$$

Therefore,

$$\text{Prob}(A \text{ is convex}) = \lim_{m\to\infty} \frac{|\mathscr{C}(G_m)|}{\binom{m^2}{n}} = \lim_{m\to\infty} \frac{|\mathscr{C}'(G_m)|}{\binom{m^2}{n}}.$$

In the estimation of the size of $\mathscr{C}'(G_m)$ we use an auxiliary set $\mathscr{S}$ defined by

$$\mathscr{S} = \{(X(R), Y(R), Q(R)) \colon R \in \mathscr{C}'(G_m)\}.$$

The following construction shows that, for every $(X, Y, Q) \in \mathscr{S}$, there are exactly $n!$ sets $R \in \mathscr{C}'(G_m)$ with $(X(R), Y(R), Q(R)) = (X, Y, Q)$:

Take any of the $n!$ one-to-one correspondences $f\colon X \to Y$ between X and Y, and define a set V of n vectors by $V = \{(x, f(x))\colon x \in X\}$. Due to the definitions of $\mathscr{C}'(G_m)$ and $\mathscr{S}$, vectors in V have distinct directions and, consequently, form the (counterclockwise oriented) boundary of the convex hull of a unique set $R \in \mathscr{C}'(G_m)$ fitting into the rectangle Q.

Thus,

$$|\mathscr{C}'(G_m)| = n! \cdot |\mathscr{S}|$$

and

$$\text{Prob}(A \text{ is convex}) = \lim_{m\to\infty} \frac{|\mathscr{C}'(G_m)|}{\binom{m^2}{n}} = \lim_{m\to\infty} \frac{n! \cdot |\mathscr{S}|}{\binom{m^2}{n}}.$$

It remains to estimate the size of the set $\mathscr{S}$ which is done in the following technical part of the proof.

For $(X, Y, Q) \in \mathscr{S}$, partition each of the two sets X and Y into two subsets containing elements with the same sign:

$$X^+ = \{x \in X\colon x > 0\}, \qquad X^- = \{x \in X\colon x < 0\},$$

$$Y^+ = \{y \in Y\colon y > 0\}, \qquad Y^- = \{y \in Y\colon y < 0\}.$$

Suppose that each of the sets X^+, X^-, Y^+, Y^- is ordered in an arbitrary way. Denote $s = |X^+|$ and $t = |Y^+|$. Thus,

$$X^+ = \{x_1, \dots, x_s\}, \qquad X^- = \{x_{s+1}, \dots, x_n\},$$

$$Y^+ = \{y_1, \dots, y_t\}, \qquad Y^- = \{y_{t+1}, \dots, y_n\}.$$

For every $(X, Y, Q) \in \mathscr{S}$, where $Q = \{(x, y)\colon\ a_1 \le x \le a_2,\ b_1 \le y \le b_2\}$, the orders on the sets X^+, X^-, Y^+, Y^- uniquely determine four sets D^-, E^-, D^+, E^+ of integers from the set $\{1, 2, \dots, m\}$ in the following way:

$$D^+ = \left\{a_1 + \sum_{i=1}^{k} x_i : k = 0, 1, \dots, s\right\}, \qquad D^- = \left\{a_2 + \sum_{i=s+1}^{k} x_i : k = s, s+1, \dots, n\right\},$$

$$E^+ = \left\{b_1 + \sum_{i=1}^{k} y_i : k = 0, 1, \dots, t\right\}, \qquad E^- = \left\{b_2 + \sum_{i=t+1}^{k} x_i : k = t, t+1, \dots, n\right\}.$$

Note that the sets D^-, E^-, D^+, E^+ satisfy the following conditions:

$$|D^+| + |D^-| = n + 2, \qquad a_1 = \min D^+ = \min D^-, \qquad a_2 = \max D^+ = \max D^-, \tag{1}$$

$$|E^+| + |E^-| = n + 2, \qquad b_1 = \min E^+ = \min E^-, \qquad b_2 = \max E^+ = \max E^-. \tag{2}$$

For any $(X, Y, Q) \in \mathscr{S}$, we obtain $|X^+|!\,|X^-|!\,|Y^+|!\,|Y^-|!$ different 4-tuples of sets D^-, E^-, D^+, E^+ corresponding to different orders on the sets X^+, X^-, Y^+, Y^-. Denote the set of all these 4-tuples (D^-, E^-, D^+, E^+) by $\mathscr{T}(X, Y, Q)$. Thus,

$$\begin{aligned} |\mathscr{T}(X, Y, Q)| &= |X^+|!\,|X^-|!\,|Y^+|!\,|Y^-|! \\ &= (|D^+| - 1)!\,(|D^-| - 1)!\,(|E^+| - 1)!\,(|E^-| - 1)!, \end{aligned}$$

where (D^-, E^-, D^+, E^+) is an arbitrary 4-tuple in $\mathscr{T}(X, Y, Q)$. For $0 \le i \le n - 2$ and $0 \le j \le n - 2$, we say that a 4-tuple (D^-, E^-, D^+, E^+) of sets of integers has property $\mathscr{P}_{i,j}$ if

($\mathscr{P}_{i,j}$) $|D^+| = i + 2$, $|E^+| = j + 2$, and the sets D^-, E^-, D^+, E^+ satisfy (1) and (2) for some $1 \le a_1 < a_2 \le m$ and $1 \le b_1 < b_2 \le m$.

