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Regular Simplices and Gaussian Samples*

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Abstract. We show that if a suitable type of simplex in R^n is randomly rotated and its vertices projected onto a fixed subspace, they are as a point set affine-equivalent to a Gaussian sample in that subspace. Consequently, affine-invariant statistics behave the same for both mechanisms. In particular, the facet behavior for the convex hull is the same, as observed by Affentranger and Schneider; other results of theirs are translated into new results for the convex hulls of Gaussian samples. We show conversely that the conditions on the vertices of the simplex are necessary for this equivalence. Similar results hold for random *orthogonal* transformations.

1. Introduction

This note was motivated by an observation made by Affentranger and Schneider [1] in their consideration of a model of J. E. Goodman and R. Pollack for a random point set. The latter takes projections of a regular simplex onto a random subspace. We adopt an essentially equivalent formulation as follows: subject a regular simplex in R^n to a random rotation and then orthogonally project its vertices onto a fixed subspace of dimension $d < n$. In the course of their study, Affentranger and Schneider observe that the convex hull of the projected set shares a feature with the convex hull of a standard Gaussian sample in R^d [5]: the expected number of facets (i.e., $(d-1)$-dimensional faces) is the same for both mechanisms. Here we show that this observation can be extended to provide a full characterization of the Goodman–Pollack model (Theorem 1).

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First we set some definitions and conventions. Recall that a *regular simplex* in R^n is the convex hull of a vertex set $\{v_i\}_{i=1}^{m+1}$, $1 \leq m \leq n$, where $\|v_i - v_j\| > 0$ is the same value for all $i \neq j$ [2, p. 121]. A regular simplex whose vertices lie on a sphere centered at the origin we call an *SR-simplex* (*S̲pherico-R̲egular*). It has

$$\|v_i\| = \|v_j\| > 0 \qquad \text{for all } i \text{ and } j, \tag{1}$$

$$(v_i, v_j) = (v_i, v_k) \qquad \text{for all } j,\ k \neq i. \tag{2}$$

Expanding $\|\sum v_i\|^2 \leq \sum \|v_i\|^2$ shows that the inner product in (2) is nonpositive ($=0 \Leftrightarrow$ the v_i's are orthogonal). The simplex is *centered* if $x = v_1 + v_2 + \cdots + v_{m+1} = 0$ and *normalized* if $\|v_i\| = 1$ for all i. We freely identify linear transformations with their associated matrix representations and shorten "orthogonal projection" to "projection."

A standard Gaussian random variable has density $(1/\sqrt{2\pi})e^{-(1/2)x^2}$, $-\infty < x < +\infty$. Independent copies provide the components of a standard Gaussian random *vector*, of which independent copies in turn compose a standard Gaussian *sample*. For properties of the Gaussian distribution, see [6]. A *random rotation* of R^n is a stochastic choice from the group $\mathscr{R}_n$ of rotations under normalized Haar measure; similarly, a *random orthogonal transformation* is a stochastic choice from the orthogonal group $\mathscr{O}_n \supset \mathscr{R}_n$ under its normalized Haar measure. Finally, *we assume throughout that underlying probability spaces are sufficiently rich as to include copies of all random variables that are needed.*

The Goodman–Pollack model treated by Affentranger and Schneider deals with projections of a regular simplex whose location is arbitrary. Without loss of generality, the simplex may be suitably translated so that it becomes an SR-simplex. With this convention, our main result characterizes the Goodman–Pollack model.

Theorem 1. *Suppose that an SR-simplex in R^n is randomly rotated and its vertices projected onto a fixed subspace. Up to an affine transformation, the resulting point set coincides in distribution with a standard Gaussian sample in that subspace. The affine transformation can be taken so that*

(i) *it is orientation-preserving and*
(ii) *its linear and translational parts and the random rotation are three mutually stochastically independent actions.*

Conversely, only *the vertices of an SR-simplex have this property.*

The following corollary, which contains the observation of Affentranger and Schneider, is immediate.

Corollary 1. *An affine-invariant functional of a point set follows the same distribution for the Goodman–Pollack model and a standard Gaussian sample.*

We turn to the proof of Theorem 1 and a closely related result in the next section. Remarks appear in Section 3. In the last section we briefly consider a related model for a random point set.

