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RESEARCH ARTICLE

Abundant Semigroups with a Multiplicative Type A Transversal

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Abstract. A complete description is given for the structure of a class of semigroups consisting of all abundant semigroups S in which the set of idempotents generates a regular subsemigroup such that S contains a multiplicative type A transversal.

Introduction

Blyth, McAlister and McFadden (see [1], [2] and [10]) have studied the structure of some classes of regular semigroups. These structures have been built on a set of idempotents E which generates a regular semigroup of an inverse semigroup S whose set of idempotents E^0 is a specified subset of E, and the Munn homomorphism $\alpha : S \to T_{E^0}$. Consider

$$W = W(E, S, \alpha) = \{(e, a, f) \in E \times S \times E : \quad e \mathcal{L} e_a, \quad f \mathcal{R} f_a\} ,$$

where e_a and f_a are idempotents associated with the element a in S.

A binary operation on W is defined as follows:

$$(e, a, f)\ (v, b, w) = (e\overline{(fv)\alpha_a^{-1}}, afvb, (fv)\overline{\alpha_b}\, w),$$

where $a\alpha = \alpha_a$ and $\overline{\alpha_a}$ is an extension of α_a.

An analogue of this semigroup has been considered in classes of abundant semigroups. Accordingly, structures of some classes of abundant semigroups are studied (see [3] and [4]). In regular semigroups, the structure of split orthodox semigroups is given in [10] by introducing the concept of the skeleton E^0 in a band E where S is an inverse semigroup whose semilattice of idempotents is E^0. In this case W is a split orthodox semigroup and any split orthodox semigroup is isomorphic to $W(E, S, \alpha)$ for some split band E whose skeleton E^0 is the set of idempotents of a certain semigroup S. This result was generalized to split quasi-adequate semigroups which satisfy an idempotent-connected property (condition A in [3]).

The result of [10] was also generalized in [2] to a class of regular semigroups by extending the concept of the skeleton E^0 in a band E to the inverse transversal of a regular semigroup S. It was shown that if E is a set of idempotents generating a regular semigroup with a multiplicative semilattice transversal E^0 and S is an inverse semigroup with a set of idempotents E^0,

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then $W(E, S, \alpha)$ is a regular semigroup with a multiplicative inverse transversal. Conversely, any regular semigroup with a multiplicative inverse transversal is of the form $W(E, S, \alpha)$. The aim of this paper is to get an analogue of this result in the abundant case which is also a generalization of the result of [3].

Recall that an *abundant* semigroup is one in which each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains an idempotent [8]. Here two elements are $\mathcal{L}^*$-related ($\mathcal{R}^*$-related) in a semigroup if they are related by Green's related $\mathcal{L}(\mathcal{R})$ in some oversemigroup. The abundant analogues of orthodox semigroups are quasi-adequate semigroups: an abundant semigroup is a *quasi-adequate* semigroup if its idempotents form a subsemigroup [6]. When the idempotents commute in an abundant semigroup, it is called an *adequate* semigroup [7]. An adequate semigroup S is called *type* A if it satisfies the following equalities:

$$Se \cap Sa = Sae, eS \cap aS = eaS$$

for any idempotent e and element a in S.

Type A semigroups are analogues of inverse semigroups in the abundant case. Some properties of inverse semigroups have been extended to type A semigroups (see [7]).

Recall that regular semigroups are abundant semigroups and in this case $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$. In [2], a description was given for the construction of all regular semigroups S with a multiplicative inverse transversal. In this paper we extend this construction to abundant semigroups S such that the set of idempotents in S generates a regular subsemigroup and S contains a multiplicative type A transversal. The "building bricks" of our construction are: an idempotent-generated regular semigroup with a multiplicative semilattice transversal E^0 and a type A semigroup whose set of idempotents is E^0. The approach adopted is similar to that used in [2].

After the preliminary results, we introduce in Section 2 the concept of multiplicative type A transversals. Sections 3 and 4 are concerned with the general construction of abundant semigroups which contain multiplicative type A transversals and includes a structure theorem for this class of semigroups.

We use the notation and terminology of [9]. Other undefined terms can be found in the preceding papers [3] and [4].

1. Preliminaries

We recall some of the basic facts about the relations $\mathcal{L}^*$ and $\mathcal{R}^*$. The relation $\mathcal{L}^*(\mathcal{R}^*)$ is defined on a semigroup S by the rule that $a \mathcal{L}^* b (a \mathcal{R}^* b)$ is and only if the elements a and b of S are related by Green's relation $\mathcal{L}(\mathcal{R})$ in some oversemigroup of S. Evidently, $\mathcal{L}^*(\mathcal{R}^*)$ is a right (left) congruence on S. The following lemma, from [5], provides us with an alternative description for $\mathcal{L}^*(\mathcal{R}^*)$:

Lemma 1.1. *Let S be a semigroup and let a and b be in S. Then the following conditions are equivalent:*

(1) $a \mathcal{L}^* b$ $(a \mathcal{R}^* b)$;

(2) *For all $s, t \in S^1$, $as = at$ $(sa = ta)$ if and only if $bs = bt$ $(sb = tb)$.*

As an easy consequence of Lemma 1.1 we have:

Corollary 1.2. *Let S be a semigroup $a \in S$ and e be an idempotent of S. Then the following conditions are equivalent:*

(1) $a\,\mathcal{L}^*\,e$ $(a\,\mathcal{R}^*\,e)$

(2) $ae = a$ $(ea = a)$ *and for all* $s, t \in S^1$, $as = at$ $(sa = ta)$ *implies* $es = et$ $(se = te)$.

Obviously, in any semigroup S we have $\mathcal{L} \subseteq \mathcal{L}^*$ and $\mathcal{R} \subseteq \mathcal{R}^*$. It is well-known and easy to see that for regular elements a and b in S, $a\,\mathcal{L}^*\,b$ $(a\,\mathcal{R}^*\,b)$ if and only if $a\,\mathcal{L}\,b$ $(a\,\mathcal{R}\,b)$. In particular, if S is a regular semigroup, then $\mathcal{L}^* = \mathcal{L}$ and $\mathcal{R}^* = \mathcal{R}$.