There are

$$\binom{n-2}{i}\binom{m}{n} \cdot \binom{n-2}{j}\binom{m}{n}$$

4-tuples (D^-, E^-, D^+, E^+) with $\mathscr{P}_{i,j}$ and $|D^+ \cap D^-| = |E^+ \cap E^-| = 2$. It follows that there are

$$(1 + o(1)) \cdot \binom{n-2}{i}\binom{m}{n} \cdot \binom{n-2}{j}\binom{m}{n}$$

4-tuples (D^-, E^-, D^+, E^+) with $\mathscr{P}_{i,j}$. (Throughout the proof, $o(1)$ denotes functions of m which tend to 0 as m tends to infinity.) Most of them (i.e., a $(1 - o(1))$-fraction of them) lie in the disjoint union

$$\bigcup_{(X,Y,Q)\in\mathscr{S}} \mathscr{F}(X,Y,Q).$$

Thus,

$$|\mathscr{S}| = \sum_{(X,Y,Q)\in\mathscr{S}} 1 = \sum_{(X,Y,Q)\in\mathscr{S}} \frac{|\mathscr{F}(X,Y,Q)|}{|X^+|!\,|X^-|!\,|Y^+|!\,|Y^-|!}$$

$$= \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} \frac{(1 - o(1)) \cdot (1 + o(1))\binom{n-2}{i}\binom{m}{n}\binom{n-2}{j}\binom{m}{n}}{(i+1)!\,(n-i-1)!\,(j+1)!\,(n-j-1)!}$$

$$= (1 + o(1))\binom{m}{n}^2 \cdot \sum_{i=0}^{n-2}\sum_{j=0}^{n-2} \frac{\binom{n}{n-i-1} \cdot \binom{n}{n-j-1}}{(n!)^2}\binom{n-2}{i}\binom{n-2}{j}$$

$$= (1 + o(1))\binom{m}{n}^2 \frac{1}{(n!)^2}\left(\sum_{i=0}^{n-2}\binom{n}{n-i-1}\binom{n-2}{i}\right)$$

$$\times\left(\sum_{j=0}^{n-2}\binom{n}{n-j-1}\binom{n-2}{j}\right)$$

$$= (1 + o(1))\binom{m}{n}^2 \frac{1}{(n!)^2}\binom{2n-2}{n-1}^2.$$

Hence,

$$\operatorname{Prob}(A \text{ is convex}) = \lim_{m\to\infty} \frac{n!\cdot|\mathscr{S}|}{\binom{m^2}{n}} = \lim_{m\to\infty} \frac{(1 + o(1))\binom{m}{n}^2 \frac{1}{n!}\binom{2n-2}{n-1}^2}{\binom{m^2}{n}}$$

$$= \left(\frac{\binom{2n-2}{n-1}}{n!}\right)^2. \qquad \square$$

Now we briefly sketch some applications of Theorem 1. More information about applications can be found in the preliminary version [V] of this paper.

If K is a bounded convex body in the plane, then, applying proper approximations of K by parallelograms, Theorem 1 implies that there are two positive constants c_1 and c_2 such that the set of n points chosen independently and uniformly from K is convex with probability at least $(c_1/n)^n$ and at most $(c_2/n)^n$.

It is not difficult to show that

$$\text{Prob}(A \text{ is convex}) + 4 \cdot E[\text{Area of } T] = 1,$$

where A is a set of four random points selected independently and uniformly from a convex body S of area 1, and T is a triangle with random vertices also selected independently and uniformly from S. If S is a parallelogram, Theorem 1 yields that the expected area of T is

$$\frac{1 - (\frac{5}{6})^2}{4} = \frac{11}{144},$$

which was also shown in [H] by a different method.

Let A be a set of n random points chosen independently and uniformly from a parallelogram. Theorem 1 yields that, for any $\lambda \geq 0$, the size of the largest convex subset of A is greater than $\lambda n^{1/3}$ with probability smaller than $(2^{4/3}e/\lambda)^{3\lambda n^{1/3}}$, which was shown in a somewhat stronger form by Talagrand.

Welzl (personal communication) pointed out that the above proof of Theorem 1 yields a fast way to construct a random convex set of size n in a square. The proof also shows that such a random set has a limit shape represented by the curve

$$\left\{(x, y) \colon \sqrt{1 - |x|} + \sqrt{1 - |y|} = 1\right\}$$

(see [B] for other limit-shape theorems).

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Note added in proof. The author has very recently proved that n random points chosen independently and uniformly from a triangle are in convex position with probability

$$\frac{2^n(3n-3)!}{((n-1)!)^3(2n)!}.$$

The proof will appear elsewhere.