2. Random Projections of Simplices

The following is of independent interest and provides a convenient route to Theorem 1.

Theorem 2. *Theorem* 1 *holds if "randomly rotated" is replaced by "subjected to a random orthogonal transformation."*

Corollary 2. *Corollary* 1 *holds with the same change.*

Our main tool connects orthogonal transformations and Gaussian random variables. A general theory of such factorizations of random matrices together with a discussion of distributional invariance properties is available in Section 7.1 of [4]. For the reader's convenience, we sketch a proof of what we need.

Lemma 1. *Let* $\mathbf{Z}$ *be an* $n \times n$ *matrix composed of independent, standard Gaussian random variables. Then* $\mathbf{Z} = \mathbf{LO}$ *in distribution, where* $\mathbf{O}$ *is a random* $n \times n$ *orthogonal matrix and* $\mathbf{L}$ *is a lower-triangular matrix that is independent of* $\mathbf{O}$ *and such that the determinant of any first* $d \times d$ *submatrix is almost surely strictly positive.*

Proof. Apply the Gram–Schmidt procedure to the rows of $\mathbf{Z}$. This can be written as $\mathbf{L}^{-1}\mathbf{Z} = \mathbf{O}_1$, or

$$\mathbf{Z} = \mathbf{LO}_1, \tag{3}$$

where $\mathbf{O}_1$ is a random orthogonal matrix and $\mathbf{L}$ is lower-triangular and almost surely nonsingular. Adjust the diagonal elements of $\mathbf{L}$ to be (almost surely) positive by postmultiplying by a diagonal matrix of ± 1's and absorb the same matrix into $\mathbf{O}_1$ from the left. Now postmultiply both sides of (3) by an independent random orthogonal matrix $\mathbf{O}_2$ and observe that distributions are preserved, but now $\mathbf{L}$ and the random orthogonal matrix $\mathbf{O} = \mathbf{O}_1\mathbf{O}_2$ are stochastically independent. □

Proof of Theorem 2. Taking the standard basis for R^n, suppose that the vertices of the simplex are arranged as columns in an $n \times (m + 1)$ matrix $\mathbf{S}$ and assume that the d-dimensional subspace onto which the projection will be done is spanned by the first d coordinates. Taking $\mathbf{Z}$ as in the lemma and $\mathbf{\Pi}$ as the $d \times n$ matrix with zero entries except $\Pi_{ii} = 1$, $i = 1, \ldots, d$, consider

$$\tilde{\mathbf{S}} = \mathbf{\Pi ZS}. \tag{4}$$

As vectors in R^d, the columns of $\tilde{\mathbf{S}}$ are distributed like a correlated Gaussian sample

$$Y_i, \qquad i, \ldots, m+1, \tag{5}$$

where the Y_i's are identically distributed, each having independent mean zero Gaussian components, and $E(Y_i, Y_j) < 0$ is the same for all $i \neq j$. Now import an independent standard Gaussian vector W. For easily computed constants a and b, $\{E[(aY_i + bW, aY_j + bW)]\}_{i,j=1}^{m+1}$ is the identity matrix and so $\{aY_i + bW\}_1^{m+1}$ is a standard Gaussian sample. Thus, up to a (deterministic) scaling and random translation, the columns of $\tilde{\mathbf{S}}$ represent a standard Gaussian sample.

Now use the lemma to rewrite $\tilde{\mathbf{S}}$, in distribution, as

$$\tilde{\mathbf{S}} = \mathbf{\Pi LOS} = \tilde{\mathbf{L}}\mathbf{OS}, \tag{6}$$

where $\tilde{\mathbf{L}}$ is the $d \times (m+1)$ matrix formed by the first d rows of $\mathbf{L}$. Comparing (4) and (6), we see that a linear image of the projection of a random orthogonal image of the simplex equals, in distribution, a translate of a standard Gaussian sample. This proves the first part of the theorem.