Let S be an abundant semigroup with sets of idempotents E, and let U be an abundant subsemigroup of S. U is called a *left* (*right*) $*$*-subsemigroup* if for any $a \in I$ there exists $e \in U \cap E$ such that $a\,\mathcal{L}^*(S)\,e$ $(a\,\mathcal{R}^*(S)\,e)$. U is called a $*$*-subsemigroup* if it is both a left and a right $*$-subsemigroup.

From [3] we have the following proposition:

Proposition 1.3. *Let S be an abundant semigroup and let U be an abundant subsemigroup of S. U is a left (right) $*$-subsemigroup if and only if*

$$\mathcal{L}^*(U) = \mathcal{L}^*(S) \cap (U \times U)\ (\mathcal{R}^*(U) = \mathcal{R}^*(S) \cap (U \times U))\ .$$

As in [5], if a is an element of S, then a^* denotes a typical element of $L^*_a(S) \cap E$ and $a^\dagger$ denotes a typical element of $R^*_a(S) \cap E$. Recall that an abundant semigroup S is adequate if its set of idempotents forms a semilattice. In this case, it follows from the commutativity of the idempotents in S that each $\mathcal{L}^*$-class and each $\mathcal{R}^*$-class contains a unique idempotent. For all a and b in S, it is easy to see that $(ab)^* = (a^*b)^*$ and $(ab)^\dagger = (ab^\dagger)^\dagger$.

Let S be an adequate semigroup with semilattice of idempotents E. For any a in S, define

$$\alpha_a : a^\dagger E \to a^* E \text{ by } e\alpha_a = (ea)^*, \quad \text{and} \quad \beta_a : a^* E \to a^\dagger E \text{ by } e\beta_a = (ae)^\dagger\ .$$

In order to use α_a and β_a to obtain a representation of S on the Munn semigroup T_E as in the inverse (case (see [9]), it seems essential to impose the condition that α_a and β_a are both injective for each a in S. The following proposition, from [7], relates this condition to others which allow one to show that S with this condition can be represented in the Munn semigroup T_E.

Proposition 1.4. *For any adequate semigroup S with semilattice of idempotents E, the following conditions are equivalent:*

(i) *for each element a of S, the mappings α_a and β_a are injective;*

(ii) *for each element a of S: the mappings α_a and β_a are inverse isomorphisms.*

(iii) *for each element a of S and any idempotent e in $a^\dagger E$ and f in $a^* E$*

$$(a(ea)^*)^\dagger = e \text{ and } ((af)^\dagger a)^* = f\ .$$

The semigroups described in this proposition are called *type* A *semigroups.* This definition coincides with the definition in the introduction [7]. Since any cancellative monoid is type A but not necessarily inverse, it is clear that the class of type A semigroups properly contains the class of inverse semigroups.

2. Adequate transversals

The aim of this section is to introduce the concept of a multiplicative type A transversal for an abundant semigroup S. We begin by the following Lemma.

Lemma 2.1. *Let S be an abundant semigroup with sets of idempotents E and $x, y \in S$. If there exist $e, f \in E$ such that $x = eyf$ and $e \,\mathcal{L}\, y^\dagger$, $f \,\mathcal{R}\, y^*$ for some $y^\dagger$, y^*, then $e \,\mathcal{R}^*\, x$ and $f \,\mathcal{L}^*\, x$.*

Proof. Clearly $ex = x$. Let $s, t \in S^1$. Then

$$\begin{aligned} sx = tx \Rightarrow\ & seyfy^* = teyfy^* \\ \Rightarrow\ & sey = tey && (f \,\mathcal{R}\, y^* \text{ and } yy^* = y) \\ \Rightarrow\ & sey^\dagger = tey^\dagger && (\text{Corollary 1.2}) \\ \Rightarrow\ & se = te && (e \,\mathcal{L}\, y^\dagger)\,. \end{aligned}$$

Now, by Corollary 1.2, $e \,\mathcal{R}^*\, x$. Similarly, $f \,\mathcal{L}^*\, x$. ∎

Let S be an abundant semigroup with a set of idempotents E. Let S^0 be an adequate $*$-subsemigroup of S and E^0 be the semilattice of idempotents of S^0. The semigroup S^0 is called an *adequate transversal* for S if for each element x in S, there are a unique element x^0 in S^0 and idempotents e, f in E such that $x = ex^0 f$ where $e \,\mathcal{L}\, x^{0\dagger}$, $f \,\mathcal{R}\, x^{0*}$ for $x^{0\dagger}$ and x^{0*} in E^0. In this case e and f are uniquely determined by x, because, if

$$e_1 x^0 f_1 = x = e_2 x^0 f_2 \quad \text{where} \quad e_i \,\mathcal{L}\, x^{0\dagger}, f_i \,\mathcal{R}\, x^{0*} \quad (i = 1, 2)\,,$$

then $e_1 \,\mathcal{L}\, x^{0\dagger} \,\mathcal{L}\, e_2$ and by Lemma 2.1, $e_1 \,\mathcal{R}^*\, x \,\mathcal{R}^*\, e_2$. Thus $e_1 \,\mathcal{H}\, e_2$ and $e_1 = e_2$. Likewise, $f_1 = f_2$. Denote e by e_x and f by f_x.

We say that the adequate transversal S^0 is *multiplicative* if for any $x, y \in S$, $f_x e_y \in E^0$. If it happens also that S^0 is type A, then it is called a *multiplicative type* A *transversal.*

We are now in a position to show that when specialized to the regular case, our definition of abundant semigroups with a multiplicative adequate transversal coincides with the definition of regular semigroups with a multiplicative inverse transversal as defined in [2]. To demonstrate this fact, let S be a regular semigroup and S^0 be an inverse subsemigroup of S. If S^0 is a multiplicative adequate transversal of S in the sense of our definition and $x \in S$, then $x = e_x y f_x$ where y is in S^0. It is clear that $y^* = y^{-1}y$ and $y^\dagger = yy^{-1}$ where y^{-1} is the inverse of y in S^0. Since $e_x \,\mathcal{L}\, y^\dagger$ and $f_x \,\mathcal{R}\, y^*$, we have $yy^{-1}e_x = yy^{-1}$ which implies that $y^{-1}e_x = y^{-1}$. Likewise, $f_x y^{-1} = y^{-1}$. It follows that

$$y^{-1}xy^{-1} = y^{-1}e_x y f_x y^{-1} = y^{-1}yy^{-1} = y^{-1}$$

and

$$xy^{-1}x = e_x y f_x y^{-1} e_x y f_x = e_x y y^{-1} y f_x = e_x y f_x = x\,.$$

Hence, $y^{-1} \in S^0 \cap V(x)$.

