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Embedding a Polytope in a Lattice

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Abstract. We present a special similarity of $\mathbf{R}^{4n}$ which maps lattice points into lattice points. Applying this similarity, we prove that if a $(4n-1)$-polytope is similar to a lattice polytope (a polytope whose vertices are all lattice points) in $\mathbf{R}^{4n}$, then it is similar to a lattice polytope in $\mathbf{R}^{4n-1}$, generalizing a result of Schoenberg [4]. We also prove that an n-polytope is similar to a lattice polytope in some $\mathbf{R}^N$ if and only if it is similar to a lattice polytope in $\mathbf{R}^{2n+1}$, and if and only if $\sin^2(\angle ABC)$ is rational for any three vertices A, B, C of the polytope.

1. Introduction

Let $Z^n, \mathbf{Q}^n$ denote the subsets of Euclidean n-space $\mathbf{R}^n$ consisting of all lattice points, and all rational points, respectively. In the following a *point-set* means a subset of a Euclidean space. The dimension of a point-set X, $\dim(X)$, means the dimension of the convex hull of X. A point-set X is said to be *embeddable* in Z^n (or $\mathbf{Q}^n$) if X is congruent to a subset of Z^n (or $\mathbf{Q}^n$). If X is similar to a subset of Z^n (or $\mathbf{Q}^n$), then X is *similarly embeddable* (*s-embeddable*) in Z^n (or in $\mathbf{Q}^n$). A polytope is said to be embeddable (or s-embeddable) in Z^n ($\mathbf{Q}^n$) if its vertex-set is. A polytope with vertices in a Z^n is called a *lattice polytope*.

It was proved in [1] that any triangle embeddable in Z^4 is s-embeddable in Z^3, and every lattice triangle is s-embeddable in Z^5. In this paper we present a special similarity of $\mathbf{R}^{4n}$ which maps Z^{4n} into Z^{4n}, and, by applying this similarity, we show the following. Every sublattice Λ of Z^{4n} of dimension $< 4n$ is s-embeddable in Z^{4n-1}. Hence, for example, any 3-polytope in Z^4 is s-embeddable in Z^4.

It is well known that a finite metric space $\{p_1, \ldots, p_n\}$ is isometrically embeddable in Euclidean space if and only if the $n \times n$ matrix (D_{ij}) is of negative-type (where $D_{ij} = d(p_i, p_j)^2$), that is, for any real numbers $v_1, \ldots, v_n$,

$$\sum_i v_i = 0 \quad \Rightarrow \quad \sum_{i,j} D_{ij} v_i v_j \leq 0,$$

see, e.g., [3]. Then, under what condition is a point-set embeddable in a $\mathbf{Q}^N$? We prove that a point-set X is embeddable in a $\mathbf{Q}^N$ for some N if and only if the square-distances among the points in X are all rationals. If X is embeddable in a $\mathbf{Q}^N$, then it is embeddable in $\mathbf{Q}^{3n+1}$, where $n = \dim(X)$. For s-embedding, we can reduce the dimension $3n + 1$ to $2n + 1$. That is, if X is finite and the square-distances in X are all rationals, then X is s-embeddable in Z^{2n+1}.

2. A Special Similarity of $\mathbf{R}^{4n}$

Lemma 1. *For any point $P \in Z^{4n}$, $P \neq (0, \ldots, 0)$, a similarity ψ: $\mathbf{R}^{4n} \to \mathbf{R}^{4n}$ fixing the origin exists such that $\psi(Z^{4n}) \subset Z^{4n}$ and*

$$\psi(P) = (*, 0, \ldots, 0).$$

Proof. (1) First, consider the four-dimensional case. Let us denote by $[[x, y, z, w]]$ the matrix

$$\begin{pmatrix} x & -y & z & w \\ y & x & -w & z \\ z & -w & -x & -y \\ w & z & y & -x \end{pmatrix}.$$

The column-vectors of this matrix are of the same length and mutually orthogonal. Hence, if $x^2 + y^2 + z^2 + w^2 \neq 0$, then the linear transformation defined by $[[x, y, z, w]]$ is a similarity.