For the converse part, suppose that a point set in R^n satisfies the hypothesis, that is, subjected to a random orthogonal transformation and projected onto a d-dimensional subspace, it is independently affine-equivalent to a standard Gaussian sample. If we write the point set as columns in the $n \times (m+1)$ matrix $\mathbf{S}$, this means that, with a random orthogonal matrix $\mathbf{O}$, there are an independent $d \times n$ matrix $\mathbf{M}$ and an independent d-vector $\mathbf{b}$ such that the columns of

$$\tilde{\mathbf{S}} = \mathbf{MOS} + \mathbf{be}^T \tag{7}$$

represent a standard Gaussian sample in R^d. Here $\mathbf{e}$ is the $(m+1)$-column vector with all ones. It follows that

$$E\tilde{\mathbf{S}}^T\tilde{\mathbf{S}} = E[\mathbf{eb}^T + \mathbf{S}^T\mathbf{O}^T\mathbf{M}^T][\mathbf{MOS} + \mathbf{be}^T] \tag{8}$$

is d times the $(m+1) \times (m+1)$ identity matrix $\mathbf{I}_{m+1}$. Expanding (8) out and observing that the cross-terms vanish (by first taking the expectation over $\mathbf{O}$) yields

$$E\tilde{\mathbf{S}}^T\mathbf{S} = \mathbf{S}^T E[\mathbf{O}^T\mathbf{M}^T\mathbf{MO}]\mathbf{S} + E\|\mathbf{b}\|^2 \cdot \mathbf{ee}^T. \tag{9}$$

In the first term, the inner (symmetric) matrix expectation is orthogonally similar to itself using *any* orthogonal similarity (again think of holding $\mathbf{M}$ fixed and taking expectations with respect to $\mathbf{O}$). In particular, as a symmetric matrix it is orthogonally similar to a diagonal matrix and hence itself must be diagonal. However, then similarity with respect to permutation matrices implies that all the diagonal elements must be the same. Thus it is a multiple λ of the $n \times n$ identity matrix (λ strictly positive owing to $\mathbf{M}^T\mathbf{M}$ having strictly positive trace almost surely; see (7)). Plugging into (9), we have

$$d\mathbf{I}_{m+1} = E\tilde{\mathbf{S}}^T\tilde{\mathbf{S}} = \lambda\mathbf{S}^T\mathbf{S} + E\|\mathbf{b}\|^2\mathbf{e}\mathbf{e}^T. \tag{10}$$

It follows that the columns of $\mathbf{S}$ satisfy conditions (1) and (2), and we are done. □

Proof of Theorem 1. The proof of Theorem 1 is the same as that of Theorem 2 except for an adjustment to take into account that random *rotations* are used. As for Theorem 1, let $\mathbf{O}$ stand for a random orthogonal matrix. Let $\mathbf{I_O}$ stand for the diagonal matrix of the same dimension which has one's on the diagonal except for the last entry which is det $\mathbf{O}$. We claim that $\hat{\mathbf{O}} = \mathbf{I_O}\mathbf{O}$ is a random rotation. It obviously has unit determinant so that it is only necessary to check that its distribution is invariant under postmultiplication by a fixed rotation matrix $\mathbf{R}$. Since det $\mathbf{R} = 1$,

$$\hat{\mathbf{O}}\mathbf{R} = \mathbf{I_O}\mathbf{O}\mathbf{R} = \mathbf{I_{OR}}\mathbf{O}\mathbf{R}, \tag{11}$$

and since $\mathbf{OR}$ is a random orthogonal matrix, the right-hand side of (11) has the same distribution as $\mathbf{I_O}\mathbf{O}$.

For the first part of the proof, observe that (6) holds with $\mathbf{O}$ replaced by $\mathbf{I_O}\mathbf{O}$ since only the first $d < n$ rows of $\mathbf{O}$ are active. The argument is then the same. The converse part holds verbatim with $\mathbf{O}$ read as a random rotation matrix. □

3. Remarks

1. Using Corollary 1, we transcribe to standard Gaussian samples the asymptotic results of Affentranger and Schneider [1] for projections of regular simplices.