Notice that $x = xy^{-1}x = xy^{-1}yy^{-1}x, e_x = xy^{-1}$ and $f_x = y^{-1}x$, which coincides with the notation of [2]. Therefore S^0 is a multiplicative inverse transversal of S in conformity with the definition in [2]. On the other hand, if S is a regular semigroup with a multiplicative inverse transversal S^0 in the sense of the definition in [2], then S is an abundant semigroup and S^0 is an adequate $*$-subsemigroup. In fact, S^0 is type A. Let $y \in S^0 \cap V(x)$ and let y^{-1} be the inverse of y in S^0. Clearly

$$x = xyx = xyy^{-1}yx \ .$$

Notice that $(y^{-1})^\dagger = y^{-1}y$, $(y^{-1})^* = yy^{-1}$, xy and yx are idempotents in S such that

$$xy \,\mathcal{L}\, (y^{-1})^\dagger, yx \,\mathcal{R}\, (y^{-1})^* \ .$$

Thus, $e_x = xy$ and $f_x = yx$. Therefore, e_x and f_x coincide with e_x and f_x as defined before. It follows that S^0 is a multiplicative type A transversal of S in conformity with our definition.

Note, however, that it is conceivable that a regular semigroup could have a type A transversal which is not inverse.

Moreover, there is an abundant semigroup with a multiplicative type A transversal which is not regular, as the following example illustrates.

Example 2.2. Let M be a cancellative monoid with an identity 1. Let E be a set of idempotents which generates a regular semiband $\langle E \rangle$ with a multiplicative semilattice transversal E^0. Consider the direct products $S = M \times \langle E \rangle$ and $S^0 = M \times E^0$. It is clear that $E(S) = \{1\} \times E$ and $E(S^0) = \{1\} \times E^0$ are the sets of idempotents of S and S^0, respectively. Therefore, S is a semigroup whose set of idempotents generates a regular subsemigroup with a multiplicative semilattice transversal $E(S^0)$. For any $(m,a) \in S$, let a' be an inverse of a in $\langle E \rangle$. It is routine to check that $(m,a)\,\mathcal{L}^*\,(1,a'a)$ and $(m,a)\,\mathcal{R}^*\,(1,aa')$. Thus S is an abundant semigroup which is not regular provided that M is not a group. Likewise, for any $(m,e) \in S^0$, $(m,e)\,\mathcal{L}^*\,(1,e)$ and $(m,e)\,\mathcal{R}^*\,(1,e)$. So, clearly S^0 is an adequate $*$-subsemigroup of S. In fact, S^0 is a type A semigroup.

Now for any $(m,a) \in S$, as $a^0 \in V(a) \cup E^0$, we have

$$a = aa^0\, a^0\, a^0\, a \ .$$

Thus, $(m,a) = (1,aa^0)(m,a^0)(1,a^0,a)$ where $(1,aa^0)$, $(1,a^0a)$ are idempotents in S, $(1,aa^0)\,\mathcal{L}\,(1,a^0)$; $(1,a^0a)\,\mathcal{R}\,(1,a^0)$, and

$$(m,a^0)^\dagger = (1,a^0) = (m,a^0)^* \ .$$

It follows that $e_{(m,a)} = (1,aa^0)$ and $f_{(m,a)} = (1,a^0a)$. Moreover, for any (m,a), (m,b) in S we have a^0abb^0 in E^0 and thus $f_{(m,a)}e_{(m,b)} \in E(S^0)$. Therefore, S^0 is a multiplicative type A transversal for S.

3. The Semigroup W(E,S)

In this section we construct an abundant semigroup S in which the set of idempotents generates a regular subsemigroup and S contains a multiplicative type A transversal. It is clear from the previous section that the class of abundant semigroups which satisfy these conditions properly includes the class of regular semigroups with a multiplicative inverse transversal studied in [2].

The components in this construction will be: type A semigroups and idempotent generated regular semibands with multiplicative semilattice transversals.

To begin, let $\langle E\rangle$ be a regular semigroup generated by a set of idempotents E and let E^0 be a multiplicative semilattice transversal of $\langle E\rangle$. Let S be a type A semigroup whose semilattice of idempotents is isomorphic to E^0. For convenience of notation, we shall identify this semilattice with E^0. Consider the set

$$W = W(E,S) = \{(g,a,h) \in E \times S \times E : g \,\mathcal{L}\, a^\dagger, h \,\mathcal{R}\, a^\dagger\} .$$

For any $(g,a,h) \in W$, $g\ell \,\mathcal{L}\, a^\dagger$ and $g = ga^\dagger a^\dagger$, where $a^\dagger \in E^0$. Thus $g^0 = a^\dagger$ and $e_g = g$. Likewise $f_h = h$. Hence for any (g,a,h) and (v,b,w) in W we have $hv = f_h e_v \in E^0$ because E^0 is a multiplicative semilattice transversal of $\langle E\rangle$.

Proposition 3.1. *The rule*

$$(g,a,h)\;(v,b,w) = (g(ahv)^\dagger, ahvb, (hvb)^*_w)$$

defines a binary operation on W.