Now, let $P = (a, b, c, d) \in Z^4$ and let ψ: $\mathbf{R}^4 \to \mathbf{R}^4$ be the linear transformation defined by

$$(x_1, x_2, x_3, x_4) \to (x_1, x_2, x_3, x_4) \cdot [[a, b, c, d]].$$

Then ψ is a similarity, and, since a, b, c, d are all integers, ψ maps Z^4 into Z^4. Further, $\psi(P) = (m, 0, 0, 0)$ with $m = a^2 + b^2 + c^2 + d^2$.

(2) Next, the eight-dimensional case. Let $P = (*, \ldots, *, a, b, c, d)$. By switching coordinates (with an orthogonal transformation) if necessary, we may suppose that $a^2 + b^2 + c^2 + d^2 \neq 0$. Then the linear transformation α of $\mathbf{R}^8$ defined by the matrix

$$\begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{A} \end{pmatrix}$$

($\mathbf{A} = [[a, b, c, d]]$) is a similarity of $\mathbf{R}^8$, and $\alpha(P)$ becomes

$$(p, q, r, s, m, 0, 0, 0) \in Z^8.$$

If $p = q = r = s = 0$, then (by switching the first and the fifth coordinates) we are done. If one of p, q, r, s is not zero, then apply further the linear transformation β of $\mathbf{R}^8$ determined by the 8×8 matrix

$$\begin{pmatrix} \mathbf{B} & -m\mathbf{I} \\ m\mathbf{I} & \mathbf{B}^t \end{pmatrix},$$

where $\mathbf{B} = [[p, q, r, s]]$, $\mathbf{B}^t$ its transpose, and $\mathbf{I}$ is the 4×4 identity matrix. Since the column-vectors of this 8×8 matrix are mutually orthogonal and have the same length, β is a similarity. Further, $\beta\alpha(P) = (*, 0, \ldots, 0)$. Thus the composition $\psi = \beta\alpha$ is a desired similarity.

(3) Now, as a general case, let us consider the 12-dimensional case. (Other cases follow analogously.) Let $P = (a_1, \ldots, a_{12})$ be a lattice point in $\mathbf{R}^{12}$ different from O. By (2), there is an integral 8×8 matrix C which induces a similarity of $\mathbf{R}^8$ such that

$$(a_5, a_6, \ldots, a_{12}) \cdot \mathbf{C} = (*, 0, \ldots, 0).$$

Let λ be the square-length of a column-vector of $\mathbf{C}$. Then, by Lagrange's four squares theorem, there are four integers x, y, z, w such that $\lambda = x^2 + y^2 + z^2 + w^2$. Let $\mathbf{D} = [[x, y, z, w]]$ and consider the linear transformation γ defined by the matrix

$$\begin{bmatrix} \mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{C} \end{bmatrix}.$$

The column-vectors of this matrix are of the same length and mutually orthogonal. Hence γ is a similarity of $\mathbf{R}^{12}$. If $a_1 = a_2 = a_3 = a_4 = 0$, then we have $\gamma(P) = (0, 0, 0, 0, *, 0, \ldots, 0)$ and we are done. Otherwise, $\gamma(P) = (*, *, *, *, *, 0, \ldots, 0)$. Then switch coordinates by an orthogonal transformation with 0, 1 entries, and apply a similar procedure. The composition of the applied transformations gives the desired similarity. □

Remark. If $n \neq 2$ and $n \not\equiv 0 \pmod 4$, then there is no analogue of Lemma 1 in dimension n. This can be seen as follows. For $n = 3$, this can be checked directly by taking $P = (1, 1, 1)$. For $n > 4$, we apply the following result proved by van Lint and Seidel [5]. (See the proof of Theorem 5.2 in [5].)

If $\mathbf{M}$ is an $n \times n$ matrix with rational entries such that $\mathbf{M} \cdot \mathbf{M}^t = m\mathbf{I}$ (m: integer), then there is a rational $(n-4) \times (n-4)$ matrix $\mathbf{L}$ such that $\mathbf{L} \cdot \mathbf{L}^t = m\mathbf{I}$. □

Suppose that, for $n = 4k + r$ $(0 < r < 4)$, there is an integral $n \times n$ matrix $\mathbf{M}$ such that $\mathbf{M} \cdot \mathbf{M}^t = m\mathbf{I}$ and $(1, 1, 1, 2, 0, \ldots, 0) \cdot \mathbf{M} = (*, 0, \ldots, 0)$. Then $m = c^2 \cdot 7$ for some integer c. By repeating the above result, we come to a rational $r \times r$ matrix $\mathbf{L}$ such that $\mathbf{L} \cdot \mathbf{L}^t = m\mathbf{I}$. This is, however, impossible since $c^2 \cdot 7$ cannot be expressed as a sum of r squares of rationals, by the *three-square theorem* of Legendre:

A positive integer N can be expressed as a sum of three integral squares if and only if N is not of the form $4^j M$ with $M \equiv 7 \pmod 8$.