Theorem 3. *For a standard Gaussian sample of size n in* R^d, *the expected number of k-dimensional faces of its convex hull is asymptotic to*

$$\frac{2^d}{\sqrt{d}}\binom{d}{k+1}\beta_{k,d-1}(\pi \log n)^{(d-1)/2}$$

as $n \to \infty$, *where* $\beta_{k,d-1}$ *is the internal angle of the regular simplex of d vertices at one of its k-dimensional faces.*

Theorem 4. *For a standard Gaussian sample of size n in* R^{n-d}, *the expected* *number of k-dimensional faces of its convex hull, $0 \le k < n - d$, is asymptotic to*

$$\binom{n}{k+1}$$

as $n \to \infty$.

The case $k = d - 1$ in Theorem 3 is the result of Raynaud [5] mentioned earlier, which was incorporated by Affentranger and Schneider into their asymptotic argument.

2. A feature of the affine transformation asserted in Theorems 1 and 2 is that it consists solely of a linear map (without a translative component) if and only if the SR-simplex has orthogonal vectors as vertices. In this case, the vectors Y_i in (5) are already independent, and the constant b is zero. On the other hand, if the number of vertices is large, then the translative component will be small (in mean square) in any case. This is significant for asymptotic questions that will be treated elsewhere.

3. It is possible to exploit other connections between randomly rotated polytopes and Gaussian samples. Consider the problem of determining the mean number of faces of a *Gaussian zonotope*, i.e., a Minkowski sum of line segments $\overline{0, X_i}$, $1 \le i \le n$, where $\{X_i\}_1^n$ is a standard Gaussian sample in R^d. A direct solution seems to be rather difficult, but the problem can be approached as follows. By Theorem 1 and Remark 2, $\{X_i\}_1^n$ is affinely equivalent (with no translative component) to a set $\{\tilde{X}_i\}_1^n$ which arises by randomly projecting n orthogonal vectors in R^n onto R^d. This implies that the given Gaussian zonotope is affinely equivalent to the zonotope which sums $\overline{0, \tilde{X}_i}$, $1 \le i \le n$. However, the latter is just the random projection of a cube in R^n, and formulas for its mean face numbers can be consulted in Section 2 of [1].

4. A Further Equivalence

We conclude with another way of relating a high-dimensional sample to Gaussian structure in lower dimension. With $N \ge 1$, think of generating a random point $X = (X_1, X_2, \ldots, X_{Nd})$ uniformly distributed on the unit sphere of $\mathsf{R}^{Nd} = (\mathsf{R}^d)^N$. Let $Y = \{Y_1, Y_2, \ldots, Y_N\}$ in R^d, where $Y_1 = (X_1, X_2, \ldots, X_d)$, $Y_2 = (X_{d+1}, X_{d+2}, \ldots, X_{2d}), \ldots, Y_N = (X_{N-d+1}, X_{N-d+2}, \ldots, X_{Nd})$.

Proposition 1. *Up to a stochastically independent scaling, Y is equivalent in distribution to a standard Gaussian sample in R^d.*

Proof. It is a standard fact that if $Z = (Z_1, Z_2, \ldots, Z_{Nd})$ is a standard Gaussian sample in R^{Nd}, then $\|Z\|$ and $Z/\|Z\|$ are independent and the latter is uniformly distributed on the unit sphere in R^{Nd}. We may as well identify $X = Z/\|Z\|$, so that

$$\begin{aligned}\|Z\| Y &= (\|Z\| Y_1, \|Z\| Y_2, \ldots, \|Z\| Y_N) = (\|Z\| X_1, \|Z\| X_2, \ldots, \|Z\| X_d)\\ &= (Z_1, Z_2, \ldots, Z_d)\end{aligned}$$

is a standard Gaussian sample in R^d. □

A related assertion, variously attributed to Poincaré and Borel (see [3] for a critical discussion) is that as $N \to \infty$, the distribution of $\sqrt{Nd}Y_1$ (or $\sqrt{Nd}Y_j$ for any fixed j) tends to that of a standard Gaussian vector in R^d. This is because $\sqrt{Nd}Y_1 = \sqrt{Nd}(X_1, X_2, \ldots, X_d) = (\sqrt{Nd}/\|Z\|)(Z_1, Z_2, \ldots, Z_d)$ and $\sqrt{Nd}/\|Z\|$ tends to 1 in probability.

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