Proof. Let (g,a,h) and (v,b,w) be in W. Since $(ahv)^\dagger = (ahv)^\dagger a^\dagger$ and $g \,\mathcal{L}\, a^\dagger$, then $g(ahv)^\dagger \in E$. Similarly, $(hvb)^* w \in E$. Furthermore,

$$\begin{aligned} g(ahv)^\dagger\;(ahvb)^\dagger &= g(ahv)^\dagger\;(ahvb^\dagger)^\dagger \\ &= g(ahv)^\dagger\;(ahv)^\dagger \qquad\qquad (v \,\mathcal{L}\, b^\dagger) \\ &= g(ahv)^\dagger\;, \end{aligned}$$

and

$$\begin{aligned} (ahvb)^\dagger g(ahv)^\dagger &= (ahvb)^\dagger a^\dagger g(ahv)^\dagger \\ &= (ahvb)^\dagger\;(ahv)^\dagger \qquad\qquad (g \,\mathcal{L}\, a^\dagger) \\ &= (ahvb)^\dagger\;. \end{aligned}$$

Hence, $g(ahv)^\dagger \,\mathcal{L}\, (ahvb)^\dagger$. Similarly, $(hvb)^* w \,\mathcal{R}\, (ahvb)^*$. ∎

The following sequence of results provides considerably more information about W.

Proposition 3.2. *W is a semigroup.*

Proof. Let (g,a,h), (v,b,w) and (x,c,y) be in W. Then

$$\begin{aligned} (g,a,h)[(v,b,w),(x,c,y)] &= (g,a,h)(v(bwx)^\dagger, bwxc, (wxc)^* y) \\ &= (g \cdot (ahv(bwx)^\dagger)^\dagger, ahv(bwx)^\dagger bwxc, (hv(bwx)^\dagger bwxc)^* \cdot (wxc)^* y) \end{aligned}$$

and

$$\begin{aligned} [(g,a,h)(v,b,w)]\;(x,c,y) &= (g(ahv)^\dagger, ahvb, (hvb)^* \cdot w)\;(x,c,y) \\ &= (g(ahv)^\dagger \cdot (ahvb(hvb)^* wx)^\dagger, ahvb(hvb)^* \cdot wxc, ((hvb)^* wxc)^* \cdot y). \end{aligned}$$

Notice that

$$\begin{aligned} g(ahv)^\dagger (ahvb(hvb)^* wx)^\dagger &= g(ahv)^\dagger (ahvbwx)^\dagger \\ &= g(ahvbwx)^\dagger = g(ahv(bwx)^\dagger)^\dagger\;. \end{aligned}$$

In a similar way, we obtain

$$(hv(bwx)^\dagger bwxc)^*(wxc)^*y = ((hvb)^*wxc)^*y \ .$$

Therefore, the first and the third components of the product coincide. Now for the second components, we have

$$ahv(bxw)^\dagger \ bwxc = ahvbwxc = ahvb(hvb)^*wxc \ ,$$

and associativity holds. ■

Proposition 3.3. *W is an abundant semigroup.*

Proof. Let (g, a, h) be in W. Consider $(g, a^\dagger, a^\dagger)$. It is easy to see that $(g, a^\dagger, a^\dagger)$ is an idempotent in W, and

$$(g, a^\dagger, a^\dagger)\ (g, a, h) = (g(a^\dagger a^\dagger g)^\dagger, a^\dagger a^\dagger ga, (a^\dagger ga)^* \cdot h) = (g, a, h) \ .$$

Now let (v, b, w) and (x, c, y) be in W. Then $(v, b, w)(g, a, h) = (x, c, y)(g, a, h)$; that is,

$$(v(bwg)^\dagger, bwga, (wga)^*h) = (x(cyg)^\dagger, cyga, (yga)^*h) \ .$$

Hence $v(bwg)^\dagger = x(cyg)^\dagger$ and $bwga = cyga$, which implies that $bwga^\dagger = cyga^\dagger$ by Corollary 1.2. Also,

$$(wga)^*h = (yga)^*h$$

gives $(wga)^*ha^* = (yga)^*ha^*$ and $(wga)^* = (yga)^*$. Thus (with α_a as in Proposition 1.4), $(wga^\dagger)\alpha_a = (yga^\dagger)\alpha_a$, and so $wga^\dagger = yga^\dagger$ since α_a is injective. It follows that

$$(wga^\dagger)^*a^\dagger = (yga^\dagger)^*a^\dagger \ .$$

Now

$$(v, b, w)(g, a^\dagger, a^\dagger) = (v(bwg)^\dagger, bwga^\dagger, (wga^\dagger)^*a^\dagger) \ ,$$

and

$$(x, c, y)(g, a^\dagger, a^\dagger) = (x(cyg)^\dagger, cyga^\dagger, (yga^\dagger)^*a^\dagger) \ .$$

Hence $(v, b, w)(g, a^\dagger, a^\dagger) = (x, c, y)(g, a^\dagger, a^\dagger)$, and so by Corollary 1.2 it follows that $(g, a, h)\ \mathcal{R}^*\ (g, a^\dagger, a^\dagger)$. Similarly, $(g, a, h)\ \mathcal{L}^*\ (a^*, a^*, h)$, and the result follows. ■

Proposition 3.4. *The set of idempotents of W is*

$$E(W) = \{(g, a, h) \in W : a^2 = a, ghg = g, hgh = h\} \ .$$

Proof. For any (g, a, h) in W,

$$(g, a, h)(g, a, h) = (g(ahg)^\dagger, ahga, (hga)^*h) \ .$$

If $a^2 = a$, $ghg = g$, $hgh = h$, then clearly $a^\dagger = a = a^*$. Thus

$$ahga = ahgha = aha = a, g(ahg)^\dagger = ghg = g \text{ and } (hga)^*h = ghg = h \ .$$

Therefore, (g, a, h) is an idempotent. Conversely, if (g, a, h) is an idempotent, then $ahga = a$ which implies, by Corollary 1.2, that $hga = a^*hga = a^*$ and $ahg = agha^\dagger = a^\dagger$. Hence

$$a = ahg\, hga = a^\dagger a^* \in E^0; \text{that is}, a^2 = a \ .$$

Also $g = g(ahg)^\dagger = ghg$, and $h = (hga)^* \cdot h = hgh$. ■

Proposition 3.5. *If $\langle E(W)\rangle$ denotes the subsemigroup of W generated by $E(W)$, then $\langle E(W)\rangle = \{(g,a,h)\in W : a^2 = a\}$.*

Proof. This is the same as in [2]. ∎

Proposition 3.6. *$\langle E(W)\rangle$ is isomorphic to $\langle E\rangle$.*

Proof. Notice that if (g,a,h) is in $\langle E(W)\rangle$ and $a^2 = a$, then $gah \in \langle E\rangle$ and $a^\dagger = a = a^*$, so that $gah = gh$. Define $\phi : \langle E(W)\rangle \to \langle E\rangle$ by $(g,a,h)\phi = gh$.