For a proof of three-square theorem, see, e.g., p. 161 of [2].

Theorem 1. *Every* $(4n-1)$*-dimensional sublattice* Λ *of* Z^{4n} *is s-embeddable in* Z^{4n-1}.

Proof. Suppose that Λ is generated by $P_1, \ldots, P_{4n-1}$. By solving the simultaneous linear equations with integral coefficients

$$\overrightarrow{OP} \cdot \overrightarrow{OP_i} = 0\,, \qquad i = 1, \ldots, 4n-1,$$

on $P = (x_1, \ldots, x_{4n})$, we can find a lattice point $P \in Z^{4n}$, $P \neq O$. Now, by Lemma 1, there is a similarity ψ of R^{4n} which maps Z^{4n} into Z^{4n}, $\psi(O) = O$ and $\psi(P) = (*, 0, \ldots, 0)$. Then, since

$$\overrightarrow{O\psi(P)} \cdot \overrightarrow{O\psi(P_i)} = 0, \qquad i = 1, \ldots, 4n-1,$$

the first coordinates of $\psi(P_i)$ must be zero. Hence $\psi(\Lambda)$ is congruent to a subset of Z^{4n-1}. □

Corollary 1. *If a polytope of dimension* $< 4n$ *is s-embeddable in* Z^{4n}, *then it is s-embeddable in* Z^{4n-1}.

Since a regular n-simplex is embeddable in Z^{n+1}, we have the following.

Corollary 2 [4]. *For* $n \equiv 3 \pmod 4$, *a regular* n*-simplex is always embeddable in* Z^n.

Schoenberg proved this result by applying Minkowski's theory of rational equivalence of quadratic forms. He completely determined those dimensions n for which a regular n-simplex is embeddable in Z^n: For even n, the embedding is possible if and only if $n+1$ is a perfect square; for $n \equiv 1 \pmod 4$, if and only if $n+1$ is a sum of two squares, and for $n \equiv 3 \pmod 4$, it is always possible.

Corollary 3 [1]. *Let* $\Theta_n = \{\theta\colon \theta = \angle ABC,\ \text{for } A, B, C \in Z^n\}$. *Then*

(1) $\Theta_3 = \Theta_4$ *and*
(2) $\theta \in \Theta_4$

if and only if $\tan^2\theta = \infty$ *or* $= (b^2 + c^2 + d^2)/a^2$ $(a, b, c, d \in Z)$.

Proof. (1) and the "if" part of (2) is clear. So, we show that if $\theta \in \Theta_4$, $\theta \neq 90°$, then $\tan^2\theta = (b^2 + c^2 + d^2)/a^2$ $(a, b, c, d \in Z)$. Let $\theta = \angle AOB$ $(A, B \in Z^4)$. By Lemma 1, there is a similarity ψ of $\mathbf{R}^4$ such that $\psi(Z^4) \subset Z^4$, $\psi(O) = O$ and $\psi(B) = (m, 0, 0, 0)$. Let $\psi(A) = (a, b, c, d)$. Then the point $F = (a, 0, 0, 0)$ is the foot of the perpendicular from $\psi(A)$ to the line $O\psi(B)$. Hence $\tan^2\theta = (b^2 + c^2 + d^2)/a^2$. □

It is known that $\Theta_2 \subsetneq \Theta_3 = \Theta_4 \subsetneq \Theta_5 = \Theta_6 = \ldots, \theta \in \Theta_2$ if and only if $\tan\theta$ is a rational or ∞, and $\theta \in \Theta_5$ if and only if $\tan^2\theta$ is a rational or ∞, see [1].

3. A Few More Lemmas

The length of a line segment AB is denoted by $|AB|$.

Lemma 2. *Let $Y_0, Y_1, \dots, Y_n$ be $n+1$ rational points which span an n-simplex, and let P be an affine combination of the Y_i's, that is,*

$$P = x_0 Y_0 + x_1 Y_1 + \cdots + x_n Y_n \qquad (x_0 + x_1 + \cdots + x_n = 1).$$

If $|Y_i P|^2 \in \mathbf{Q}$, $i = 0, 1, \dots, n$, then P is a rational point.