Since for any $x\in\langle E\rangle$, there exists $x^0 \in E^0$ such that $x = e_x x^0 f_x$ where $e_x \,\mathcal{L}\, x^{0\dagger}$, $x^{0\dagger} = x^0 = x^{0*}$, $x^{0*} \,\mathcal{R}\, f_x$. Thus $x = e_x f_x$, $(e_x, x^0, f_x)\in\langle E(W)\rangle$, and $(e_x,x^0,f_x)\phi = e_x f_x = x$. Therefore, ϕ is a surjective map. To show that ϕ is also injective, let (g,a,h) and (v,b,w) be in $\langle E(W)\rangle$ such that $gh = vw$. Since $a^2 = a$, $b^2 = b$ we have $gh = gah$ and $vw = vbw$. Thus $gab = vbw$; $a,b\in E^0$ and $a = b$ follows from the uniqueness of the element in E^0 associated with a given element in $\langle E\rangle$. Therefore $gha = vwa$ which implies $g = v$. Also $agh = avw$ which implies $h = w$. Thus $(g,a,h) = (v,b,w)$ and ϕ is injective. Finally, for any (g,a,h) and (v,b,w) in $\langle E(W)\rangle$ with a and b in E^0, we have

$$\begin{aligned}(g,a,h)(v,b,w)\phi &= (g(ahv)^\dagger, ahvb, (hvb)^*\cdot w)\phi = (g(ahv), ahvb, (hvb)w)\phi\\ &= (ghv, hv, hvw)\phi = ghvw = (g,a,h)\phi(v,b,w)\phi\ ;\end{aligned}$$

that is, ϕ is a homomorphism. Hence ϕ is an isomorphism.

As $\langle E\rangle$ is an idempotent-generated regular semigroup, we have the following corollary as an immediate consequence of Proposition 3.6.

Corollary 3.7. *The set of idempotents of W generates a regular semigroup.*

We now proceed to a characterization of the relations $\mathcal{L}^*$ and $\mathcal{R}^*$ on W. To do this, we need the following proposition.

Proposition 3.8. *For any (g,a,h) and (v,g,w) in $E(W)$,*

(i) *$(g,a,h)\,\mathcal{L}\,(v,b,w)$ if and only if $a = b$ and $h = w$,*

(ii) *$(g,a,h)\,\mathcal{R}\,(v,b,w)$ if and only if $a = b$ and $g = v$.*

Proof. Since (ii) is the dual of (i), it suffices to prove (i). If $(g,a,h)\,\mathcal{L}\,(v,b,w)$, then

$$(g,a,h)(v,b,w) = (g,a,h) \text{ and } (v,b,w)(g,a,h) = (v,b,w)\ .$$

That is, $(g(ahv)^\dagger,\ ahvb,\ (hvb)^*\cdot w) = (g,a,h)$, and

$$(v(awg)^\dagger, bwga, (wga)^*\cdot h) = (v,b,w)\ .$$

Since $a,b\in E^0$, we get $(ghv, hv, hvw) = (g,a,h)$ and $(vwg, wg, wgh) = (v,b,w)$. Therefore, $hv = a$, $hvw = h$; that is, $aw = h$. Also, $wg = b$ and $bh = w$. Now $aw = h$ and $bh = w$ imply $abh = aw = h$, which with $a\,\mathcal{R}\,h$ implies $aba = a$. Also $baw = bh = w$ implies $bab = b$. But $a,b\in E^0$. Hence $a = b$. Since $h\,\mathcal{R}\,a$ and $b\,\mathcal{R}\,w$, then $h\,\mathcal{R}\,w$. Now $aw = h$ implies $hw = h$ and $bh = w$ implies $wh = w$, so $h\,\mathcal{L}\,w$. Thus $h\,\mathcal{H}\,w$ and $h = w$.

On the other hand, if (g,a,h) and (v,b,w) are in $E(W)$ such that $a = b$ and $h = w$, then $g\,\mathcal{L}\,a = b\,\mathcal{L}\,v$. It follows that

$$\begin{aligned}(g,a,h)(v,b,w) &= (g(ahv)^\dagger, ahvb, (hvb)^*\cdot w)\\ &= (ghv, ahvb, hvw) = (gvhv, hv, hvw) \qquad (g\,\mathcal{L}\,v)\\ &= (g, hv, h) \qquad (h = w\in V(v)).\end{aligned}$$

Since $a, hv \in E^0$, then by using the properties of elements of W we get $a \mathcal{L} g \mathcal{L} hv$; that is, $hv = a$ and thus $(g,a,h)(v,b,w) = (g,a,h)$. Similarly; $(v,b,w)(g,a,h) = (v,b,w)$, and so $(g,a,h) \mathcal{L} (v,b,w)$. ∎

Now let (g,a,h) be in $W = W(E,S)$. From the proof of Proposition 3.3, we have

$$(g,a,h) \mathcal{L}^* (a^*,a^*,h) \text{ and } (g,a,h) \mathcal{R}^* (g,a^\dagger,a^\dagger) .$$

Thus, for any (g,a,h) and (v,b,w) in W,

$$\begin{aligned}(g,a,h) \mathcal{L}^*(W) (v,b,w) &\Leftrightarrow (a^*,a^*,h)(\mathcal{L} (b^*,b^*,w) \\ &\Leftrightarrow a^* = b^* \text{ and } h = w \qquad \text{(Proposition 3.8)} \\ &\Leftrightarrow a \mathcal{L}^*(S) b \text{ and } h = w .\end{aligned}$$

and

$$\begin{aligned}(g,a,h) \mathcal{R}^*(W) (v,b,w) &\Leftrightarrow (g,a^\dagger,a^\dagger) \mathcal{R} (v,b^\dagger,b^\dagger) \\ &\Leftrightarrow a^\dagger = b^\dagger \text{ and } g = v \qquad \text{(Proposition 3.8)} \\ &\Leftrightarrow a \mathcal{R}^*(S) b \text{ and } g = v .\end{aligned}$$

Consequently, if we put $\mathcal{H}^* = \mathcal{L}^* \cap \mathcal{R}^*$, then $(g,a,h)\ \mathcal{H}^*(W)\ (v,b,w) \Leftrightarrow a \mathcal{H}^*(S) b$, $h = w$ and $g = v$.

Finally, define $W^0 = \{(a^\dagger,a,a^*); a \in S)\}$. Since for any a and b in S,

$$(a^\dagger,a,a^*)(b^\dagger,b,b^*) = (a^\dagger(aa^*b^\dagger)^\dagger, aa^*b^\dagger b, (a^*b^\dagger b)^*b^*) = ((ab)^\dagger, ab, (ab)^*) ,$$

we deduce that W^0 is a subsemigroup of W. In fact, $a \to (a^\dagger,a,a^*)$ describes an isomorphism of S onto W^0, and so W^0 is a type A semigroup. Clearly – as in the proof of Proposition 3.3 – for any $(a^\dagger,a,a^*)$ in W^0

$$(a^*,a^*,a^*) \mathcal{L}^*(a^\dagger,a,a^*) \text{ and } (a^\dagger,a^\dagger,a^\dagger) \mathcal{R}^* (a^\dagger,a,a^*) .$$

Thus W^0 is an adequate $*$-subsemigroup of W.

Now let (g,a,h) be in W. Since

$$\begin{aligned}(g,a^\dagger,a^\dagger)(a^\dagger,a,a^*)(a^*,a^*,h) &= (ga^\dagger, a, (a^\dagger a)^*a^*)(a^*,a,h) \\ &= (g,a,a^*)(a^*,a^*,h) = (g(aa^*)^\dagger, a, a^*h) = (g,a,h),\end{aligned}$$

it is clear that $(g,a^\dagger,a^\dagger)$ and (a^*,a^*,h) are in $E(W)$. By Proposition 3.8,

$$(g,a^\dagger,a^\dagger) \mathcal{L} (a^\dagger,a^\dagger,a^\dagger) \text{ and } (a^*,a^*,h) \mathcal{R} (a^*,a^*,a^*) .$$

Further, if there exists $(b^\dagger,b,b^*)$ in W^0 such that

$$(g,a,h) = (e,x,f)(b^\dagger,b,b^*)(i,y,j) ,$$

where $(e,x,f) \mathcal{L} (b^\dagger,b^\dagger,b^\dagger)$, $(i,y,j) \mathcal{R} (b^*,b^*,b^*)$ for (e,x,f) and (i,y,j) in $E(W)$, that is, for $x,y \in E^0$, then in particular

$$(e,x,f)(b^\dagger,b^\dagger,b^\dagger) = (e,x,f) \text{ and } (b^\dagger,b^\dagger,b^\dagger)(e,x,f) = (b^\dagger,b^\dagger,b^\dagger) .$$

It follows that $xfb^{\dagger} = x$ and as $x \mathcal{R} f$ so $fb^{\dagger} = x$. Also

$$(b^{\dagger}(b^{\dagger}eb^{\dagger}), b^{\dagger}ex, xb^{\dagger}exb^{\dagger}) = (b^{\dagger}, b^{\dagger}, b^{\dagger}) ,$$

which implies $xb^{\dagger}exb^{\dagger} = b^{\dagger}$ and hence $xb^{\dagger} = b^{\dagger}$. Moreover, we have $(i,y,j)(b^*,b^*,b^*) = (b^*,b^*,b^*)$ and $(b^*,b^*,b^*)(i,y,j) = (i,y,j)$. By an argument similar to the previous one, we obtain $b^*i = y$ and $b^*y = b^*$. Recall that $(g,a,h) = (e(xfb^{\dagger})^{\dagger}, xfb^{\dagger}b, (fb^{\dagger}b)^*b^*)(i,y,j)$, which implies that

$$\begin{aligned} a = xfb^{\dagger}b(fb^{\dagger}b)^*b^*iy = xb(xb)^*y \qquad & (fb^{\dagger} = x, b^*i = y) \\ = b \qquad & (xb^{\dagger} = b^{\dagger}, b^*y = b^*) . \end{aligned}$$

Hence $(a^{\dagger}, a, a^*) = (b^{\dagger}, b, b^*)$. Therefore, $(a^{\dagger}, a, a^*)$ in W^0 is uniquely determined by (g,a,h) in W and W^0 is a type A transversal for W. In fact, W^0 is multiplicative since for any $w = (e,a,f)$ and $v = (g,b,h)$ in W, we have

$$f_w = (a^*, a^*, f), e_v = (g, b^{\dagger}, b^{\dagger})$$

and

$$\begin{aligned} f_w e_v &= (a^*, a^*, f)(g, b^{\dagger}, b^{\dagger}) = (a^*(fg), a^*fgb^{\dagger}, (fg)b^{\dagger}) \\ &= (a^*fg, a^*fgb^{\dagger}, fgb^{\dagger}) = (fg, fg, fg); \end{aligned}$$

that is, $f_w e_v \in E(W^0)$. Hence W^0 is a multiplicative type A transversal for W.

Summing up, we have the following theorem:

Theorem 3.9. *Let $\langle E \rangle$ be an idempotent generated regular semiband with a multiplicative semilattice transversal E^0, and let S be a type* A *semigroup whose semilattice of idempotents is (isomorphic to) E^0. Then $W = W(E,S)$ is an abundant semigroup in which the set of idempotents generates a regular subsemigroup isomorphic to $\langle E \rangle$ and W contains a multiplicative type* A *transversal W^0 which is isomorphic to S.*

4. The Characterization

In this section we shall prove the following converse of Theorem 3.9. Let S be an abundant semigroup in which the set of idempotents generates a regular subsemigroup. Furthermore, let S contain a multiplicative type A transversal. Then S has the same form as the semigroup constructed in Section 3.