Proof. To prove the lemma, we may suppose that $Y_0 = O$, the origin. Then $P = x_1 Y_1 + \cdots + x_n Y_n$, and

$$\overrightarrow{OP} \cdot \overrightarrow{OY_i} = x_1 \overrightarrow{OY_1} \cdot \overrightarrow{OY_i} + \cdots + x_n \overrightarrow{OY_n} \cdot \overrightarrow{OY_i} \qquad (i = 1, \dots, n). \qquad (*)$$

Note that $\overrightarrow{OP} \cdot \overrightarrow{OY_i} = (|OP|^2 + |OY_i|^2 - |Y_i P|^2)/2$ $(i = 1, \dots, n)$ are all rationals. Now, let us regard $(*)$ as a system of linear equations on $x_1, \dots, x_n$. Then, since the coefficients are all rationals, $x_1, \dots, x_n$ must all be rationals. Hence P is a rational point. □

Lemma 3. *For any two integers $a, b > 0$, five integers $x \neq 0, y, u, v, w$ exist such that $ax^2 - by^2 = u^2 + v^2 + w^2$.*

Proof. By the three-square theorem of Legendre, it is enough to show that two integers $x \neq 0$, y exist such that $ax^2 - by^2$ is positive and not of the form $4^j(8k+7)$. If a is not of the form $4^j(8k+7)$, then we may put $x = 1$, $y = 0$. So, suppose that $a = 4^j(8k+7)$. Let $b = 4^i c$, $c \not\equiv 0 \pmod 4$. Choose m so that $(2m+1)^2 > 4c$.

If c is even, then put $x = 2^i(2m+1)$, $y = 2^j$. Then

$$ax^2 - by^2 = 4^{j+i}((2m+1)^2(8k+7) - c)$$

and $(2m+1)^2(8k+7) - c$ is odd. Since $(2m+1)^2(8k+7) \equiv 7 \pmod 8$ and $c \not\equiv 0 \pmod 8$,

$$(2m+1)^2(8k+7) - c \not\equiv 7 \pmod 8.$$

If c is odd, then put $x = 2^i(2m+1)$, $y = 2^{j+1}$. Then

$$ax^2 - by^2 = 4^{j+i}((2m+1)^2(8k+7) - 4c)$$

and similarly we have

$$(2m+1)^2(8k+7) - 4c \not\equiv 7 \pmod 8. \qquad \square$$

Corollary 4. *For any two rationals $a, b > 0$, there are five rationals $x \neq 0, y, u, v, w$ such that $ax^2 - by^2 = u^2 + v^2 + w^2$.*

For an n-simplex Σ, $|\Sigma|$ denotes its content (i.e., the n-dimensional volume). If $X_0, X_1, \ldots, X_n$ are the vertices of Σ, then

$$|\Sigma|^2 = \frac{\det(a_{ij})}{n!}, \qquad \text{where} \quad a_{ij} = X_0X_i \cdot X_0X_j.$$

Therefore, if $|X_iX_j|^2$ $(i, j = 0, 1, \ldots, n)$ are all rationals, then $|\Sigma|^2$ is also a rational. A *facet* of a simplex is a maximal proper face.

Lemma 4. *Let Σ be a simplex such that $|AB|^2 \in \mathbf{Q}$ for every edge AB. Suppose that a facet Δ is embeddable in $\mathbf{Q}^n$, $n >$ $\dim(\Delta)$. Then:*

(1) *Σ is embeddable in $\mathbf{Q}^{n+3}$.*
(2) *If n is odd, then Σ is s-embeddable in $\mathbf{Q}^{n+2}$.*