Let S be an abundant semigroup in which the set of idempotents E generates a regular subsemigroup $\langle E \rangle$. Suppose that S contains a multiplicative type A transversal S^0 whose semilattice of idempotents E^0 is the corresponding semilattice transversal of $\langle E \rangle$. Our objective is to prove that S is isomorphic to $W = W(E, S^0)$. We begin by providing a technical result.

Lemma 4.1. *For any x and y in S, we have:*
(i) $(xy)^0 = x^0 f_x e_y y^0$; (ii) $e_{xy} = e_x(x^0 f_x e_y)^{\dagger}$; (iii) $f_{xy} = (f_x e_y y^0)^* f_y$.

Proof. Since $x = e_x x^0 f_x$ and $y = e_y y^0 f_y$, we have

$$xy = e_x x^0 f_x e_y y^0 f_y = e_x(x^0 f_x e_y)^{\dagger}\, x^0 f_x e_y y^0 \, (f_x e_y y^0)^* f_y .$$

It is clear that

$$(x^0 f_x e_y y^0)^\dagger = (x^0 f_x e_y y^{0\dagger})^\dagger = (x^0 f_x e_y)^\dagger \ \mathcal{R}^* \ x^0 f_x e_y \ ,$$

which implies $x^{0\dagger}(x^0 f_x e_y)^\dagger = (x^0 f_x e_y)^\dagger$. But $e_x \ \mathcal{L} \ x^{0\dagger}$, and so

$$e_x(x^0 f_x e_y)^\dagger \ \mathcal{L} \ x^{0\dagger}(x^0 f_x e_y)^\dagger = (x^0 f_x e_y)^\dagger = (x^0 f_x e_x y^0)^\dagger \ .$$

Similarly, $(f_x e_y y^0)^* f_y \ \mathcal{R} \ (x^0 f_x e_y y^0)^*$. Now $x^0 f_x e_y y^0 \in S^0$ and so (i), (ii) and (iii) will follow from the uniqueness of $(xy)^0$, e_{xy} and f_{xy} if we show that $e_x(x^0 f_x e_y)^\dagger$ and $(f_x e_y y^0)^* f_y$ are idempotents.

For any element a and idempotent e of any type A semigroup we have

$$(ae)^\dagger a^\dagger = a^\dagger (ae)^\dagger = (ae)^\dagger \ .$$

Hence, since $e_x \ \mathcal{L} \ x^{0\dagger}$, we have

$$\begin{aligned}(x^0 f_x e_y)^\dagger e_x (x^0 f_x e_y)^\dagger &= (x^0 f_x e_y)^\dagger x^{0\dagger} e_x (x^0 f_x e_y)^\dagger = (x^0 f_x e_y)^\dagger \ x^{0\dagger} (x^0 f_x e_y)^\dagger \\ &= (x^0 f_x e_y)^\dagger \ (x^0 f_x e_y)^\dagger = (x^0 f_x e_y)^\dagger \ .\end{aligned}$$

Thus $e_x(x^0 f_x e_y)^\dagger$ is an idempotent. Similarly, so is $(f_x e_y y^0) f_y$ and so the proof is complete. ■

Theorem 4.2. *S is isomorphic to $W = W(E, S^0)$.*

Proof. For any $x \in S$, $x^0 \in S^0$ and $x = e_x x^0 f_x$ where e_x and f_x are idempotents uniquely determined by x and $e_x \ \mathcal{L} \ x^{0\dagger}$, $f_x \ \mathcal{R} \ x^{0*}$. Therefore $(e_x, x^0, f_x) \in W = W(E, S^0)$.

Define $\Theta : S \to W$ by $x\Theta = (e_x, x^0, f_x)$ for any x in S. If for any x, y in S, $(e_x, x^0, f_x) = (e_y, y^0, f_y)$, then $x = e_x x^0 f_x = e_y y^0 f_y = y$ and we have that Θ is an injective map.

To show that Θ is also surjective, let (g, a, h) be in W, and consider $b = gah$. It is easy to see that $b \in S$ and $e_b = g$, $b^0 = a$ and $f_b = h$. Consequently,

$$b\Theta = (e_b, b^0, f_b) = (g, a, h)$$

and Θ is surjective.

Finally, for any x and y in S,

$$\begin{aligned} x\Theta y\Theta &= (e_x, x^0, f_x)(e_y, y^0, f_y) = (e_x(x^0 f_x e_y)^\dagger, x^0 f_x e_y y^0, (f_x e_y y^0)^* f_y) \\ &= (e_{xy}, (xy)^0, f_{xy}) \qquad\qquad \text{(Lemma 4.1)} \\ &= (xy)\Theta \ . \end{aligned}$$

So Θ is a homomorphism. Hence Θ is an isomorphism.

In general, a semigroup S which is isomorphic to $W(E, S^0)$, as in the previous theorem, need not be a quasi-adequate semigroup (see Example 5.3 in [2]) and it is of interest to see when this is the case.

Proposition 4.3. *If the function $\Psi : S \to S^0$ defined by $x\Psi = x^0$ is a homomorphism from S into S^0, then S is a quasi-adequate semigroup.*

Proof. By Lemma 4.1, we have $x^0y^0 = (xy)^0 = x^0 f_x e_y y^0$, which implies by Corollary 1.2 that $x^{0*}y^0 = x^{0*}f_x e_y y^0 = f_x e_y y^0$. Then $x^{0*}y^0 = f_x e_y y^{0\dagger} = f_x e_y$ for any x and y in S. In particular, if $v, w \in E$, then

$$(vw)^0 = v^0 w^0 = f_v e_w \text{ where } v^0, w^0 \in E \ ,$$

because E^0 is a multiplicative semilattice transveral of $\langle E\rangle$. It follows that

$$\begin{aligned} vw &= e_{vw}(vw)^0 f_{vw} = e_v(v^0 f_v e_w)^\dagger v^0 w^o (f_v e_w w^o)^* f_w && \text{(Lemma 4.1)} \\ &= e_v f_v e_w w^0 v^0 f_v e_w f_w && (v^0 w^0, f_v e_w \in E^0) \\ &= e_v f_v e_w f_v e_w f_w = e_v f_v e_w f_w = e_v v^0 w^0 f_w && (v^0 w^0 = f_v e_w) \\ &= e_v f_w \ . \end{aligned}$$

Now it is clear from above that we have

$$e_v f_w = e_w w^0 v^0 f_v = e_w f_w e_v f_v, e_v f_v e_v = e_v v^0 v^0 \text{ and } f_v e_v f_v = v^0 v^0 f_v = f_v \ .$$

Likewise, $e_w f_w e_w = e_w$ and $f_w e_w f_w = f_w$, and so

$$\begin{aligned} vw = e_v f_w &= e_v f_v e_v f_w e_w f_w = e_v f_w e_w f_v e_v f_w && (f_v e_v, f_w \in E^0) \\ &= e_v f_w e_w f_w e_v f_v e_v f_w && (e_w f_v = e_w f_w e_v f_v) \\ &= e_v f_w e_v f_w = (vw)(vw) \ . \end{aligned}$$

Thus vw is an idempotent and E is a band. Thus, S is a quasi-adequate semigroup. ■

If S is a quasi-adequate semigroup, that is if $E = \langle E\rangle$, we follow [9] and denote the $\mathcal{J}$-class in E of an idempotent e by $E(e)$. In this case, as in [6], we define the relation δ on S by the rule:

$$a \ \delta \ b \text{ if and only if } b = eaf \text{ for some } e \in E(a^\dagger), f \in E(a^*) \ .$$

Proposition 4.4. *Let S be a quasi-adequate semigroup. The function Ψ of Proposition 4.3 is a homomorphism from S into S^0 if and only if δ is a congruence on S. In this case,* Ker $\Psi = \delta$.

Proof. If δ is a congruence, then for any x and y in S, $x \ \delta \ x^0$ and $y \ \delta \ y^0$ we have $xy \ \delta \ x^0y^0$. But x^0y^0 and $(xy)^0 \in S^0$. Thus $(xy)^0 = x^0y^0$ and Ψ is a homomorphism.

Conversely, suppose that Ψ is a homomorphism and let x, y and c be in S such that $x \ \delta \ y$. Then $x^0 = y^0$ so that $x^0c^0 = y^0c^0$. Now because Ψ is a homomorphism, $(xc)^0 = (yc)^0$; that is, $xc \ \delta \ yc$. Likewise, $cx\delta cy$. Therefore, δ is a congruence.

Finally, if Ψ is a homomorphism and $x, y \in S$, then

$$(x, y) \in \text{Ker } \Psi \Leftrightarrow \mathrm{x}\Psi = \mathrm{y}\Psi \Leftrightarrow \mathrm{x}^0 = \mathrm{y}^0 \Leftrightarrow (\mathrm{x}, \mathrm{y}) \in \delta \ .$$

It now follows that if Ψ is a homomorphism, then S is a quasi-adequate semigroup on which δ is a congruence and thus δ is the minimum adequate good congruence on S [6]. Consider the mapping $\pi : S/\delta \to S$ defined by $(x\delta)\pi = x^0$. For any x and y in S,

$$(x\delta)\pi(y\delta)\pi = x^0y^0 = (xy)^0 = (xy)\delta\pi = (x\delta y\delta)\pi$$

so π is a homomorphism. Moreover, $x = e_x x^0 f_x$ and for any x^* there exists $(x^*)^0 \in E^0$ such that $x^* = e_{x^*}(x^*)^0 f_{x^*}$. Let $x^{0*} \in L^*_{x^0}(S) \cap E^0$. Then

$$x^{0*} \,\mathcal{R}\, f_x \,\mathcal{L}^*\, x \,\mathcal{L}^* f_{x^*} \,\mathcal{R}\, (x^*)^0 ,$$

and so $x^{0*} \,\mathcal{D}\, (x^*)^0$. But E is a band, and x^{0*}, $(x^*)^0 \in E^0$. Hence $x^{0*} = (x^*)^0$, and $(x^*\delta)\pi = (x^*)^0 = x^{0*} = ((x\delta)\pi)^*$. Similarly, for $x^\dagger$ we get $(x^\dagger)^0 = (x^0)^\dagger$ and $(x^\dagger\delta)\pi = ((x\delta)\pi)^\dagger$. It follows that for any x and y in S, if $x \,\mathcal{L}^*\, y$, then $(x^*\delta)\pi \,\mathcal{L}\, (y^*\delta)\pi$, but from above we have for any x^*, y^*, $(x^*\delta)\pi = ((x\delta)\pi)^*$, $(y^*\delta)\pi = ((y\delta)\pi)^*$. Therefore:

$$(x\delta)\pi \,\mathcal{L}^*\, ((x\delta)\pi)^* = (x^*\delta)\pi \,\mathcal{L}\, (y^*\delta)\pi = ((y\delta)\pi)^* \,\mathcal{L}^*\, (y\delta)\pi .$$

Similarly, $x \,\mathcal{R}^*\, y$ implies $(x\delta)\pi \,\mathcal{R}^*(y\delta)\pi$.

Hence, π is a splitting homomorphism and S is a split quasi-adequate semigroup [3].

Further, for any (g, a, h), (v, b, w) in W, we have $a^* \,\mathcal{R}\, h$, $b^\dagger \,\mathcal{L}\, v$ and $b^\dagger a^* \,\mathcal{L}\, va^* \,\mathcal{R}\, vh \,\mathcal{L}\, b^\dagger h$. But $\mathcal{D}$ is a congruence on E and the $\mathcal{D}$-classes in E are the rectangular bands in E. Therefore, $va^*b^\dagger hva^* = va^*$ so that $vhva^* = va^*$ which implies $b^\dagger vhva^* = b^\dagger va^*$ and thus $b^\dagger h \cdot va^* = b^\dagger a^*$. Then

$$\begin{aligned} ahvb &= aa^*hvb^\dagger b = a(a^*b^\dagger b^\dagger hva^*)b \\ &= a(b^\dagger hva^*)b = ab^\dagger a^*b \\ &= aa^*b^\dagger b = ab . \end{aligned}$$

Hence the products in W coincide with those in [3]. Therefore the result in this paper extends the result in the previous paper [3]. ∎

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