Proof. (1) We may suppose that $\Sigma \subset R^{n+3}$ and $\Delta \subset \mathbf{Q}^n \times \{(0,0,0)\}$. Let P be the opposite vertex of Δ. Since the squares of edge-lengths are all rationals, $|\Sigma|^2$ and $|\Delta|^2$ are rationals. Let $X_1, \ldots, X_k$ be the vertices of Δ, and let F be the foot of the perpendicular from P to the flat $L(\Delta)$ spanned by Δ. Then F is represented by an affine combination of $X_1, \ldots, X_k$. Since

$$|\Sigma| = \frac{1}{k}|\Delta| \cdot |PF|,$$

we have $|PF|^2 \in \mathbf{Q}$. Hence, by the Pythagorean theorem, $|X_iF|^2$ are all rationals for $i = 1, \ldots, k$, and hence, by Lemma 2, F is a rational point. Since $n > \dim(\Delta)$ and the vertices of Δ are all rational points, there is a rational point $Q = (*, \ldots, *, 0, 0, 0)$ such that OQ is perpendicular to the flat $L(\Delta)$. Let $|PF|^2 = a$ and $|OQ|^2 = b$ $(a, b \in \mathbf{Q})$. Then, by Corollary 4, there are rationals $x \neq 0, y, u, v, w$ such that

$$ax^2 - by^2 = u^2 + v^2 + w^2.$$

Then

$$a = \left(\frac{y}{x}\right)^2 b + \left(\frac{u}{x}\right)^2 + \left(\frac{v}{x}\right)^2 + \left(\frac{w}{x}\right)^2.$$

Let $P' = F + (y/x)Q + (0, \ldots, 0, u/x, v/x, w/x)$. Then $P' \in \mathbf{Q}^{n+3}$, $\overrightarrow{FP'}$ is perpendicular to $L(\Delta)$, and $|P'F|^2 = |PF|^2$. Hence the convex hull of $\Delta \cup \{P'\}$ is congruent to Σ.

(2) Note that in the above congruent embedding of Σ in $\mathbf{Q}^{n+3}$, points X_i and P' are of the following form:

$$X_i = (*, \ldots, *, 0, 0, 0) \qquad (i = 1, \ldots, k),$$

$$P' = (*, \ldots, *, p, q, r).$$

Now, consider the linear transformation $\psi\colon \mathbf{R}^{n+3} \to \mathbf{R}^{n+3}$ defined by $(x_1, \ldots, x_{n+3}) \to (x_1, \ldots, x_{n+3}) \cdot \mathbf{M}$, where $\mathbf{M}$ is the matrix

$$\begin{pmatrix} \begin{bmatrix} q & -r \\ r & q \end{bmatrix} & & 0 \\ & \ddots & \\ 0 & & \begin{bmatrix} q & -r \\ r & q \end{bmatrix} \end{pmatrix}.$$

Then $\psi(X_i) = (*, \ldots, *, *, 0, 0)$ and $\psi(P') = (*, \ldots, *, *, *, 0)$. Hence, neglecting the last coordinate, we can see $\psi(\Delta \cup \{P'\}) \subset \mathbf{Q}^{n+2}$. □

Remark. If a facet Δ of a simplex Σ is embeddable in $\mathbf{Q}^{4m}$, and $|AB|^2 \in \mathbf{Q}$ for every edge AB of Σ, then Σ is s-embeddable in $\mathbf{Q}^{4m+1}$.

4. Embeddings in $\mathbf{Q}^n$ and Z^n

Theorem 2. *Let Σ be an n-simplex such that $|AB|^2 \in \mathbf{Q}$ for every edge AB. Then*:

(1) *Σ is embeddable in $\mathbf{Q}^{3n+1}$.*
(2) *Σ is s-embeddable in Z^{2n+1}.*

Proof. Since a point (0-simplex) is embeddable in $\mathbf{Q}^1$, it follows by induction on n and Lemma 4 that Σ is embeddable in $\mathbf{Q}^{3n+1}$ and s-embeddable in $\mathbf{Q}^{2n+1}$. Thus, by dilating suitably, Σ is s-embeddable in Z^{2n+1}. □

Problem 1. Is there a triangle which is embeddable in $\mathbf{Q}^7$ but not embeddable in $\mathbf{Q}^6$?

There is a triangle which is s-embeddable in $Z^5 = Z^{2\cdot 2+1}$ but not s-embeddable in Z^4. For example, the triangle with side-lengths $1, \sqrt{7}, \sqrt{8}$ is such a one. (By Corollary 3, $\arctan\sqrt{7}$ does not belong to Θ_4.) By Theorem 2, any lattice tetrahedron is s-embeddable in Z^7.

Problem 2. Is there a lattice tetrahedron which is not s-embeddable in Z^6?

Theorem 3. *For a point-set X of dimension n, the following three conditions are equivalent*:

(1) *X is embeddable in $\mathbf{Q}^{3n+1}$.*
(2) *X is embeddable in a $\mathbf{Q}^N$ for some N.*
(3) *For any $A, B \in X$, $|AB|^2 \in \mathbf{Q}$.*

Proof. (1) ⇒ (2) ⇒ (3) is obvious. So, we show (3) ⇒ (1). Since $\dim(X) = n$, X contains (the vertex set of) an n-simplex Σ. By Theorem 2, there is an embedding $\Sigma \to \mathbf{Q}^{3n+1}$. This embedding can be extended to an embedding $X \to \mathbf{R}^{3n+1}$. Then, by (3) and Lemma 2, all points of X are automatically sent to rational points. □

Theorem 4. *For a finite point-set X of dimension $n \geq 2$, the following three conditions are equivalent*:

(1) *X is s-embeddable in Z^{2n+1}.*
(2) *X is s-embeddable in a Z^N for some N.*
(3) *For any $A, B, C \in X$, $\sin^2(\angle ABC)$ is a rational.*

Remark. Suppose that three points A, B, C are collinear, and $|AB| = 1$, $|BC| = \pi$. Then, though $X = \{A, B, C\}$ satisfies (3), it is not s-embeddable in Z^N. The restriction $\dim(X) \geq 2$ in Theorem 4 excludes such cases.

Proof. (1) $\Rightarrow$ (2) is clear. To see (2) $\Rightarrow$ (3), suppose that X is a subset of some Z^N. Let A, B, C be three points of X. If A, B, C are collinear, then $\sin^2(\angle ABC) = 0$. Suppose that ABC forms a triangle. Let F be the foot of perpendicular from A to the line BC. Then, since $|ABC| = |BC| \cdot |AF|/2$, $|AF|^2$ is a rational, and hence $\sin^2(\angle ABC) = |AF|^2/|AB|^2$ is a rational.

Now we show (3) $\Rightarrow$ (1). Since $\dim(X) = n$, X contains (the vertex set of) an n-simplex Σ. By dilating X, we may suppose that an edge AB of Σ has rational length. Then, for any two vertices C, D of Σ, $|CD|^2$ is rational. This can be seen as follows. Applying the law of sine to the triangle ABC, we have $|AB|/\sin C = |BC|/\sin A$. Hence $|BC|^2 = |AB|^2(\sin A/\sin C)^2$, which is a rational by (3). Similarly, from the triangle BCD, we have

$$|CD|^2 = |BC|^2\left(\frac{\sin B}{\sin D}\right)^2,$$

which is a rational.

Then, by Theorem 2, Σ is s-embeddable in Z^{2n+1}. Therefore, there is an injection φ: $X \to \mathbf{R}^{2n+1}$ such that $\varphi(X)$ is similar to X, and $\varphi(\Sigma)$ is a lattice simplex. Let Y be an arbitrary point of X, and let P be a vertex of Σ. We can choose a vertex Q of Σ so that P, Q, Y are not collinear. Then applying the law of sine to the triangle $\varphi(PQY)$, we have $|\varphi(PY)|^2 = |\varphi(PQ)|^2(\sin Q/\sin Y)^2$, which is a rational by (3). Thus the square-distances from $\varphi(Y)$ to the vertices $\varphi(P)$ of $\varphi(\Sigma)$ are all rationals. Hence $\varphi(Y)$ is a rational point by Lemma 2, and hence $\varphi(X)$ is a subset of $\mathbf{Q}^{2n+1}$. Now, since X is a finite set, we can dilate $\varphi(X)$ so that it becomes a subset of Z^{2n+1}. Therefore X is s-embeddable in Z^{2n+1}. □

References

1. M. J. Beeson, Triangles with vertices on lattice points, *Amer. Math. Monthly*, **99** (1992), 243–252.
2. W. Narkiewicz, *Classical Problems in Number Theory*, PWN—Polish Scientific Publishers, Warszawa, 1986.
3. I. J. Schoenberg, Remarks to Maurice Frechet's article ..., *Ann. Math.* **36** (1935), 64–70.
4. I. J. Schoenberg, Regular simplices and quadratic forms, *J. London Math. Soc.*, **12** (1937), 48–55.
5. J. H. van Lint and J. J. Seidel, Equilateral point sets in elliptic geometry, *Nederl. Akad. Wetensch. Indag. Math.* **28** (1926), 335–348.

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